INTERMEDIATE SIMULTANEOUS INFERENCE PROCEDURES

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Institute of Statistics Mimeo Series No. 998

MARCH 1975

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ABSTRACT

Two well-known simultaneous inference procedures for the balanced one-way layout are Tukey's T-method and Scheffè's S-method. The T-method gives short simultaneous confidence intervals for all pairwise comparisons while paying a high price for high-order comparisons. The S-method is appropriate when one has uniform interest in all contrasts, but gives long confidence intervals for pairwise and other low-order comparisons. In this paper we propose a family of procedures which are intermediate between the T- and S-methods. The resulting simultaneous intervals are shorter than Scheffè's for low-order comparisons, shorter than Tukey's for high-order comparisons, and shorter than both Tukey's and Scheffè's for some intermediate comparisons of interest.

1. INTRODUCTION

Consider a balanced one-way layout with k groups and n observations on each group, with $\underset{\sim}{\mu} = (\mu_1,\ldots,\mu_k)'$ as the population vector of means, $\hat{\underset{\sim}{\mu}} = (\hat{\mu}_1,\ldots,\hat{\mu}_k)'$ as the corresponding sample estimates where $\hat{\underset{\sim}{\mu}} \sim N(\underset{\sim}{\mu}, \frac{\sigma^2}{n}\underset{\sim}{I})$, and s_ν^2 with $\nu = k(n-1)$ degrees of freedom as the usual unbiased estimator of σ^2. We use $\mathcal{C}^k$ to denote the contrast subspace of $\mathcal{R}^k$ and $\underset{\sim}{c} = (c_1,\ldots,c_k)'$ as a generic notation for a contrast in $\mathcal{C}^k$.

Two well-known simultaneous interval estimation procedures for all $\underset{\sim}{c}'\underset{\sim}{\mu}$ are:

Tukey's T-method:

$$\underset{\sim}{c}'\underset{\sim}{\mu} \in [\underset{\sim}{c}'\hat{\underset{\sim}{\mu}} \pm n^{-\frac{1}{2}} s_\nu q_{k,\nu}^{(\alpha)} T(\underset{\sim}{c})],$$

where $q_{k,\nu}^{(\alpha)}$ is the $(1-\alpha)$-th quantile of the studentized range distribution with parameters k and ν, and

$$T(\underset{\sim}{c}) = \tfrac{1}{2}\sum_i |c_i|.$$

Scheffe's S-method:

$$\underset{\sim}{c}'\underset{\sim}{\mu} \in \left[\underset{\sim}{c}'\hat{\underset{\sim}{\mu}} \pm n^{-\frac{1}{2}} s_\nu [(k-1)F_{k-1,\nu}^{(\alpha)}]^{\frac{1}{2}} S(\underset{\sim}{c})\right]$$

where $F_{k-1,\nu}^{(\alpha)}$ is the $(1-\alpha)$-th quantile of the central F distribution with $k-1$ and ν degrees of freedom and

$$S(\underset{\sim}{c}) = (\underset{\sim}{c}'\underset{\sim}{c})^{\frac{1}{2}}.$$

The T-method gives shorter confidence intervals than the S-method for all pairwise comparisons, but longer confidence intervals for most other contrasts. Note that the length of the confidence intervals obtained by the T- and S-methods is proportional to $T(\underset{\sim}{c})$ and $S(\underset{\sim}{c})$, respectively.

That the T-method favors the pairwise comparisons is clearly demonstrated by the fact that for any $d > 0$

$$\min_{\underset{\sim}{c} \in C^k : S(\underset{\sim}{c}) = d} \{T(\underset{\sim}{c})\} = d/\sqrt{2} ,$$

which is obtained when $\underset{\sim}{c}$ is a pairwise comparison.

As a test procedure, the S-method has superior average power when one has uniform interest in all alternatives (see Scheffè (1959) p. 48 and pp. 76-7), while the T-method is more resolvent (see Gabriel (1969)) than the S-method when the testing family consists only of the null hypotheses on subsets of treatments.

In spite of the confusion that seems to prevail in evaluating the various multiple comparison procedures (unistage, multistage, etc.), the existence of a variety of techniques from which the statistician can choose one to fit the nature of the problem at hand is essential. This variety of choice should ameliorate the discomfort felt by statisticians using multiple comparison procedures that do not sufficiently suit the nature of the loss functions associated with the problem at hand (see Anscombe (1965), Duncan (1965), O'Neill and Wetherill (1971)).

The well-known multistage multiple comparison procedures (see Miller (1966, Ch. 2)) provide considerable flexibility, as they can easily be modified to fit more closely one's interest in distributing error rates in any given problem. Such flexibility is not provided, however, by the simultaneous procedures. From our earlier discussion it is clear that problems for which the T- and S-methods would be appropriate are very distinct. In using the T-method one goes out of his way to obtain short confidence intervals for pairwise comparisons while paying a high price on other contrasts (see Scheffè (1959, Ch. 3)). On the other hand, the S-method fits situations where one has uniform interest in all contrasts, but gives long confidence intervals for pairwise and other low-order comparisons.

In this paper we propose a family of simultaneous inference procedures that are intermediate between the T- and S-methods. These procedures give confidence intervals shorter than the S-method (but longer than the T-method) for pairwise comparisons, shorter than the T-method (but longer than the S-method) for some high-order comparisons, and shorter than both the T- and S-methods for other comparisons of interest, such as some one-to-many comparisons. These properties depend on the specific procedure used. Since a whole range of procedures is developed, the proposed approach allows the statistician to distribute error rates in a way that fits more closely the nature of his loss function.

In Section 2 we give the theory needed, derive the family of intermediate procedures and discuss their application in generality. In Section 3 we apply the theory of Section 2 to obtain two important intermediate procedures and provide tables for their implementation.

2. INTERMEDIATE SIMULTANEOUS INFERENCE PROCEDURES

2.1. A Class of Intermediate Procedures.

The T- and S-methods are both special applications of Roy's Union-Intersection (UI) principle (see Gabriel (1970)). As test procedures they are obtained writing

$$H_0: \mu_1 = \cdots = \mu_k$$

as the following intersections:

T-method: $$H_0 = \bigcap_{1 \le i < j \le k} |\mu_i - \mu_j| = 0$$

S-method: $$H_0 = \bigcap_{\underset{\sim}{c} \in C^k} |\underset{\sim}{c}'\underset{\sim}{\mu}| = 0.$$

The overall statistic is defined as the maximum of all likelihood ratio test statistics for the individual hypotheses in the intersection set.

Let M be a subset of C^k that includes all pairwise comparisons. A wide class of intermediate procedures can be obtained by generating different subsets M and applying the UI principle with H_0 expressed as

$$H_0 = \bigcap_{\underset{\sim}{c} \in M} |\underset{\sim}{c}'\underset{\sim}{\mu}| = 0.$$

Let us define for each $\underset{\sim}{c} \in M$

$$u(\underset{\sim}{c}) = n^{\frac{1}{2}}\underset{\sim}{c}'(\hat{\underset{\sim}{\mu}} - \underset{\sim}{\mu})/\sigma S(\underset{\sim}{c}), \text{ and}$$

$$t(\underset{\sim}{c}) = u(\underset{\sim}{c})\frac{\sigma}{s_\nu} .$$

Then $|t(\underset{\sim}{c})|$ is the likelihood ratio test statistic for $H_0^{\underset{\sim}{c}}$: $|\underset{\sim}{c}'\underset{\sim}{\mu}| = 0$ and has a central $|t|$ distribution with ν degrees of freedom.

The test statistic $\xi(M)$ and critical value $\xi_\alpha(M)$ for an α-level simultaneous test of the family

$$\{H_0^{\underset{\sim}{c}}: |\underset{\sim}{c}'\underset{\sim}{\mu}| = 0: \underset{\sim}{c} \in M\}$$

are defined by

$$\xi(M) = \sup_{\underset{\sim}{c} \in M} |t(\underset{\sim}{c})|$$

and

$$\Pr\{|t(\underset{\sim}{c})| > \xi_\alpha(M) \text{ for at least one } \underset{\sim}{c} \in M | H_0\} = \alpha. \tag{2.1}$$

Simultaneous confidence intervals for all $\underset{\sim}{c}'\underset{\sim}{\mu}$ such that $\underset{\sim}{c} \in M$

are given by

$$\underset{\sim}{c}'\underset{\sim}{\mu} \in [\underset{\sim}{c}'\hat{\underset{\sim}{\mu}} \pm n^{-\frac{1}{2}} s_\nu \xi_\alpha(M) S(\underset{\sim}{c})]. \tag{2.2}$$

Note that

$$F^{(\alpha)}_{1,\nu} < \xi^2_\alpha(M) < (k-1)F^{(\alpha)}_{k-1,\nu}, \tag{2.3}$$

thus (2.2) provides simultaneous confidence intervals for contrasts $\underset{\sim}{c}'\underset{\sim}{\mu}$ with $\underset{\sim}{c} \in M$ which are shorter than those obtained by the S-method.

There are two difficulties in constructing these procedures. (1) obtaining the distribution of the test criterion, and (2) extending the resolution of the procedure from M to $C^k - M$.

The way in which these difficulties can be solved depends on the nature of the set M. We shall now discuss how they can be solved when M is a finite set and in particular when M consists of comparisons between the average of a subset of I means and the average of a disjoint set of J means, for some integers I,J. This approach leads to a meaningful class of intermediate simultaneous inference procedures.

2.2. Approximate Distribution of the Test Statistic Under H_0.

The first difficulty can be handled for any finite M containing distinct contrasts $\underset{\sim}{c}_i$, $i = 1,\ldots,m$ as follows. Let $u_i = u(\underset{\sim}{c}_i)$ and $t_i = t(\underset{\sim}{c}_i)$, and note that the test statistic is of the form

$$\xi(M) = \max_{1\le i\le m} |t_i|,$$

where the $|t_i|$ are dependent random variables, each distributed as a central $|t|$ under the null hypothesis.

We will approximate $\xi_\alpha(M)$ using a modified second-order Bonferroni approximation due to Siotani (1964). Let $\bar{\xi}_\alpha(M)$ be the first-order

Bonferroni approximation, i.e.

$$m \Pr\{|t_i| > \bar{\xi}_\alpha(M)\} = \alpha,$$

and put

$$\delta_\alpha(M) = \sum_{1\le i<j\le m} \Pr\{\min(|t_i|,|t_j|) > \bar{\xi}_\alpha(M)\}. \tag{2.4}$$

Then Siotani's second-order approximation $\hat{\xi}_\alpha(M)$ is given by

$$m \Pr\{|t_i| > \hat{\xi}_\alpha(M)\} = \alpha + \delta_\alpha(M). \tag{2.5}$$

Let $\bar{\bar{\xi}}_\alpha(M)$ be the second order Bonferroni approximation, i.e.

$$m \Pr\{|t_i| > \bar{\bar{\xi}}_\alpha(M)\} - \sum_{1\le i<j\le m} \Pr\{\min(|t_i|,|t_j|) > \bar{\bar{\xi}}_\alpha(M)\} = \alpha;$$

then it can be shown that (see Siotani, 1964)

$$\bar{\bar{\xi}}_\alpha(M) \le \hat{\xi}_\alpha(M) \le \bar{\xi}_\alpha(M).$$

Siotani (1964) has argued that in several applications his approximation should be closer to the true critical value than either the first- or second-order Bonferroni approximations. We have evaluated the approximation when M is the set of all pairwise comparisons, in which case the procedure reduces to the T-method and the exact distribution is known, and found that $\hat{\xi}_\alpha(M)$ was slightly conservative but closer to $\xi_\alpha(M)$ than either $\bar{\xi}_\alpha(M)$ or $\bar{\bar{\xi}}_\alpha(M)$.

To obtain the probabilities required in (2.4) we note that $(|t_i|,|t_j|)$ has a bivariate $|t|$ distribution (see Dunnett and Sobel (1954), for example) with parameters ν and $|\rho_{ij}|$ where

$$\rho_{ij} = \frac{\underset{\sim}{c}_i'\underset{\sim}{c}_j}{S(\underset{\sim}{c}_i)S(\underset{\sim}{c}_j)}$$

is the correlation between u_i and u_j.

Although the number $M = \frac{1}{2}m(m-1)$ of pairs of contrasts in M is usually large, the number D of distinct correlations may be reasonably small.

Let M_d be the number of pairs (i,j) for which $|\rho_{ij}| = \rho_d$, $d = 1,\ldots,D$.

Then $\sum_{d=1}^{D} M_d = M$ and

$$\delta_\alpha(M) = \sum_{d=1}^{D} M_d P_d$$

where P_d is the probability that the two components of a bivariate $|t|$ distribution with ν degrees of freedom and correlation ρ_d will both exceed $\bar{\xi}_\alpha(M)$.

Although tables of the bivariate $|t|$ distribution are available (Krishnaiah, et.al. 1969), we have preferred to write a computer program to do all necessary calculations. The program evaluates the bivariate $|t|$ distribution function using Gaussian quadrature formulae. A description and listing of this program are available from the authors.

2.3. Extension of the Resolution of the Procedure.

We now consider the second difficulty, namely extending the resolution of the procedure from M to $C^k - M$. Let $\underset{\sim}{c}(I,J)$ denote a contrast that is a comparison between the average of a set of I means and the average of a disjoint set of J means. These contrasts are sometimes called standard comparisons. The set of all I: J comparisons in C^k is denoted by G^k. This set usually contains most, if not all, contrasts of interest. Note that for a given I,J

$$S[\underset{\sim}{c}(I,J)] = (1/I + 1/J)^{\frac{1}{2}},$$

and does not depend on the particular I: J comparison.

We will first extend the resolution of the procedure from M to G^k. Let $P(\underset{\sim}{c})$ denote the set of indices i for which c_i is positive and $N(\underset{\sim}{c})$ the set for which c_i is negative. We say that $\underset{\sim}{c}_1$ dominates $\underset{\sim}{c}_2$

if and only if

$$\mathcal{P}(\underset{\sim}{c}_1) \supseteq \mathcal{P}(\underset{\sim}{c}_2) \quad \text{and} \quad \mathcal{N}(\underset{\sim}{c}_1) \supseteq \mathcal{N}(\underset{\sim}{c}_2).$$

Thus the 2:1 comparison $\frac{1}{2}(\mu_1 + \mu_2) - \mu_3$ dominates the 1:1 comparisons $\mu_1 - \mu_3$ and $\mu_2 - \mu_3$ but not $\mu_1 - \mu_2$.

For given I,J, let $\bar{I},\bar{J}$ be a generic notation for any integers such that

$$\min(\bar{I},\bar{J}) \le \min(I,J) \quad \text{and} \quad \max(\bar{I},\bar{J}) \le \max(I,J).$$

Let $\mathcal{D}_{\bar{I},\bar{J}}[\underset{\sim}{c}(I,J)]$ be the set of all $\underset{\sim}{c}(\bar{I},\bar{J})$ dominated by $\underset{\sim}{c}(I,J)$. Also let $\mathcal{D}_{\bar{I},\bar{J}}[\underset{\sim}{c}(J,I)]$ be the set of all $\underset{\sim}{c}(\bar{I},\bar{J})$ dominated by $-\underset{\sim}{c}(I,J)$.

For a given $\underset{\sim}{c}(I,J)$ and M let I^*,J^* be any two integers such that $I^* + J^* \le k$, $I^* \ge 1$, $J^* \ge 1$ and

$$\min_{\bar{I},\bar{J}}\{S[\underset{\sim}{c}(\bar{I},\bar{J})]: \text{ either } \mathcal{D}_{\bar{I},\bar{J}}[\underset{\sim}{c}(I,J)] \subseteq M \text{ or } \mathcal{D}_{\bar{I},\bar{J}}[\underset{\sim}{c}(J,I)] \subseteq M\}$$
$$= S[\underset{\sim}{c}(I^*,J^*)]. \tag{2.6}$$

The empty set is not considered to be contained in M. Since by definition M contains all pairwise comparisons, (2.6) is always well defined. Note that for any $\underset{\sim}{c}(I,J) \in G^k$ we let I^*,J^* be the pair of integers such that all $I^*: J^*$ comparisons dominated by $\underset{\sim}{c}(I,J)$ or $-\underset{\sim}{c}(I,J)$ are in M and $1/I^* + 1/J^*$ is a minimum.

<u>Lemma 2.1</u>. With probability at least $1 - \alpha$ we have simultaneously for all $\underset{\sim}{c}(I,J) \in G^k$

$$\underset{\sim}{c}'(I,J)\underset{\sim}{\mu} \in \left[\underset{\sim}{c}'(I,J)\hat{\underset{\sim}{\mu}} \pm n^{-\frac{1}{2}} s_\nu \xi_\alpha(M) S[\underset{\sim}{c}(I^*,J^*)]\right] \tag{2.7}$$

<u>Proof</u>: Put $\hat{\underset{\sim}{\mu}} - \underset{\sim}{\mu} = \underset{\sim}{y}$. With no loss in generality assume that the minimum in (2.6) takes place when

$$\mathcal{D}_{I^*,J^*}[\underset{\sim}{c}(I,J)] \subseteq \mathcal{M} .$$

Then, since we have

$$\underset{\sim}{c}'y = \frac{1}{\binom{I}{I^*}\binom{J}{J^*}} \sum_{\bar{\underset{\sim}{c}}:\bar{\underset{\sim}{c}}\in\mathcal{D}_{I^*,J^*}(I,J)} \bar{\underset{\sim}{c}}'\underset{\sim}{y}, \tag{2.8}$$

the lemma follows.

Thus each comparison $\underset{\sim}{c}(I,J)$ in $\mathcal{G}^k$ is written as an optimal linear combination of lower-order comparisons in $\mathcal{M}$ that are dominated by $\underset{\sim}{c}(I,J)$ or $-\underset{\sim}{c}(I,J)$. Note that if $\underset{\sim}{c}(\tilde{I},\tilde{J})$ is dominated by $\underset{\sim}{c}(I,J)$, the confidence interval for the former cannot be shorter than for the latter.

It seems reasonable to confine attention to procedures that for any given I: J give the same length of confidence interval for all $\underset{\sim}{c}(I,J)$. Also, if $\mathcal{G}^k$ is indeed the part of $\mathcal{C}^k$ of most interest we should restrict $\mathcal{M}$ to contain only pure I: J comparisons. Based on these considerations, we can now define I^* and J^* in (2.6) simply as those integers such that all I^*: J^* comparisons are in $\mathcal{M}$ and $1/I^* + 1/J^*$ is minimized under the restrictions

$$\min(I^*,J^*) \le \min(I,J) \quad \text{and} \quad \max(I^*,J^*) \le \max(I,J).$$

The method also provides confidence intervals for contrasts in $\mathcal{C}^k - \mathcal{G}^k$, since any general contrast $\underset{\sim}{c} = (c_1,\ldots,c_k)'$ can be written as a positive linear combination of I: J comparisons that are contained in $\mathcal{M}$.

An optimum linear combination may be found using the following algorithm. Find the largest order I: J comparison $\underset{\sim}{c}(I,J)_1$ in $\mathcal{M}$ that involves the same means as the contrast of interest $\underset{\sim}{c}$. (The largest order comparison is the one minimizing $1/I + 1/J$.) Let w_1 be the largest possible positive number such that none of the components in $w_1\underset{\sim}{c}(I,J)_1$

exceeds the corresponding component in $\underset{\sim}{c}$. Then obtain the new contrast $\underset{\sim}{c} - w_1\underset{\sim}{c}(I,J)_1$, and apply the same procedure to this contrast. This is repeated L times, say, until one gets

$$\underset{\sim}{c} = \sum_{i=1}^{L} w_i \underset{\sim}{c}(I,J)_i \ .$$

Then a confidence interval for $\underset{\sim}{c}'\underset{\sim}{\mu}$ is

$$\underset{\sim}{c}'\underset{\sim}{\mu} \in [\underset{\sim}{c}'\hat{\underset{\sim}{\mu}} \pm n^{-\frac{1}{2}} s_\nu \xi_\alpha(M) \sum_{i=1}^{L} S[\underset{\sim}{c}(I,J)_i]] \ .$$

For example suppose $k = 4$ and M contains all 1:1, all 2:1 and all 2:2 comparisons. The lengths of the confidence intervals for these comparisons are proportional to $\sqrt{2}$, $\sqrt{1.5}$ and $\sqrt{1}$ respectively. (The multiplier is $2n^{-\frac{1}{2}} s_\nu \xi_\alpha(M)$.) Suppose $\underset{\sim}{c} = (.6,.4,-.7,-.3)'$ and let $\hat{\underset{\sim}{\mu}} - \underset{\sim}{\mu} = \underset{\sim}{y}$. Then the algorithm described above gives

$$.6y_1 + .4y_2 - .7y_3 - .3y_4$$

$$= .6\left[\frac{y_1 + y_2}{2} - \frac{y_3 + y_4}{2}\right] + .2\left[\frac{y_1 + y_2}{2} - y_3\right] + .2\left[y_1 - y_3\right] .$$

The resulting LCI is thus proportional to

$$.6\sqrt{1} + .2\sqrt{1.5} + .2\sqrt{2} = 1.13 \ .$$

3. TWO INTERMEDIATE PROCEDURES.

3.1. A Procedure Based on All 1:1 and 1:2 Comparisons.

To illustrate the proposed approach suppose it is decided to let M be the set of all 1:1 and 1:2 comparisons. For $k \geq 3$ there are $\binom{k}{2}$ possible 1:1 comparisons and $3\binom{k}{3}$ possible 1:2 comparisons, so there are $m = k(k - 1)^2/2$ comparisons in all, and

$$M = \binom{m}{2} = k(k-1)^2(k-2)(k^2+1)/8$$

distinct pairs of comparisons.

These M pairs may be classified into 9 classes according to the correlation between $\underset{\sim}{c}_i$ and $\underset{\sim}{c}_j$, with frequencies as given in Table 3.1.

TABLE 3.1. CORRELATIONS AND FREQUENCIES FOR FIRST INTERMEDIATE PROCEDURE

d	ρ_d	M_d
1	0	$3\binom{k}{3} + 3\binom{k}{4} + 30\binom{k}{5} + 90\binom{k}{6}$
2	$\frac{1}{6}$	$24\binom{k}{4} + 60\binom{k}{5}$
3	$\frac{1}{\sqrt{12}}$	$24\binom{k}{4}$
4	$\frac{1}{3}$	$6\binom{k}{4} + 60\binom{k}{5}$
5	$\frac{1}{2}$	$6\binom{k}{3}$
6	$\frac{1}{\sqrt{3}}$	$12\binom{k}{4}$
7	$\frac{2}{3}$	$12\binom{k}{4} + 15\binom{k}{5}$
8	$\frac{5}{6}$	$12\binom{k}{4}$
9	$\sqrt{\frac{3}{4}}$	$6\binom{k}{3}$

To compute $\hat{\xi}_\alpha$ for a given k and n one needs to compute 9 bivariate $|t|$ probabilities. Tables A.1 and A.2 in the appendix give values of $\hat{\xi}_\alpha$ for the procedure based on 1:1 and 1:2 comparisons for $\alpha = .10$ and .05, $k = 4(1)10$ and $n = 2(1)20$. In a few cases the approximation given in the tables is known to be too conservative, as it exceeds the critical value for the S-method (see eq. 2.3). These cases are indicated with a star in the tables.

Simultaneous confidence intervals for all I: J comparisons are given by

$$\underset{\sim}{c}'(I,J)\underset{\sim}{\mu} \in \left[\underset{\sim}{c}'(I,J)\hat{\underset{\sim}{\mu}} \pm n^{-\frac{1}{2}} s_\nu \xi_\alpha(M) S[c(I^*,J^*)]\right],$$

where $S[\underset{\sim}{c}(I^*,J^*)] = S[\underset{\sim}{c}(1,1)] = \sqrt{2}$ for all 1:1 comparisons and $S[\underset{\sim}{c}(I^*,J^*)] = S[\underset{\sim}{c}(1,2)] = \sqrt{3/2}$ for all other (higher order) comparisons.

Let us now consider the application of this procedure in an example with $k = 8$ treatments, $n = 6$ observations per cell and $\alpha = .10$. From Table A.1, $\hat{\xi}_{.10} = 3.2135$. In Table 3.2 we compare the lengths of the confidence intervals obtained in this example using the T-, intermediate and S-methods for several comparisons of interest. The factors shown should be multiplied by $2n^{-\frac{1}{2}} s_\nu$ to obtain actual lengths of confidence intervals. The critical values used for the T- and S-methods are $q_{8,40}^{(.10)} = 4.10$ and $F_{7,40}^{(.10)} = 1.87$.

TABLE 3.2. COMPARISON OF LENGTHS OF CONFIDENCE INTERVALS OBTAINED BY THREE METHODS

Comparison	Tukey	Intermediate	Scheffè
1:1	4.10	4.54	5.12
1:2	4.10	3.94	4.43
1:3	4.10	3.94	4.18
1:4	4.10	3.94	4.05
1:5	4.10	3.94	3.96
1:6	4.10	3.94	3.91
1:7	4.10	3.94	3.87
2:2	4.10	3.94	3.62
2:3	4.10	3.94	3.30
2:4	4.10	3.94	3.13

In this case the proposed procedure is better than the T- and S-methods for the 1:2, 1:3, 1:4 and 1:5 comparisons and is intermediate in the other cases. The T-method is best only for 1:1 comparisons while the S-method is best for comparisons of order higher than 1:5. (Note also that T is better than S up to order 1:3, from there on S is better than T.) Thus the intermediate procedure would be recommended in this case if the statistician is interested mainly in 1:1 and one-to-many comparisons.

3.2. A Procedure Based on All 1:1 and 2:2 Comparisons.

An alternative type of intermediate procedure is obtained by letting $\mathcal{M}$ be the set of all 1:1 and 2:2 comparisons. For $k \geq 4$ there are $\binom{k}{2}$ possible 1:1 comparisons and $3\binom{k}{4}$ possible 2:2 comparisons. Hence $m = k(k-1)[k^2 - 5k + 10]/8$.

The $M = \binom{m}{2}$ pairs of comparisons may be classified into 6 classes according to their correlation, with frequencies as given in Table 3.3.

TABLE 3.3. CORRELATIONS AND FREQUENCIES FOR SECOND INTERMEDIATE PROCEDURE

d	ρ_δ	M_d
1	0	$12\binom{k}{4} + 225\binom{k}{6} + 315\binom{k}{8}$
2	$\frac{1}{4}$	$60\binom{k}{5} + 630\binom{k}{7}$
3	$\frac{1}{\sqrt{8}}$	$60\binom{k}{5}$
4	$\frac{1}{2}$	$3\binom{k}{3} + 225\binom{k}{6}$
5	$\frac{1}{\sqrt{2}}$	$12\binom{k}{4}$
6	$\frac{3}{4}$	$30\binom{k}{5}$

Tables A.3 and A.4 in the appendix give values of $\hat{\xi}_\alpha$ for this procedure for $\alpha = .10$ and $.05$, $k = 4(1)10$ and $n = 2(1)20$. Again a few cases where the approximation is too conservative are starred.

Simultaneous confidence intervals for all I: J comparisons are given by

$$\underset{\sim}{c}'(I,J)\underset{\sim}{\mu} \in \left[\underset{\sim}{c}'(I,J)\hat{\underset{\sim}{\mu}} \pm n^{-\frac{1}{2}} s_\nu \xi_\alpha(M) S[\underset{\sim}{c}(I^*,J^*)]\right],$$

where $S[\underset{\sim}{c}(I^*,J^*)] = S[\underset{\sim}{c}(1,1)] = \sqrt{2}$ for 1:1 or one-to-many comparisons and $S[\underset{\sim}{c}(I^*,J^*)] = S[\underset{\sim}{c}(2,2)] = \sqrt{1}$ for all other comparisons.

Let us consider the performance of this procedure in an example with $k = 6$, $n = 6$ and $\alpha = .10$. From Table A.3 $\hat{\xi}_{.10} = 3.0239$. Critical values for the T- and S-methods are $q_{6,30}^{(.10)} = 3.85$ and $F_{5,30}^{(.10)} = 2.05$. Table 3.4 shows a comparison of the lengths of the confidence intervals for all I: J comparisons. The factors shown should be multiplied by $2n^{-\frac{1}{2}} s_\nu$ to obtain actual lengths of confidence intervals.

TABLE 3.4. COMPARISON OF LENGTHS OF CONFIDENCE INTERVALS OBTAINED BY THREE METHODS

Comparison	Tukey	Intermediate	Scheffè
1:1	3.85	4.28	4.53
1:2	3.85	4.28	3.92
1:3	3.85	4.28	3.70
1:4	3.85	4.28	3.58
1:5	3.85	4.28	3.51
2:2	3.85	3.02	3.20
2:3	3.85	3.02	2.67
2:4	3.85	3.02	2.77
3:3	3.85	3.02	2.61

In this case the proposed procedure is better than the T- and S-methods for the 2:2 comparisons and is intermediate in all other cases. The T-method is best for 1:1 and 1:2 comparisons while the S-method is best for 1:3, 1:4, 1:5 and comparisons of order higher than 2:2, (note also that T is better than S for 1:1 and 1:2, from there on S is better). Thus the intermediate procedure might be recommended in this case if the statistician is particularly interested in 1:1 and 2:2 comparisons.

ACKNOWLEDGEMENT

The authors are grateful to Professor D. Quade of the Biostatistics Department of the University of North Carolina for several suggestions for improving the original manuscript.

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TABLE A.1

CRITICAL VALUES FOR INTERMEDIATE SIMULTANEOUS INFERENCE PROCEDURE
(BASIC SET: 1-1 AND 1-2 COMPARISONS)

$\alpha = 0.10$

OBSERVATIONS PER CELL	TREATMENTS						
	4*	5	6	7	8	9	10
2.	3.7240	3.6995	3.7042	3.7238	3.7500	3.7769	3.8044
3.	3.0454	3.1753	3.2726	3.3514	3.4185	3.4753	3.5185
4.	2.8527	3.0134	3.1318	3.2253	3.3035	3.3684	3.4251
5.	2.7626	2.9356	3.0631	3.1638	3.2470	3.3161	3.3759
6.	2.7104	2.8899	3.0225	3.1267	3.2135	3.2851	3.3470
7.	2.6763	2.8598	2.9954	3.1024	3.1906	3.2647	3.3280
8.	2.6523	2.8385	2.9763	3.0849	3.1743	3.2495	3.3138
9.	2.6345	2.8226	2.9619	3.0716	3.1620	3.2376	3.3020
10.	2.6207	2.8102	2.9506	3.0612	3.1531	3.2284	3.2955
11.	2.6098	2.8004	2.9418	3.0533	3.1457	3.2220	3.2892
12.	2.6009	2.7924	2.9346	3.0464	3.1391	3.2168	3.2829
13.	2.5935	2.7857	2.9285	3.0410	3.1342	3.2109	3.2789
14.	2.5873	2.7801	2.9234	3.0363	3.1299	3.2072	3.2752
15.	2.5820	2.7753	2.9190	3.0324	3.1257	3.2040	3.2728
16.	2.5774	2.7711	2.9152	3.0288	3.1227	3.2013	3.2681
17.	2.5734	2.7675	2.9118	3.0256	3.1199	3.1986	3.2656
18.	2.5698	2.7643	2.9089	3.0230	3.1175	3.1967	3.2642
19.	2.5667	2.7614	2.9063	3.0205	3.1153	3.1932	3.2619
20.	2.5639	2.7589	2.9040	3.0185	3.1134	3.1914	3.2605

*All critical values in this column are too conservative and exceed the corresponding value for the S-method.

TABLE A.2

CRITICAL VALUES FOR INTERMEDIATE SIMULTANEOUS INFERENCE PROCEDURE
(BASIC SET: 1-1 AND 1-2 COMPARISONS)

$\alpha = 0.05$

OBSERVATIONS PER CELL	TREATMENTS						
	4	5	6	7	8	9	10
2.	4.3103	4.1529	4.1083	4.1058	4.1191	4.1366	4.1533
3.	3.5318*	3.6135	3.6767	3.7297	3.7770	3.8187	3.8522
4.	3.2491*	3.3794	3.4749	3.5503	3.6142	3.6644	3.7088
5.	3.1195*	3.2686	3.3785	3.4649	3.5368	3.5958	3.6448
6.	3.0452	3.2045	3.3219	3.4139	3.4901	3.5517	3.6062
7.	2.9971	3.1624	3.2845	3.3802	3.4584	3.5263	3.5779
8.	2.9633	3.1327	3.2578	3.3565	3.4371	3.5055	3.5597
9.	2.9384	3.1106	3.2380	3.3384	3.4225	3.4905	3.5447
10.	2.9192	3.0937	3.2231	3.3243	3.4089	3.4776	3.5353
11.	2.9040	3.0801	3.2107	3.3137	3.3977	3.4698	3.5251
12.	2.8915	3.0691	3.2010	3.3041	3.3907	3.4597	3.5188
13.	2.8813	3.0599	3.1925	3.2970	3.3836	3.4546	3.5149
14.	2.8726	3.0523	3.1856	3.2910	3.3772	3.4501	3.5068
15.	2.8653	3.0457	3.1797	3.2849	3.3726	3.4457	3.5027
16.	2.8589	3.0398	3.1744	3.2805	3.3682	3.4395	3.5009
17.	2.8534	3.0349	3.1699	3.2764	3.3650	3.4369	3.4972
18.	2.8484	3.0305	3.1660	3.2727	3.3618	3.4342	3.4959
19.	2.8441	3.0266	3.1625	3.2697	3.3572	3.4315	3.4933
20.	2.8402	3.0232	3.1594	3.2668	3.3553	3.4295	3.4865

TABLE A.3

CRITICAL VALUES FOR INTERMEDIATE SIMULTANEOUS INFERENCE PROCEDURE
(BASIC SET: 1-1 AND 2-2 COMPARISONS)

$\alpha = 0.10$

OBSERVATIONS PER CELL	TREATMENTS						
	4	5	6	7	8	9	10
2.	3.6412*	3.6994	3.7568	3.7948	3.8250	3.8557	3.8876
3.	2.9205	3.1243	3.2768	3.3929	3.4877	3.5691	3.6411
4.	2.7283	2.9594	3.1336	3.2688	3.3790	3.4723	3.5477
5.	2.6398	2.8812	3.0646	3.2089	3.3268	3.4272	3.5096
6.	2.5889	2.8355	3.0239	3.1729	3.2948	3.3980	3.4863
7.	2.5558	2.8055	2.9969	3.1492	3.2746	3.3816	3.4736
8.	2.5327	2.7843	2.9779	3.1322	3.2597	3.3672	3.4603
9.	2.5155	2.7686	2.9636	3.1194	3.2479	3.3571	3.4513
10.	2.5023	2.7564	2.9526	3.1095	3.2385	3.3480	3.4408
11.	2.4919	2.7467	2.9437	3.1017	3.2320	3.3430	3.4387
12.	2.4833	2.7388	2.9365	3.0950	3.2267	3.3390	3.4350
13.	2.4763	2.7322	2.9306	3.0897	3.2210	3.3325	3.4262
14.	2.4703	2.7266	2.9255	3.0852	3.2171	3.3301	3.4252
15.	2.4653	2.7219	2.9211	3.0814	3.2137	3.3266	3.4231
16.	2.4609	2.7178	2.9174	3.0780	3.2111	3.3243	3.4221
17.	2.4570	2.7143	2.9141	3.0748	3.2084	3.3220	3.4146
18.	2.4537	2.7111	2.9113	3.0723	3.2053	3.3184	3.4132
19.	2.4507	2.7083	2.9087	3.0701	3.2033	3.3162	3.4146
20.	2.4480	2.7058	2.9063	3.0679	3.2017	3.3156	3.4124

TABLE A.4

CRITICAL VALUES FOR INTERMEDIATE SIMULTANEOUS INFERENCE PROCEDURE
(BASIC SET: 1-1 AND 2-2 COMPARISONS)

$\alpha = 0.05$

OBSERVATIONS PER CELL	TREATMENTS						
	4	5	6	7	8	9	10
2.	4.5019*	4.4227	4.3886	4.3606	4.3418	4.3326	4.3394
3.	3.4014	3.5626	3.6831	3.7754	3.8512	3.9145	3.9806
4.	3.1229	3.3264	3.4792	3.5977	3.6934	3.7654	3.8417
5.	2.9968	3.2162	3.3828	3.5142	3.6213	3.7014	3.7843
6.	2.9251	3.1524	3.3263	3.4640	3.5760	3.6661	3.7441
7.	2.8788	3.1108	3.2891	3.4309	3.5469	3.6404	3.7279
8.	2.8465	3.0815	3.2628	3.4077	3.5267	3.6212	3.7130
9.	2.8226	3.0598	3.2432	3.3902	3.5110	3.6077	3.6960
10.	2.8043	3.0430	3.2280	3.3763	3.4978	3.5941	3.6832
11.	2.7898	3.0297	3.2158	3.3659	3.4887	3.5878	3.6842
12.	2.7780	3.0188	3.2061	3.3567	3.4792	3.5782	3.6732
13.	2.7682	3.0098	3.1979	3.3495	3.4744	3.5759	3.6681
14.	2.7600	3.0022	3.1908	3.3436	3.4698	3.5704	3.6684
15.	2.7530	2.9957	3.1850	3.3375	3.4641	3.5680	3.6542
16.	2.7469	2.9901	3.1799	3.3328	3.4589	3.5596	3.6531
17.	2.7417	2.9852	3.1755	3.3291	3.4556	3.5564	3.6548
18.	2.7370	2.9809	3.1714	3.3256	3.4528	3.5555	3.6542
19.	2.7329	2.9771	3.1679	3.3226	3.4499	3.5532	3.6516
20.	2.7292	2.9736	3.1648	3.3198	3.4482	3.5520	3.6486