A FAST MULTILEVEL ALGORITHM FOR THE SOLUTION OF NONLINEAR SYSTEMS OF CONDUCTIVE-RADIATIVE HEAT TRANSFER EQUATIONS

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Abstract. In this paper we describe and analyze a fast multilevel algorithm for the solution of a system of nonlinear integro-differential equations that model steady-state combined conductive-radiative heat transfer. This system of equations for radiative intensity and temperature can be formulated as a compact fixed point problem in temperature alone with a fixed point map that requires both a solution of the linear transport equation and the linear heat equation for its evaluation. We obtain an efficient evaluation of the fixed point map by coupling a finite element diffusion solver with a fast transport solver developed by the second author. As a solver we apply a modification of the Atkinson-Brakhage method, with Newton-GMRES as the coarse mesh solver, to the full nonlinear system. We compare our discretization/solver pair with Newton-GMRES and the classical Atkinson-Brakhage algorithm.

Key words. conductive-radiative heat transfer, multilevel algorithm, compact fixed point problems

AMS subject classifications. 45G10, 45L10, 65H10, 65J15, 82A70

1. Introduction. The purpose of this paper, an expanded version of [3], is to describe and analyze a fast solver for a system of nonlinear integro-differential equations that model steady-state combined conductive-radiative heat transfer. Such models arise in the study of porous materials such as fibers, powders, and foams used in insulations at either very low or very high temperatures, heat treatment of ceramics, and the thermal properties of coated materials, [21], [28], [31]. We refer to [21], [28], and [31] for derivation of the equations. The equations can be formulated as a compact fixed point problem with a fixed point map that requires both a solution of the linear transport equation and the linear heat equation for its evaluation.

By using a weak formulation and evaluating certain projections exactly, we create a norm-convergent sequence of approximate maps. We obtain an efficient evaluation of these fixed point maps by coupling a finite element diffusion solver with a fast transport solver developed by the second author [16], [14]. We can then apply the fast algorithm from [14] to solve the full system. This algorithm requires only one fine mesh fixed point map evaluation at each level if used in a nested iteration. We compare this approach to two more classical methods, Newton-GMRES [15], and the Atkinson-Brakhage [2], [4] algorithm. These two methods require less of the sequence of approximate maps and we apply them using a different and less expensive discretization. However, both the Newton-GMRES and Atkinson-Brakhage solvers require at least two fine mesh function evaluations at each level and therefore one would expect them to be more expensive overall than the approach proposed here. The numerical results in § 4 confirm this. Any of these solvers is more efficient and robust than the relaxation approaches suggested in [21], [28], and [30], and more general than the spline-based approach suggested in [30] and [29].

1.1. The Continuous Problem. We consider the normalized and dimensionless form of the equations for conductive-radiative heat transfer, [21], [29], [30]. The unknowns are the dimension-
less intensity $\psi(x, \mu)$ of radiation at a point $x$ in the direction having cosine $\mu$ with the positive $x$ axis and the dimensionless temperature $\Theta(x)$ at $x$.

The radiative transport equation is

\begin{equation}
\mu \frac{\partial \psi}{\partial x}(x, \mu) + \psi(x, \mu) = \frac{c(x)}{2} \int_{-1}^{1} \psi(x, \mu') d\mu' + (1 - c(x))\Theta^4(x),
\end{equation}

for $x \in (0, \tau)$ with boundary conditions

\begin{equation}
\psi(0, \mu) = \varepsilon_l \Theta^4_l + \rho^s_l \psi(0, -\mu) + 2\rho^d_l \int_0^1 \psi(0, -\mu') d\mu', \mu > 0
\end{equation}

and

\begin{equation}
\psi(\tau, \mu) = \varepsilon_r \Theta^4_r + \rho^s_r \psi(\tau, -\mu) + 2\rho^d_r \int_0^1 \psi(\tau, -\mu') d\mu', \mu < 0.
\end{equation}

In the boundary conditions (1.2) and (1.3) $\varepsilon_i \geq 0$ for $i = l, r$. The coefficients for specular ($\rho^s_i, \rho^s_r$) and diffuse ($\rho^d_i, \rho^d_r$) reflection at the left and right boundaries satisfy

\begin{equation}
\rho^d_i, \rho^s_i \geq 0 \text{ and } \varepsilon_i + \rho^s_i + \rho^d_i = 1 \text{ for } i = l, r.
\end{equation}

The local albedo $c(x)$ satisfies

\begin{equation}
0 \leq c(x) \leq 1 \text{ for all } x \in [0, \tau].
\end{equation}

In this paper we will assume that $c$ is continuous. The results for piecewise continuous $c$ are the same provided that spatial mesh points are placed at discontinuities of $c$.

Now let

\begin{equation}
f(x) = \frac{1}{2} \int_{-1}^{1} \psi(x, \mu') d\mu'
\end{equation}

be the scalar flux. It is well known that (1.1) is equivalent to a linear compact fixed point problem for $f$

\begin{equation}
f - Lf = M(\Theta^4) + g
\end{equation}

where $L$ and $M$ are compact operators on $C[0, \tau]$ and $g \in C[0, \tau]$ depends on the boundary data. The operators $L$ and $M$ are related by

\begin{equation}
Lw = K(cw) \text{ and } Mw = K((1 - c)w),
\end{equation}

where $K$ is a compact operator on $C[0, \tau]$. Because of this relation between $L$ and $M$, the analysis of the convergence properties of the discretizations will be the same for both operators and in § 2 we need consider only $L$ and its discretizations.

As an example, consider the case of Dirichlet boundary conditions ($\rho^d_i = \rho^s_i = 0$ for $i = l, r$). In this case

\begin{equation}
L(f)(x) = K(cf)(x) = \int_0^\tau k(x, y)c(y)f(y) dy
\end{equation}

where

\begin{equation}
k(x, y) = \frac{1}{2} \int_0^1 \exp(|x - y|/\mu) \frac{d\mu}{\mu},
\end{equation}
We refer the reader to [7] or [5] for the details.

The temperature $\Theta$ satisfies the boundary value problem

$$\frac{\partial^2 \Theta}{\partial x^2} = Q(x), \ x \in [0, \tau], \ \Theta(0) = \Theta_0, \ \Theta(\tau) = \Theta_r,$$

and couples to the radiative transport equation by

$$Q(x) = \frac{1}{2N_c} \frac{d}{dx} \int_{-1}^{1} \mu' \psi(x, \mu') \, d\mu'.$$

In (1.9) $N_c$ is the conduction to radiation parameter [21].

The order of integration and differentiation in (1.9) can be changed [17] and we obtain, using the transport equation (1.1) and the definition of $f$, (1.6),

$$\frac{d}{dx} \int_{-1}^{1} \mu' \psi(x, \mu') \, d\mu' = \int_{-1}^{1} \mu' \frac{\partial}{\partial x} \psi(x, \mu') \, d\mu'$$

$$= -2(1 - c(x))f(x) + 2(1 - c(x))\Theta^4(x).$$

Hence, 

$$Q(x) = \alpha(x)(\Theta^4(x) - f(x)), \ 0 < x < \tau$$

where

$$\alpha(x) = (1 - c(x))/N_c.$$

**1.2. Formulation as a Fixed Point Problem.** The full system, (1.1),(1.2), (1.3), (1.8), can be expressed as a fixed point problem

$$\Theta = \mathcal{T}(\Theta)$$

in $\Theta$ alone [17]. The fixed point map $\Theta \rightarrow \mathcal{T}(\Theta)$ is computed as follows:

1. Solve the transport problem (1.1), (1.2), (1.3) and obtain $f$. This problem has a unique solution for any $\Theta$ because of (1.5) and (1.4), [5].
2. Use $\Theta$ and $f$ to form the right hand side of (1.8) using (1.11).
3. Solve $T_{xx} = Q$ subject to the boundary conditions from (1.8) and set $\mathcal{T}(\Theta) = T$.

Let $\mathcal{G}$ be the affine map that takes $Q$ to the solution of the boundary problem (1.8), and let $\mathcal{S}$ be the affine map that takes a function $u$ to

$$f(x) = \frac{1}{2} \int_{-1}^{1} \psi(x, \mu') \, d\mu'$$

where

$$\mu' \frac{\partial}{\partial x} \psi(x, \mu) + \psi(x, \mu) = \frac{c(x)}{2} \int_{-1}^{1} \psi(x, \mu') \, d\mu' + (1 - c(x))u(x)$$

for $x \in (0, \tau)$ with boundary conditions (1.2) and (1.3). Using the notation of (1.7) we have

$$\mathcal{S}(u) = (I - \mathcal{L})^{-1} (Mu + g)$$

and

$$\mathcal{T}(\Theta) = \mathcal{G}(\alpha[I - \mathcal{S}]\Theta^4),$$
where $\alpha$ is defined by (1.12).

The map $\mathcal{T}$ is a completely continuous map on $C[0, \tau]$, [17], and it is natural to apply fast algorithms for compact fixed point problems, such as Newton-GMRES (fast because of the compactness of $\mathcal{T}$'s [24], [6]) or one of the multilevel methods from [2], [4], [11], [16], or [14]. The most efficient of these, when applicable, is the algorithm described in [14] and [16], which requires only one fine mesh fixed point map evaluation at each level. In order to apply that algorithm the discretization and fine-to-coarse intergrid transfers must be managed carefully so that the Fréchet derivatives of the approximate maps converge in the operator norm. We show how to do that in § 2 and compare that approach in § 4 with two other algorithms, Newton-GMRES and the classical Atkinson-Brakhage method, both of which require less of the sequence of approximate maps.

2. Discretizations. In this section we briefly review the discretization of the transport equation from [26] and the solver from [16], which we use for step 1 of the computation of $\mathcal{T}$. We then show how finite element (finite difference) approximations for the diffusion equation in step 3 lead to operator norm (strongly) convergent sequences of approximations to $\mathcal{T}'$.

2.1. Discretization of the Transport Equation. We will state our convergence results for $L$ and its approximations with the understanding that similar results hold for $\tilde{M}$. We begin with a linear transport boundary value problem.

\[
\mu \frac{\partial \psi}{\partial x}(x, \mu) + \psi(x, \mu) = \frac{c(x)}{2} \int_{-1}^{1} \psi(x, \mu') \, d\mu' + q(x)
\]

for $x \in (0, \tau)$ with boundary conditions

\[
\psi(0, \mu) = \varepsilon_l q_l + \rho_l \psi(0, -\mu) + 2 \mu \int_{0}^{1} \psi(0, -\mu') \, d\mu', \mu > 0
\]

and

\[
\psi(\tau, \mu) = \varepsilon_r q_r + \rho_r \psi(\tau, -\mu) + 2 \mu \int_{0}^{1} \psi(\tau, -\mu') \, d\mu', \mu < 0.
\]

This boundary value problem can be expressed as a linear integral equation for $f$ using the notation of (1.7). We seek to solve

\[
f - L f = K q + g
\]

where $g$ depends on the boundary conditions and $q$ is the source. Compactness of $L$ will allow the use of fast solvers such as those in [2] or [14] for the transport problem.

As in both [20] and [26], we use a discrete ordinates approximation to the transport equation. This means that we discretize the integrals in (1.13), (1.2), and (1.3) with a standard quadrature rule and then solve the resulting two-point boundary value problem in the spatial variable. We will use a double Gaussian quadrature rule on $[-1, 1]$, which is the optimal choice [26], and let $\{\mu_j\}_{j=1}^{\nu}$ and $\{w_j\}_{j=1}^{\nu}$ denote the nodes and weights. The resulting semi-discrete (i.e. discrete only in angle) transport equation is

\[
\mu_j \frac{\partial \psi_j}{\partial x}(x) + \psi_j(x) = S(x)
\]

where

\[
S(x) = \frac{c(x)}{2} \sum_{k=1}^{\nu} \psi_k(x) w_k + q(x),
\]
and 
\[ \psi_j(x) \approx \psi(x, \mu_j). \]

We will let \( L(\nu) \) and \( g(\nu) \) denote the approximations to the operator \( L \) and the function \( g \) in (1.14) that result from solving this semi-discrete form of the transport equation. We refer the reader to [20], [16], or [26] for the details as to how this semi-discrete system and the fully discrete systems described below are actually solved. As an example, in the case of Dirichlet boundary conditions \( (\rho_i^t = \rho_i^s = 0 \text{ for } i = l, r) \) we have

\[ (L(\nu)f)(x) = \int_0^\tau k_\nu(x, y)c(y)f(y) \, dy \]

and

\[ k_\nu(x, y) = \frac{1}{2} \sum_{j=1}^\nu \exp(|x - y|/\mu_j)w_j/\mu_j. \]

The approach in [26] differs from previous discrete ordinates approaches such as [20] in its weak formulation of the approximate equation. The intensities \( \psi_j \) are taken to be piecewise linear functions with nodes at \( \{x_i\} \). The discrete transport equation is

\[ \int_{x_i}^{x_{i+1}} \mu_j \psi_j(x) + \psi_j(x) \, dx = \int_{x_i}^{x_{i+1}} S(x) \, dx \]  

for all \( 1 \leq i \leq N \). (2.6) is the same as the classical diamond difference approximation if the forcing term \( q \) is also a piecewise linear function with nodes at \( \{x_i\} \). If, however, \( q \) is an arbitrary continuous function or even a piecewise linear function on a finer mesh, as it can be if multilevel methods of the type in [16] or [14] are used, it is different in that the cell averages on the right hand side of (2.6) are computed exactly while they are approximated by the trapezoid rule in the classical approach. Note that the map from \( q \to f \) determined by this discretization is a continuous affine map on \( C[0, \tau] \); this fact makes it easy to state convergence results. If \( c \) is not piecewise constant with jumps at the nodes \( \{x_i\} \), these discretizations approximate \( c \) on \( (x_{i-1}, x_i) \) by \( .5(c(x_i) + c(x_{i-1})) \). We do not make this approximation explicit. As pointed out in [16], continuity of \( c \) and the error estimates in [26] imply that this approximation preserves all of the convergence properties that would be obtained if the integrals involving \( c \) were done exactly.

Let \( L(N, \nu) \) and \( g(N, \nu) \) denote the approximations to \( L \), and \( g \) that arise from (2.6). Convergence of \( g(N, \nu) \) to \( g \) is trivial. Operator norm convergence of \( L(N, \nu) \) is proved in [26] using the fact that the cell averages are computed exactly. We let \( \text{COM}(L^2[0, \tau], C[0, \tau]) \) denote the space of compact operators from \( L^2[0, \tau] \) to \( C[0, \tau] \). The theorem we will use here is

**Theorem 2.1.** Let the spatial mesh be equally spaced with \( x_0 = 0 \) and \( x_N = \tau \). Let the angular mesh be double Gaussian quadratures on \([-1, 1]\) with \( \nu \) nodes. Then, in the norm of \( \text{COM}(L^2[0, \tau], C[0, \tau]) \) (and hence also in the uniform operator topology on \( C[0, \tau] \)),

\[ \lim_{N, \nu \to \infty} L(N, \nu) = L, \]

and, for fixed \( \nu \)

\[ \lim_{N \to \infty} L(N, \nu) = L(\nu), \]
As indicated above, the difference between the algorithm from [26] and the traditional approach analyzed in [20] is the way in which the cell averages
\[ \int_{x_i}^{x_{i+1}} q(x) \, dx \]
are computed. In the work of [26] the view is taken that they are computed exactly. In the classical approach the cell averages are approximated by the trapezoid rule. We will require yet another method, namely to project \( q \) onto the space of piecewise linear functions before computing the cell averages of the result exactly. Our reason for this is to make the discretization of the transport equation from [26], which projects the source term onto the space of piecewise constants, fully compatible with a finite element solution of (1.8), which projects the right side onto the space of piecewise linear functions.

Let \( P_N \) be the \( L^2 \) projection onto the piecewise linear functions based on the mesh \( \{x_i\}_{i=1}^N \). Our new discretization simply replaces \( S \) by \( SP_N \) in (2.6). This does not change the operator \( L(N, \nu) \) or the function \( g(N, \nu) \), but only changes \( q \) to \( P_N q \). Theorem 2.2 states that the convergence properties are unchanged.

**Theorem 2.2.** Let the spatial mesh be equally spaced with \( x_0 = 0 \) and \( x_N = \tau \). Let the angular mesh be double Gaussian quadratures on \([-1, 1]\) with \( \nu \) nodes. Then, in the norm of \( C^0(M) \) (and hence also in the uniform operator topology on \( C[0, \tau] \)),

\[ \lim_{N, \nu \to \infty} L(N, \nu) P_N = L \]

and, for fixed \( \nu \)

\[ \lim_{N \to \infty} L(N, \nu) P_N = L(\nu). \]

**Proof.** Our proof does not depend on the angular discretization in the sense that \( L(\nu) \to L \) in the norm of \( C^0(M) \) and so proving the result for a fixed \( \nu \) will suffice. Moreover, in view of Theorem 2.1, it suffices to show that \( L(\nu) P_N \to L(\nu) \).

It is known (see [26]) that the kernel \( k_L(x, y) \) of the integral operator \( L(\nu) \) is continuous on the set where it is bounded. Hence, by using a suitable cutoff function \( L \) can be approximated in the norm of \( C^0(M) \) by an integral operator with a continuous kernel.

More precisely, given an arbitrary \( \epsilon > 0 \) there is \( k_0(x, y) \in C([0, \tau] \times [0, \tau]) \) such that if \( L_0 \in C^0(M) \) is defined by

\[ L_0 u(x) = \int_0^\tau k_0(x, y) u(y) \, dy \]

then, in the norm of \( C^0(M) \)

\[ \|L_0 - L(\nu)\| < \epsilon/2. \]

By the Weierstrass approximation theorem [27] we may assume that \( k_0 \) is a polynomial and hence, for some \( p > 0 \)

\[ k_0(x, y) = \sum_{i,j=0}^p \alpha_{ij} \phi_i(x) \phi_j(y) \]
where \( \phi_i(x) = x^i \). Now, let \( N_0 \) be large enough so that
\[
\| (I - P_N) \phi_j \|_2 \leq \frac{\epsilon}{2(p^2 + 1)\tau^p \max_{0 \leq i,j \leq p} |\alpha_{ij}|}
\]
for all \( N \geq N_0 \).

Now, \( I - P_N \) is self-adjoint in \( L^2[0, \tau] \) and hence, for all \( N \geq N_0, i, j \leq p, \) and \( x \in [0, \tau], \)
\[
\left| \phi_i(x) \int_0^\tau \phi_j(y) [(I - P_N)u](y) dy \right| = \left| \phi_i(x) \int_0^\tau [(I - P_N) \phi_j](y) u(y) dy \right|
\leq \tau^p \|u\|_2 \| (I - P_N) \phi_j \|_2
\leq \frac{\epsilon \|u\|_2}{2(p^2 + 1) \max_{0 \leq i,j \leq p} |\alpha_{ij}|}
\]
Hence, for all \( N \geq N_0 \) and \( x \in [0, \tau], \)
\[
|L_0(I - P_N)u(x)| \leq \sum_{i,j=0}^p |\alpha_{ij}| \left| \phi_i(x) \int_0^\tau \phi_j(y) [(I - P_N)u](y) dy \right|
\leq (p^2 + 1)\tau^p \max_{0 \leq i,j \leq p} |\alpha_{ij}| \max_{0 \leq j \leq p} \left\| \int_0^\tau \phi_j(y) [(I - P_N)u](y) dy \right\|
\leq \epsilon \|u\|_2 / 2.
\]
Hence, in the norm of \( \mathcal{COM}(L^2[0, \tau], C[0, \tau]) \),
\[
\| L_0(I - P_N) \| < \epsilon / 2
\]
and therefore, since \( I - P_N \) has norm 1 as an operator on \( L^2 \),
\[
\| L(\nu) - L(\nu)P_N \| = \| (L(\nu) - L_0)(I - P_N) + L_0(I - P_N) \|
\leq \| L_0(I - P_N) \| + \| (L(\nu) - L_0)(I - P_N) \|
\leq \| L_0(I - P_N) \| + \| (L(\nu) - L_0) \| \leq \epsilon.
\]
This completes the proof since \( \epsilon \) was arbitrary. \( \square \)

As we noted at the beginning of this section, if \( M(\nu) \) and \( M(N, \nu) \) are constructed by replacing \( \epsilon \) with \( 1 - \epsilon \) in the construction of \( L(\nu) \) and \( L(N, \nu) \), similar convergence results hold. In summary, therefore, we have two fully discrete approximations to the solution operator for (1.1) which are defined for \( q \in [0, \tau] \) by
\[
S^n(N, \nu)(q) = (I - L(N, \nu))^{-1}(M(N, \nu)q + g(N, \nu)) \quad \text{and} \quad S^n(N, \nu)(q) = (I - L(N, \nu))^{-1}(M(N, \nu)P_Nq + g(N, \nu)).
\]
We will also use the semi-discrete approximation
\[
S(\nu) = (I - L(\nu))^{-1}(M(\nu)q + g(\nu)).
\]
Finally, we have, noting \( S'(\nu) = (I - L(\nu))^{-1} M(\nu) \)

**Theorem 2.3.** Let the spatial mesh be equally spaced with \( x_0 = 0 \) and \( x_N = \tau \). Let the angular mesh be double Gaussian quadratures on \([-1, 1]\) with \( \nu \) nodes. Let \( S(N, \nu) \) be either of \( S^n(N, \nu) \) or \( S^e(N, \nu) \). Then, in the operator norm on \( C[0, \tau] \),

\[
\lim_{N, \nu \to \infty} S'(N, \nu) = S'
\]

and

\[
\lim_{N \to \infty} S'(N, \nu) = S'(\nu) = (I - L(\nu))^{-1} M(\nu).
\]

Because of the computation of \( P_N \), the approximation of \( S^e \) is slightly more costly to compute than \( S^n \). The advantage of \( S^e \) is that it is more directly compatible with a norm convergent approximation to the diffusion solver.

**2.2. Discretization of the Diffusion Equation.** To complete our approximation of \( \mathcal{T} \) we must define approximate solution operators for \((1.8)\). We may apply central differences and use piecewise linear interpolation to recover an approximation of the solution of \((1.8)\), which will generate a sequence of approximations \( \mathcal{G}^n_k \) to \( \mathcal{G} \) for which \( \{((\mathcal{G}^n_k)'\} \) is a strongly convergent collectively compact family of functions [1].

Alternatively we may apply a finite element discretization, using the usual piecewise linear basis functions on the same spatial mesh \( \{x_i\}_{i=1}^N \) that we used in § 2.1. We will denote the sequence of finite-element approximations to \( \mathcal{G} \) by \( \{\mathcal{G}^e_k\} \). The standard estimates [13] imply that \( \{((\mathcal{G}^e_k)'\} \) converges in the operator norm on \( C[0, \tau] \). The cost of the finite element approximation is the integration of \( Q \) against the basis functions in the right hand side of the weak formulation

\[
(T_x, v_x) = (Q, v) \quad \text{for all } v \in V_h.
\]

Here \( V_h \) is the space of piecewise linear functions on the equally spaced mesh on \([0, \tau]\) with mesh width \( h \) and \( v_x = dv/dx \).

We summarize the convergence of the diffusion approximations as

**Theorem 2.4.** Let the spatial mesh be equally spaced with \( x_0 = 0 \) and \( x_N = \tau \). Then \( \mathcal{G}^n(N)u \to \mathcal{G}u \) and \( \mathcal{G}^e(N)u \to \mathcal{G}u \) for all \( u \in C[0, \tau] \). Moreover \( \{((\mathcal{G}^e_k)'(N) \to \mathcal{G}' \) in the operator norm of \( C[0, \tau] \) and \( \{((\mathcal{G}^n_k)'(N) \} \) is a collectively compact strongly convergent sequence of operators on \( C[0, \tau] \).

**2.3. Discretization of the Full Problem.** We have two discretizations of \( \mathcal{T} \), corresponding to a choice of \( (\mathcal{G}^n, S^n) \) or \( (\mathcal{G}^e, S^e) \)

\[
\mathcal{T}^e(N, \nu)(\Theta) = G^e(N, \nu)(\alpha[I - S^e(N, \nu)]\Theta^1)
\]

and

\[
\mathcal{T}^n(N, \nu)(\Theta) = G^n(N, \nu)(\alpha[I - S^n(N, \nu)]\Theta^1).
\]

The convergence result for these discretizations follows directly from the discussion above.

**Theorem 2.5.** Let the spatial mesh be equally spaced with \( x_0 = 0 \) and \( x_N = \tau \). Let the angular mesh be double Gaussian quadratures on \([-1, 1]\) with \( \nu \) nodes. Then

\[
\lim_{(N, \nu) \to \infty} \mathcal{T}^n(N, \nu)u = \mathcal{T}u \quad \text{and} \quad \lim_{(N, \nu) \to \infty} \mathcal{T}^e(N, \nu)u = \mathcal{T}u
\]
for all \( u \in C[0, \tau] \). Moreover \( (T^e)'(N, \nu) \to T' \) in the operator norm of \( C[0, \tau] \) and \( \{(T^e)'(N, \nu)\} \) is a collectively compact strongly convergent sequence of operators on \( C[0, \tau] \).

3. Multilevel Algorithms. In § 2 we gave two different discretizations of \( T \) corresponding to the finite element and finite difference discretizations of the diffusion equation. In this section we discuss algorithmic options. To simplify notation we will suppress mention of the angular mesh and, rather than refer to \( N \) directly, index discretizations by levels, with \( m = 0 \) denoting the coarsest spatial mesh with \( N_0 \) spatial mesh points and mesh \( m \) having \( N_m \) spatial mesh points.

As in § 2 we denote the sequence based on finite differences by \( fT_m \) and that based on finite elements by \( fT^e_m \).

We discuss three algorithms, two that are valid for either discretization and one only for \( fT^e_m \).

The first approach, which we will use as a coarse mesh solver in all cases, is the nested iteration (grid sequencing or multigrid slash cycle) form of Newton-GMRES. Newton-GMRES is an inexact Newton method \([8]\) with GMRES as an approximate linear solver. We define the nonlinear residual as

\[ \mathcal{R}(\Theta) = \Theta - T(\Theta), \]

and use \( \epsilon \) and \( \eta \) as termination constants for the outer and inner iterations. In this context we can express the algorithm as a solver on some mesh level \( m \) as

**Algorithm 3.1. nngmres(\( T_m, \Theta, \epsilon, \eta \))**

1. While \( \| \mathcal{R}_m(\Theta) \| > \epsilon \)
   (a) Solve \( \mathcal{R}_m'(\Theta)s = -\mathcal{R}_m(\Theta) \) with GMRES
      Terminate when \( \| \mathcal{R}_m'(\Theta)s + \mathcal{R}_m(\Theta) \| < \eta \| \mathcal{R}_m(\Theta) \| \)
   (b) \( \Theta = \Theta + s \)

Sometimes it is most efficient for the parameter \( \eta \) to vary as the nonlinear iteration progresses \([10], [15]\). In the present context, however, we found that simply fixing \( \eta \) was sufficient. The reason for this is that the nested iteration form of the algorithm solves a sequence of discrete problems on a sequence of meshes, using the converged result from one mesh as the initial iterate for the next. In this context, a high accuracy solution to the equation for the Newton step is of little use. The nested Newton-GMRES solution for a mesh \( m > 0 \) can be written.

**Algorithm 3.2. nestgmb(\( T, M \))**

1. Solve \( \Theta_0 = T_0(\Theta_0) \).
2. For \( m = 1, \ldots, M \)
   Initialize \( \Theta_m = \Theta_{m-1} \)
   nngmres(\( T_m, \Theta_m, \epsilon_m, \eta \))

The second and third methods are both modified Newton or chord methods \([9], [25], [15]\), in which the inverse Jacobian \( (\mathcal{R}_m')^{-1} \) is approximated via coarse mesh solve. The first of these is the Atkinson-Brakhage \([2], [4]\), iteration. For a given level \( m > 0 \) a single iterate can be written as

(3.1) \[ \Theta_+ = \Theta_c - (I + (\mathcal{R}_0')^{-1}T'_m(\Theta_c))\mathcal{R}_m(\Theta_c). \]

The Jacobian approximation is

\[ (\mathcal{R}_m')^{-1} = (I - T'_m)^{-1} \approx I + (\mathcal{R}_0')^{-1}T'_m = I + (I - T'_0)^{-1}T'_m \]

The action of \( (I - T'_0)^{-1} \) on a vector, which is required in the right side of (3.1), is approximated by a coarse-mesh GMRES iteration which terminates on a sufficiently small relative residual. Collective compactness and strong convergence of the sequence \( \{T_m\} \) imply that if the coarse mesh is
sufficiently fine, this iteration will converge [16], [14], [2], [4]. More precisely there is a sequence $\xi(N) \to 0$ such that

$$\|(R_m^r)^{-1} - I - (I - T_0^r)^{-1} T_m^r\| \leq \xi(N)$$

independently of $m > 0$. Moreover, the convergence can be made sufficiently rapid so that if a nested iteration approach is used then only a single iteration need be taken at each level (i.e. the update of $\Theta_m$ in step 2b in Algorithm nestab is only done once).

Denoting $m = 0$ as the coarse level and $m = M$ as the level of approximation at which a solution is required, the nested iteration form of the algorithm can be expressed as

**Algorithm 3.3. nestab($T, M$)**

1. Solve $\Theta_0 = T_0(\Theta_0)$.
2. For $m = 1, \ldots, M$
   (a) Initialize $\Theta_m = \Theta_{m-1}$
   (b) While $\|R_m(\Theta_m)\| \geq \epsilon_m$

   $$\Theta_m = \Theta_m - [I + (R_0^r)^{-1} T_m^r] R_m(\Theta_m)$$

Typically the operator vector product $T_m^r R_m$ is approximated by a forward difference,

$$T_m^r R_m \approx \frac{T_m(\Theta_m + \Delta R_m) - T_m(\Theta_m)}{\Delta}$$

requiring a fine mesh function evaluation ($T_m(\Theta_m + \Delta R_m)$) in addition to the one needed to compute $R_m$. If the coarse mesh is sufficiently fine, the while loop will terminate after one iterate and the total cost will be two fine mesh function evaluations for each level. The action of $(R_0^r)^{-1}$ on a vector may be computed by a fine-to-coarse mesh projection, a coarse mesh GMRES solve, and then an interpolation back to the fine mesh [14]. This method is not particularly sensitive to the choice of intergrid transfers.

In [16] and [14] approximations, such as $T^c$, that converged in the uniform operator norm, rather than in the strong operator topology, were used to make the Atkinson-Brakhage iteration more efficient. In both the integral equations case [14] and the application to the transport equation [16], [26], the difference in the operators was that integrals over mesh intervals, rather than point evaluations at mesh points, were used to compute the right sides of the equations. If $T_m^r \to T'$ in the operator norm, step 2b in Algorithm nestab can be simplified.

**Algorithm 3.4. nestnorm($T, M$)**

1. Solve $\Theta_0 = T_0(\Theta_0)$.
2. For $m = 1, \ldots, M$
   (a) Initialize $\Theta_m = \Theta_{m-1}$
   (b) While $\|R_m(\Theta_m)\| \geq \epsilon_m$

   $$\Theta_m = \Theta_m - (R_0^r)^{-1} R_m(\Theta_m)$$

Note that only one fine mesh function evaluation need be done in step 2b of Algorithm nestnorm because of the norm convergence.
4. Numerical Results. In this section we present two simple examples that illustrate the performance of the proposed method. The underlying transport problems differ in that one has reflecting boundary conditions and the other does not. We refer the reader to [16] for reports on a larger variety of transport problems.

In our implementation, level \( m \) corresponds to a uniform mesh of \( N_m = 2^{m+m_0} + 1 \) points, where \( m_0 \) varies with the method/problem combination for the computations reported here. We let \( M \) denote the finest mesh. The approximation scheme for the transport solve is second order accurate [20], [26] for a fixed angular mesh. The cost of an evaluation of \( T_m \) is \( O(N_m) \) floating point operations and therefore the cost of a complete solve using Algorithm \texttt{nestab} is \( O(N_M) \) floating point operations. We compare Newton-GMRES and Atkinson-Brakhage, using \( T^n \) as the discretization of \( T \), with algorithm \texttt{nestnorm}. Recall that each algorithm should require a fixed number of fine mesh function evaluations (and hence \( O(N_M) \) floating point operations) to approximate \( \Theta \) to fine mesh truncation error.

The transport solve uses the method from [16]. On the higher levels, we use an angular mesh of 40 points, corresponding to a double 20 point Gaussian quadrature on \([-1, 1]\). On the coarsest level we use an 8 point (double 4 point) angular mesh. This solver is more efficient than an Atkinson-Brakhage solver or a second-kind multigrid [12] approach. Transport solvers of this type require only the source iteration map and do not need a diffusion solver as a preconditioner as do some other multilevel approaches [22], [23], which are based on diffusion-synthetic acceleration, [18], [19]. However these other methods extend directly to more than one space dimension and Atkinson-Brakhage solvers have not, as yet, been applied to other than one dimensional transport problems.

Coarse mesh solves were done with Newton-GMRES and the coarse mesh residual was reduced by a factor of \( 10^{-6} \) from that for the initial iterate of

\[
\Theta = \Theta_l(r - x)/\tau + \Theta_r x/\tau.
\]

Newton-GMRES was implemented in a straightforward manner and converged without any need for globalization of the nonlinear iteration.

We report on computations for two problems, the data for which are summarized in Table 4.1. The finest spatial mesh had 2049 points for both problems. The coarse mesh had 9 points for problem A. For problem B, which has reflecting boundary conditions and is more difficult, the choice of coarse mesh depended on the solver. The best performance was obtained for a 9 point mesh for Newton-GMRES, a 17 point mesh for the Atkinson-Brakhage method, and a 33 point coarse mesh for the modified Atkinson-Brakhage method. The coarse mesh solver was Newton-GMRES for problem A and the nested iteration form of Newton-GMRES for problem B. These meshes were the coarsest possible that allowed the nonlinear solver in Algorithms \texttt{nestgmb}, \texttt{nestab}, and \texttt{nestnorm} to terminate after a single iteration. Since we expect (and obtain) second order accuracy [20], [26], we set

\[
e_m = \|\Theta_{m-1} - T_m(\Theta_{m-1})\|/10,
\]

with a view that a reduction by a factor of 10 in the nonlinear residual will safely ensure a reduction by a factor of 4 in the error.

We report both function evaluations at the fine mesh levels (FMF) and timings in Table 4.2. The number of function evaluations at the fine mesh levels did not vary as the mesh was refined.
consistent with the theory from [2] and [6]. One can see a clear relation between the function evaluations and the timings, indicating that for both algorithms the cost of the work internal to the solver itself was minor.

The computations reported here were done on a SUN SPARCstation 5 workstation running SunOS 4.1.3_U1 version 1 with the SUN f77 compiler version 3.0.1. Some of the preliminary computations for this work were done at the North Carolina Supercomputing Center.

5. Concluding Remarks. We show how a fast solver based on work in [14] and [16] can be applied to the fixed-point form of a system of equations for conductive-radiative heat transfer in one space dimension. We compare this solver with two alternatives. The critical theoretical and algorithmic issue is the compatibility of the discretizations of the transport and diffusion equations. Implementation of either form of the Atkinson-Brakhage algorithm requires more attention to the functional analytic setting and especially to intergrid transfers than does Newton-GMRES. The Modified Atkinson-Brakhage is very recent and this paper is the first application in which the compatibility of discretizations is an issue. We plan future work on conductive-radiative heat transfer in two space dimensions in which we hope to understand better how difficult it is to deal with this problem.

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