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BANDLIMITED PROCESSES AND CERTAIN NONLINEAR TRANSFORMATIONS

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1. INTRODUCTION

This paper considers the problem of the uniqueness and invertibility of a certain class of nonlinear transformations acting on stochastic processes. Specifically, we consider a system consisting of an instantaneous nonlinearity followed by a linear system. The input to the system is a bandlimited stochastic process in the sense of Zakai [8] which may be, in particular, conventionally bandlimited stationary or harmonizable process. We study the one to one correspondence between the input and output processes and the reconstruction of the input process from the output process. Some partial answers to these problems were obtained in [6] for conventionally bandlimited stationary input processes.

In Section 2, the notions of bandlimited function and process are defined, their basic properties and characterization are summarized and some additional properties needed here are proved. Section 3 provides a brief outline of the structure of the system considered throughout this paper. The main results of the paper are collected in Sections 4 and 5. In Section 4 we consider the one to one correspondence between input and output for Gaussian bandlimited input processes and a broad class of nonlinear systems. Some general uniqueness theorems are obtained and are applied to derive some new results on the curve crossings of bandlimited Gaussian processes. Section 5 considers the reconstruction of the sample paths of the input process from, possibly noisy, observations of the output process. This is done for a specific class of instantaneous nonlinearities followed by a variety of linear systems.

2. ZAKAI'S CLASS OF BANDLIMITED FUNCTIONS AND PROCESSES

Throughout this paper the following notation is used. m is the Lebesgue measure on the real line, μ is the finite measure on the Borel sets of the real line defined by $[\frac{d\mu}{dm}](t) = \frac{1}{1+t^2}$, and $L_2(\mu)$ is the Hilbert space of Borel measurable complex valued functions on the real line satisfying $\int_{-\infty}^{\infty}|f(t)|^2 d\mu(t) < \infty$. The inner product and the norm in $L_2(\mu)$ are denoted by $\langle\circ,\circ\rangle$ and $||\circ||$ respectively.

For $W,\delta > 0$ let

$$H(\lambda) = H(\lambda;W,\delta) = \begin{cases} 1 & \text{for} \quad |\lambda| \le W \\ 1 - \frac{|\lambda|-W}{\delta} & \text{for} \quad W < |\lambda| \le W+\delta \\ 0 & \text{for} \quad W+\delta < |\lambda| \end{cases} \tag{1}$$

and denote its inverse Fourier transform by $h(t) = h(t;W,\delta) =$ $= \frac{2}{\pi\delta^2} \sin(W + \frac{\delta}{2})t \sin \frac{\delta t}{2}$. Zakai [8] defines the class $B(W,\delta)$ of functions "bandlimited to (W,δ)" as the set of all functions f in $L_2(\mu)$ satisfying

$$f(t) = \int_{\infty}^{\infty} f(\tau)h(t-\tau)d\tau = (f*h)(t) \tag{2}$$

for almost all t (with respect to the Lebesgue measure). $B(W,\delta)$ is a subspace of $L_2(\mu)$ and every function in $B(W,\delta)$ is equal almost everywhere to a continuous function (the right hand side of (2) as shown in [8]). Henceforth only these continuous modifications will be considered and thus (2) is valid for all real numbers t.

If $CB(W)$ is the class of functions which are "conventionally bandlimited to W", i.e., the class of all functions f in $L_2(m)$ whose

Fourier transform $F(\lambda) = 0$ for $|\lambda| > W$, then $CB(W)$ is a subspace of $L_2(m)$ and $CB(W) \subset B(W,\delta)$ for all $\delta > 0$. Thus Zakai's notion of band-limited function generalizes the conventional one.

In the sequel we will need the following recently derived characterization of $B(W,\delta)$.

LEMMA 1[2]. $f \in B(W,\delta)$ if and only if $f(t) = f(0) + tg(t)$ where $g \in CB(W)$.

It follows from Lemma 1 that the class $B(W,\delta)$ is independent of $\delta > 0$ and we shall therefore denote it by $B(W)$ and call its members functions "bandlimited to W."

We will also use the following simple properties.

LEMMA 2 [8]. (a) If $f \in L_2(\mu)$ and u is the convolution of f with $\frac{1}{1+t^2}$, then $u \in L_2(\mu)$ and $||u|| \leq \pi||f||$.

(b) If $f \in L_2(\mu)$ and $u(t) = (f*h)(t)$ for all t, then u is continuous and $|u(t)| \leq k\sqrt{1+t^2}||f||$ for all t.

(c) If $f, f_n \in B(W)$, $n = 1,2,\ldots,$ and $f = \lim_n f_n$ in $L_2(\mu)$, then $f(t) = \lim_n f_n(t)$ for all t.

PROOF. (a) is Lemma 1 of [8], (b) is from Lemma 2 of [8], and (c) follows from the fact that $f-f_n \in B(W)$, i.e., $f-f_n = (f-f_n) * h$, and thus by (b): $|f(t) - f_n(t)| \leq k\sqrt{1+t^2}||f-f_n||$ for all t. □

LEMMA 3 [2]. If ϕ is a real valued function on the real line such that $\phi(t) \in L_1(m)$, $t\phi(t) \in L_1(m)$, and its Fourier transform $\Phi(\lambda) = 1$

for $-W \leq \lambda \leq W$, then $f(t) = (f*\phi)(t)$ for all $f \in B(W)$.

In section 5 we will need to construct complete orthonormal sets in $B(W)$. Such sets can be obtained by orthonormalizing the complete set given in Proposition 1 in terms of a complete set in $CB(W)$; complete sets in $CB(W)$ are well known.

PROPOSITION 1. If $\{g_n\}_{n=1}^{\infty}$ is a complete sequence of functions in $CB(W)$, then the sequence $\{f_n\}_{n=0}^{\infty}$ defined by

$$f_0(t) = 1 \ , \quad f_n(t) = tg_n(t) \ , \quad n = 1,2,\ldots$$

is complete in $B(W)$.

PROOF. It follows from Lemma 1 that $f_n \in B(W)$, $n=0,1,2,\ldots$. Now fix an arbitrary function f in $B(W)$. By Lemma 1, $f(t) = f(0) + tg(t)$ for some $g \in CB(W)$. Then

$$\begin{aligned} ||f-f(0)f_0 - \sum_{n=1}^{N} a_{N,n}f_n||_{L_2(\mu)} &= ||tg - t\sum_{n=1}^{N} a_{N,n}g_n||_{L_2(\mu)} \\ &= ||\frac{t}{1+t^2}(g - \sum_{n=1}^{N} a_{N,n}g_n)||_{L_2(m)} \\ &\leq ||g - \sum_{n=1}^{N} a_{N,n}g_n||_{L_2(m)} \end{aligned}$$

(m is the Lebesgue measure) and the right hand side can be made arbitrarily small for appropriate N and constants $\{a_{N,n}\}_{n=1}^{N}$, since $g \in CB(W)$ and $\{g_n\}_{n=1}^{\infty}$ is complete in $CB(W)$. It follows that $\{f_n\}_{n=0}^{\infty}$ is complete in $B(W)$. □

Let $\{\phi_n\}_{n=1}^{\infty}$ be a complete orthonormal set in $B(W)$, which can be obtained by orthonormalizing the sequence given in Proposition 1. Let P be the projection of $L_2(\mu)$ onto $B(W)$ and P_N the projection of $L_2(\mu)$ onto the subspace M_N generated by $\{\phi_1,\ldots,\phi_N\}$, $N = 1,2,\ldots$. Then for every $f \in L_2(\mu)$ we have (for almost all t)

$$(P_N f)(t) = \sum_{n=1}^{N} \langle f,\phi_n\rangle \phi_n(t) = \int_{-\infty}^{\infty} \frac{K_N(t,s)}{1+s^2} f(s)\, ds \tag{3}$$

where

$$K_N(t,s) = \sum_{n=1}^{N} \phi_n(t)\phi_n(s) \tag{4}$$

and $P_N f \to Pf$ in $L_2(\mu)$. This approximation of P by P_N will be used at the end of this section and in Section 5.

Throughout this paper all stochastic processes defined on a probability space (Ω,F,P) are assumed to be real valued. Now let $X = \{X(t,\omega),\ -\infty<t<\infty\}$ be a second order stochastic process on (Ω,F,P) and let $R(t,s)$ be its correlation function. The process X is called "bandlimited to W" [8], and we write $X \in B(W)$, if

$$\int_{-\infty}^{\infty} R(t,t)\,d\mu(t) < \infty \tag{5}$$

and with probability one its sample functions are bandlimited to W. Then X is measurable and mean square continuous as can be seen from the following argument: Since the sample functions of X satisfy (2) with probability one, there exists a set $\Omega_0 \in F$ with $P(\Omega_0) = 0$ such that for each fixed $\omega \in \Omega - \Omega_0$, $X(t,\omega)$ is continuous in t. Moreover, for each fixed t, $X(t,\omega)$ is F-measurable. Define $X'(t,\omega) = X(t,\omega)$ for $(t,\omega) \in (-\infty,\infty) \times (\Omega - \Omega_0)$ and $X'(t,\omega) \equiv 0$ for $(t,\omega) \in (-\infty,\infty) \times \Omega_0$. Then [5, p. 122]

X' is product measurable. Since $(m \times P)\{(-\infty,\infty) \times \Omega_0\} = 0$, we make no distinction between X and X' and conclude therefore that X is product measurable. Now the representation (2) for the sample functions of $X \in B(W)$ and Fubini's theorem imply that $R(t,s)$ has the representation

$$R(t,s) = \iint_{-\infty}^{\infty} R(\tau,\sigma)h(t-\tau)h(s-\sigma)d\tau \, d\sigma .$$

Also by (5) $R(\tau,\sigma) \in L_2(\mu) \times L_2(\mu)$, and the continuity of $R(t,s)$ as a function of t and s follows in the same manner as the continuity of u in Lemma 2(b). Thus $X \in B(W)$ is mean square continuous. It is shown in [8] that wide sense stationary and harmonizable processes that are "conventionally bandlimited to W", i.e., whose spectral measure is concentrated on $[-W,W]$ and $[-W,W] \times [-W,W]$ respectively, are bandlimited to W. The following characterization of bandlimited processes will be used in the sequel.

LEMMA 4 [8]. $X \in B(W)$ if and only if its correlation function R satisfies (5) and $R(t,\circ) \in B(W)$ for all real numbers t.

The following property will be used throughout this paper.

PROPOSITION 2. Let $Y = \{Y(t,\omega), -\infty<t<\infty\}$ be a measurable second order process on the probability space (Ω,F,P) whose correlation function satisfies (5), and thus with probability one $Y(\circ,\omega) \in L_2(\mu)$. For each such $\omega\in\Omega$, define $Z(\cdot,\omega) = PY(\cdot,\omega)$, where P is the projection onto $B(W)$. Then $Z = \{Z(t,\omega), -\infty<t<\infty\}$ is a stochastic process on (Ω,F,P) and it is second order and bandlimited to W.

PROOF. We first show that Z is a product measurable function in (t,ω), i.e., Z is a measurable process, which clearly implies that for each t, $Z(t,\omega)$ is F-measurable and hence Z is a stochastic process.

Let $Y(\cdot,\omega) \in L_2(\mu)$ for all $\omega\in\Omega-\Omega_0$ where $\Omega_0 \in F$ and $P(\Omega_0) = 0$. For every $\omega\in\Omega-\Omega_0$ let $Z_N(\cdot,\omega) = P_N Y(\cdot,\omega)$, where P_N is defined by (3). Then for all $t\in(-\infty,\infty)$ and $\omega\in\Omega-\Omega_0$ we have

$$Z_N(t,\omega) = \sum_{n=1}^{N} \phi_n(t) \int_{-\infty}^{\infty} Y(s,\omega)\phi_n(s)d\mu(s) .$$

Clearly Z_N is a measurable process. Also for every $\omega\in\Omega-\Omega_0$, $Z_N(\cdot,\omega) \to Z(\cdot,\omega)$ in $L_2(\mu)$ and by Lemma 2(b), $Z_N(t,\omega) \to Z(t,\omega)$ for all t. Hence $Z(t,\omega) = \lim Z_N(t,\omega)$ for all $t \in (-\infty,\infty)$ and $\omega\in\Omega-\Omega_0$ and thus Z is a measurable process.

Next we show that Z is second order and bandlimited to W. Since $||P|| = 1$ we have $\int_{-\infty}^{\infty} Z^2(t,\omega)d\mu(t) \leq \int_{-\infty}^{\infty} Y^2(t,\omega)d\mu(t)$ with probability one, and

$$\int_{-\infty}^{\infty} E[Z^2(t)]d\ (t) \leq \int_{-\infty}^{\infty} E[Y^2(t)]d\mu(t) < \infty$$

since the correlation of Y satisfies (5) (where E denotes expectation). Hence $E^{\frac{1}{2}}[Z^2(t)] \in L_2(\mu)$ and it follows from $Z(t,\omega) = \int_{-\infty}^{\infty} Z(\tau,\omega)h(t-\tau)d\tau$ almost surely, Fubini's theorem, and Lemma 2(b), that for all t

$$\begin{aligned} E[Z^2(t)] &\leq \iint_{-\infty}^{\infty} E[|Z(\tau)Z(\sigma)|]|h(t-\tau)||h(t-\sigma)|d\tau d\sigma \\ &\leq \left(\int_{-\infty}^{\infty} E^{\frac{1}{2}}[Z^2(\tau)]|h(t-\tau)|d\tau\right)^2 \\ &\leq K^2(1+t^2) \int_{-\infty}^{\infty} E[Z^2(\tau)]d\mu(\tau) < \infty . \end{aligned}$$

Hence Z is a second order process, its correlation function satisfies (5) and with probability one its sample functions are bandlimited to W; and thus Z is bandlimited to W. $\square$

3. DESCRIPTION OF THE SYSTEM

We shall consider a system consisting of an instantaneous nonlinearity A followed by a linear system L. The input X to the system will be a stochastic process bandlimited to W and the output Z a process bandlimited to W or to some $W' \geq W$, and we shall consider the one to one correspondence between input and output processes and the reconstruction of the input from the output.

The nonlinearity is determined by a real valued Borel measurable function A on the real line such that if the input X is a process bandlimited to W, then the output Y defined by $Y(t,\omega) = A(X(t,\omega))$ is a second order process with autocorrelation function satisfying (5), and thus with almost all its sample functions in $L_2(\mu)$. For convenience we shall use the notation $Y = AX$ and $Y(t) = (AX)(t)$.

The linear system L is a linear map from $L_2(\mu)$ to some $B(W')$ with $W' \geq W$, i.e., it maps the sample functions of Y into functions in $B(W')$. If we denote by Z the output we have $Z(\cdot,\omega) = L(Y(\cdot,\omega))$, and it is assumed that L is such that Z is a second order stochastic process bandlimited to W'. For convenience we shall use the notation $Z = LY$ and $Z(t) = (LY)(t)$. Specific linear systems L will be considered in the following sections.

For convenience we shall also denote the entire transformation by LA and we shall write $Z = LAX$ and $Z(t) = (LAX)(t)$.

4. THE ONE TO ONE CORRESPONDENCE BETWEEN INPUT AND OUTPUT PROCESSES

In this section we study the one to one correspondence between the input and output processes of the system described in Section 3, for bandlimited Gaussian input processes and a broad class of nonlinearities. The results are given for two kinds of linear systems (Theorems 1 and 2, and Theorems 1' and 2' respectively). Also applications to curve crossings of bandlimited Gaussian processes are derived (Corollaries 1 and 2).

We consider bandlimited Gaussian input processes X and we denote by $\mathcal{A}(X)$ the class of all instantaneous nonlinearities A, i.e., real valued Borel measurable functions A on the real line, such that the output $Y = AX$ satisfies

$$E[Y^2(t)] < \infty \quad \text{for all} \quad t \tag{6a}$$

$$\int_{-\infty}^{\infty} E[Y^2(t)]d\mu(t) < \infty \tag{6b}$$

$$E[X(t)Y(t)] > 0 \quad (\text{or} < 0) \text{ for all} \quad t \tag{6c}$$

where E denotes expectation. It is easily seen that if A is a hard limiter, $A(x) = \text{sgn } x$, then $A \in \mathcal{A}(X)$ for all bandlimited Gaussian processes X. Further examples of nonlinearities A in $\mathcal{A}(X)$ can be easily constructed. For instance, if $A(0) = 0$ and A is monotonic and Lipschitz (i.e. $|A(x)| \leq M|x|$ for all x and some finite M), then $A \in \mathcal{A}(X)$ for all bandlimited Gaussian processes X.

Conditions (6a) to (6c) depend only on the one dimensional distributions of the Gaussian process X. Thus for all Gaussian processes X with the same first and second moments for all t, the class $\mathcal{A}(X)$ does not depend on X and will be denoted by $\mathcal{A}$.

We say that two processes U and V on a probability space (Ω, F, P) are indistinguishable if with probability one their sample functions are the same, i.e.,

$$P\{U(t) = V(t) \quad \text{for all} \quad t\} = 1.$$

If U and V have continuous sample functions with probability one, then they are indistinguishable if and only if

$$P\{U(t) = V(t)\} = 1 \quad \text{for all} \quad t.$$

This is obviously a necessary condition. In order to see that it is also sufficient, choose any countable dense subset S of the real line (for instance the rationals). Since $P\{U(t) = V(t)\} = 1$ for all t, and S is countable, it follows that $P\{U(t) = V(t)$ for all $t \in S\} = 1$. But continuous functions that agree on dense subsets, agree everywhere, and thus the sample function continuity of U and V implies $P\{U(t) = V(t)$ for all $t\} = 1$.

We first assume that the linear system L is the projection operator P from $L_2(\mu)$ onto $B(W)$ and establish a one to one correspondence between input and output processes.

THEOREM 1. Let X_1 and X_2 be two jointly Gaussian processes bandlimited to W (i.e., $X_1, X_2 \in B(W)$) with zero mean and equal nonzero second moment for all t, $A \in \mathcal{A}$, and $Z_i = PAX_i$, $i = 1,2$. Then the processes Z_1 and Z_2 are indistinguishable if and only if the processes X_1 and X_2 are indistinguishable.

PROOF. It follows from $A \in \mathcal{A}$ and Proposition 2 that Z_1, Z_2 are second order processes bandlimited to W. The "if" part of the conclusion is obvious, and in the following we establish the "only if" part. Since all processes X_1, X_2 and Z_1, Z_2 have continuous sample functions with probability one, it suffices to show that (8) implies (7), where

$$P\{X_1(t) = X_2(t)\} = 1 \quad \text{for all} \quad t \tag{7}$$

$$P\{Z_1(t) = Z_2(t)\} = 1 \quad \text{for all} \quad t\ . \tag{8}$$

Since $X_1, X_2, Z_1, Z_2 \in B(W)$, it is clear that $\int_{-\infty}^{\infty} E\{|X_1(t)-X_2(t)|\, |Z_1(t)-Z_2(t)|\}d\mu(t) < \infty$ and by Fubini's theorem we have

$$J = \int_{-\infty}^{\infty} E\{[X_1(t)-X_2(t)][Z_1(t)-Z_2(t)]\}d\mu(t)$$

$$= E\langle X_1-X_2,\ Z_1-Z_2\rangle$$

where for simplicity, we denote $\langle X(\circ,\omega),\ Z(\circ,\omega)\rangle$ by $\langle X,Z\rangle$. Since $\langle X_1-X_2,\ Z_1-Z_2\rangle = \langle X_1-X_2,\ P(Y_1-Y_2)\rangle = \langle P(X_1-X_2),\ Y_1-Y_2\rangle = \langle X_1-X_2, Y_1-Y_2\rangle$, we have, again by Fubini's theorem,

$$J = \int_{-\infty}^{\infty} E\{[X_1(t)-X_2(t)][Y_1(t)-Y_2(t)]\}d\mu(t)$$

$$= \int_{-\infty}^{\infty} \sum_{i,j=1}^{2} (-1)^{i+j} R_{X_iY_j}(t,t)\ d\mu(t)$$

where $R_{X_iY_j}(t,s) = E[X_i(t)Y_j(s)]$. The processes X_1, X_2 have zero means, and thus $R_{X_iY_j}(t,s) = C_{X_iY_j}(t,s)$, the covariance of $X_i(t)$ and $Y_j(s)$. Also the jointly Gaussian processes X_1 and X_2 have the cross-covariance property [1], i.e.,

$$C_{X_iY_j}(t,s) = a_j(s)C_{X_iX_j}(t,s)$$

where
$$a_j(s) = \frac{E[X_j(s)Y_j(s)]}{C_{X_jX_j}(s,s)}\ .$$

Since the Gaussian processes X_1 and X_2 have zero means and equal variance, they have identical one dimensional distributions and hence $a_1(s) = a_2(s) = a(s)$ (independent of the index). Also, by (6c), $a(t) > 0$ (or < 0) for all real t. Thus we finally have $R_{X_iY_j}(t,s) = a(s)R_{X_iX_j}(t,s)$ and

$$J = \int_{-\infty}^{\infty} a(t)\; E[X_1(t)-X_2(t)]^2 d\mu(t).$$

Now (8) and the definition of J, imply that $J = 0$ and thus $E[X_1(t)-X_2(t)]^2 = 0$ for almost all t. This is true for all t, since X_1 and X_2 are mean square continuous, and thus (7) is shown. □

It should be remarked that Theorem 1 remains true if the nonlinearity $A \epsilon \mathcal{A}$ is time varying; in this case $Y(t,\omega) = A(X(t,\omega),t)$ where $A(x,t)$ is a real valued Borel measurable function on the plane. As an application of Theorem 1 we have the following

COROLLARY 1. Let X_1 and X_2 be two jointly Gaussian processes band-limited to W (i.e., $X_1, X_2 \in B(W)$), with zero mean and equal nonzero second moment for all t, and let $u(t)$ be a real valued function on the real line which has a continuous derivative. Then the sample functions of X_1 and X_2 have the same upcrossings and downcrossings of u if and only if X_1 and X_2 are indistinguishable. Moreover, the u-crossings of X_1 and X_2 are genuine, i.e., u-tangencies occur with probability zero.

PROOF. According to a theorem of Bulinskaya [3, p. 76], if the process $X(t,\omega) - u(t)$, $a \le t \le b$, has, with probability one, sample functions with continuous derivative and if its one dimensional density $f_t(x)$ is bounded in x and $a \le t \le b$, then the process $X(t)$, $a \le t \le b$ has u-tangencies with probability zero, and with probability one the number of u-crossings in $[a,b]$ is finite. The first condition is satisfied in our case since u is assumed to have a continuous derivative and $X \in B(W)$ implies that its sample functions are analytic with probability one. Since X is mean square continuous and Gaussian with $R(t,t) > 0$ for all t, we have $\inf_{a \le t \le b} R(t,t) > 0$ and thus the second condition is satisfied. Thus there is zero probability of u-tangencies over any compact interval $[a,b]$, and the same is true over $(-\infty,\infty)$ as follows from $(-\infty,\infty) = \bigcup_{n=1}^{\infty} [-n,n]$. Moreover, the number of u-crossings of X in any compact interval is finite with probability one.

It then follows that the statement "X_1 and X_2 have the same upcrossings and downcrossings of u" is equivalent to

$$P[Y_1(t) = Y_2(t) \quad \text{for almost all} \quad t] = 1 \tag{9}$$

where $Y_i(t,\omega) = \text{sgn}[X_i(t,\omega) - u(t)]$. Applying Fubini's theorem we have

$$E \int_{-\infty}^{\infty} |Y_1(t)-Y_2(t)|^2 dt = \int_{-\infty}^{\infty} E|Y_1(t)-Y_2(t)|^2 dt$$

and thus (9) is equivalent to

$$P[Y_1(t) = Y_2(t)] = 1 \quad \text{for almost all} \quad t. \tag{9'}$$

Now let $A(x,t) = \text{sgn}[x-u(t)]$. Then $Y(t,\omega) = A(X(t,\omega),t)$ and $A \in \mathcal{A}$ since (6a) and (6b) are obvious and (6c) follows from the simple calculation

$$E[X(t)Y(t)] = \sqrt{\frac{2}{\pi} R(t,t)} \exp\{-\frac{u^2(t)}{2R(t,t)}\} .$$

A careful examination of the proof of Theorem 1, with P replaced by the identity operator in $L_2(\mu)$, shows that X_1 and X_2 are indistinguishable if and only if (9') holds, and thus if and only if X_1 and X_2 have the same upcrossings and downcrossings of u. □

The result of Theorem 1 can be extended to the case where there is additive noise at the input of the system, i.e., $X = S+N$ where S is the signal and N the noise. In this case we denote by $A'(X)$ the class of instantaneous nonlinearities satisfying (6a), (6b) and

$$E[S(t)Y(t)] > 0 \quad (\text{or} < 0) \quad \text{for all} \quad t. \tag{6c)'}$$

The fact that X is not anymore bandlimited has no effect on the definition of $A'(X)$. Under the assumptions of Theorem 2, the class $A'(X)$ does not depend on X and is denoted by A'.

THEOREM 2. Let S_1 and S_2 be two jointly Gaussian processes band-limited to W (i.e., $S_1, S_2 \in B(W)$), with zero mean and equal nonzero second moment for all t. Let N be a zero mean Gaussian process whose almost all sample functions are in $L_2(\mu)$ and which is independent of S_1 and S_2. Let $A \in A'$, $X_i = S_i + N$ and $Z_i = PAX_i$, $i = 1,2$. Then the processes Z_1 and Z_2 are indistinguishable if and only if the processes X_1 and X_2 are indistinguishable.

PROOF. As in Theorem 1 it suffices to prove that (8) implies (7). Again as in Theorem 1 we have

$$J = \int_{-\infty}^{\infty} E\{[S_1(t)-S_2(t)][Z_1(t)-Z_2(t)\} d\mu(t)$$

$$= \int_{-\infty}^{\infty} \sum_{i,j=1}^{2} (-1)^{i+j} R_{S_i Y_j}(t,t) d\mu(t) .$$

Since S_i has zero mean, $R_{S_i Y_j}(t,s) = C_{S_i Y_j}(t,s)$, and since S_i and X_j are jointly Gaussian they have the cross-covariance property, i.e., $C_{S_i Y_j}(t,s) = b_j(s) C_{S_i X_j}(t,s)$ where

$$b_j(t) = \frac{E[S_j(t)Y_j(t)]}{C_{S_jS_j}(t,t)} .$$

By the assumptions of the theorem for every t, the joint distribution of $S_j(t)$ and $N(t)$ does not depend on j and thus $b_1(t) = b_2(t) = b(t)$. Also, by (6c)', $b(t) > 0$ (or < 0) for all t. Thus we finally have

$$R_{S_iY_j}(t,s) = b(s)C_{S_iX_j}(t,s) = b(t)R_{S_iS_j}(t,s)$$

and substituting in the expression for J we find

$$J = \int_{-\infty}^{\infty} b(t)E[S_1(t)-S_2(t)]^2 d\mu(t) .$$

Now (8) implies $J = 0$ and since $b(t) > 0$ (or < 0) for all t, we obtain (7). □

In connection with Theorem 2 it should be pointed out that the class A' is fairly rich. The crucial condition, among (6a), (6b) and (6c)', is clearly (6c)' and it is easily seen that for independent S and N

$$E[S(t)Y(t)] = \frac{R_{SS}(t,t)}{R_{SS}(t,t)+R_{NN}(t,t)} \int_{-\infty}^{\infty} xA(x)\ f_t(x)dx$$

where $f_t(x)$ is the normal density with mean 0 and variance $R_{SS}(t,t) + R_{NN}(t,t)$. Hence (6c)' is satisfied whenever the integral is > 0 (or < 0) for all t and this is the case for a large class of A's, including monotonic odd functions A. It should be also remarked that the conclusion of Theorem 2 remains valid when the noise N does not have zero mean and is not independent of the signals S_1 and S_2, provided that $E[S_1(t)N(t)] = E[S_2(t)N(t)]$ for all t; in this case, however, condition (6c)' is a severe restriction on A and the class $A'(X)$ is not necessarily (always) nonempty.

A statement corresponding to the corollary of Theorem 1 can be derived from Theorem 2 in a similar manner. We have

COROLLARY 2. Let S_1 and S_2 be two jointly Gaussian processes band-limited to W (i.e., $S_1, S_2 \in B(W)$) with zero mean and equal nonzero second moment for all t. Let N be a zero mean Gaussian process, independent of S_1 and S_2, whose sample functions have continuous derivative and are in $L_2(\mu)$ with probability one. Let $u(t)$ be a real valued function on the real line which has a continuous derivative, and let $X_i = S_i + N$, $i = 1,2$. Then the sample functions of X_1 and X_2 have the same upcrossings and down-crossings of u if and only if S_1 and S_2 are indistinguishable.

In Theorems 1 and 2 the linear system L was the projection operator from $L_2(\mu)$ onto $B(W)$. From the practical point of view it would be of interest to obtain similar results with L a bounded linear integral operator in $L_2(\mu)$ with a simple kernel. In the following we consider the bounded linear operator L from $L_2(\mu)$ to $B(W+\delta)$, $\delta > 0$, defined by

$$z(t) = (Ly)(t) = \int_{-\infty}^{\infty} y(\tau)h(t-\tau)\,\frac{d\tau}{1+\tau^2} \tag{10}$$

for all $y \in L_2(\mu)$ and t, where $h(t) = h(t;W,\delta)$ is the inverse Fourier transform of (1). Since h is uniformly bounded and μ is finite

$$||L||^2 = \iint_{-\infty}^{\infty} h^2(t-\tau)\,d\mu(t)\,d\mu(\tau) < \infty$$

and it is easily seen from (10) that

$$||z|| = ||Ly|| \leq ||L|| \cdot ||y|| \,. \tag{11}$$

Hence $z \in L_2(\mu)$ and L is a bounded linear operator in $L_2(\mu)$. Also, an application of Fubini's theorem shows that $z(t) = (z*h')(t)$ where $h'(t) = h(t;W+\delta,\delta')$, $\delta' > 0$. Hence $z \in B(W+\delta)$ and L maps $L_2(\mu)$ into $B(W+\delta)$. Results similar to Theorems 1 and 2 can be derived, and we prove here only the corresponding result to Theorem 1.

THEOREM 1'. Let X_1 and X_2 be two jointly Gaussian processes band-limited to W (i.e., $X_1,X_2 \in B(W)$) with zero mean and equal nonzero second moment for all t, $A \in \mathcal{A}$, L defined by (10), and $Z_i = LAX_i$, $i = 1,2$. Then the processes Z_1 and Z_2 are indistinguishable if and only if the processes X_1 and X_2 are indistinguishable.

PROOF. It follows from (10) that for almost all ω we have

$$Z(t,\omega) = \int_{-\infty}^{\infty} Y(\tau,\omega)h(t-\tau)d\mu(\tau)$$

for all t. Hence Z is a measurable stochastic process and for all t,

$$E[Z^2(t)] \leq \iint_{-\infty}^{\infty} |R_{YY}(\tau,\sigma)||h(t-\tau)||h(t-\sigma)|d\mu(\tau)d\mu(\sigma) .$$

Since h is seen by (1) to be uniformly bounded on the real line, R_{YY} satisfies (5), and μ is finite, it follows that for all t, $E[Z^2(t)] \leq C < \infty$. Thus Z is of second order, R_{ZZ} satisfies (5), and since its sample functions are in $B(W+\delta)$ a.s., it follows that $Z \in B(W+\delta)$.

As in Theorem 1, it suffices to prove that (8) implies (7). Define for every t,

$$F(t) = E\{[X_1(t)-X_2(t)][Z_1(t)-Z_2(t)]\} = \sum_{i,j=1}^{2} (-1)^{i+j} R_{X_iZ_j}(t,t)$$

where $R_{X_iZ_j}(t,s) = E[X_i(t)Z_j(s)] = \int_{-\infty}^{\infty} R_{X_iY_j}(t,\tau)h(s-\tau)d\mu(\tau)$. As in Theorem

1 we have $R_{X_iY_j}(t,s) = a(s)R_{X_iX_j}(t,s)$ and thus

$$F(t) = \int_{-\infty}^{\infty} a(t)R_{ee}(t,\tau)h(t-\tau)d\mu(\tau)$$

where $e(t) = X_1(t) - X_2(t)$. Now if we can apply Fubini's theorem we obtain

$$\int_{-\infty}^{\infty} F(t)dt = \int_{-\infty}^{\infty} a(\tau)\left\{\int_{-\infty}^{\infty} R_{ee}(t,\tau)h(t-\tau)dt\right\}d\mu(\tau)$$

and since $e \in B(W)$ implies by Lemma 4 that $R_{ee}(\cdot,\tau) \in B(W)$ for all τ, we finally have

$$\int_{-\infty}^{\infty} F(t)dt = \int_{-\infty}^{\infty} a(\tau)R_{ee}(\tau,\tau)d\mu(\tau) .$$

Then (8) implies that $F(t) = 0$ for all t, and since by (6c), $a(t) > 0$ (or < 0) for all t, and e is mean square continuous, we have $R_{ee}(t,t) = 0$ for all t and hence (7). Thus the theorem is proved if the application of Fubini's theorem is justified, i.e. if we show that

$$I = \iint_{-\infty}^{\infty} |a(\tau)||R_{ee}(t,\tau)||h(t-\tau)|d\mu(\tau)dt < \infty .$$

This is shown as follows. We have $|R_{ee}(t,\tau)| \le R_{ee}^{\frac{1}{2}}(t,t)R_{ee}^{\frac{1}{2}}(\tau,\tau)$, and since $e \in B(W)$, $R_{ee}^{\frac{1}{2}}(t,t) \in L_2(\mu)$, and $|h(t)| \le \frac{C}{1+t^2}$ (see Eqs. (11) and (18) of [8]) we obtain

$$f(\tau) = \int_{-\infty}^{\infty} R_{ee}^{\frac{1}{2}}(t,t)|h(t-\tau)|dt \in L_2(\mu) .$$

Then $I \le \int_{-\infty}^{\infty} |a(\tau)|R_{ee}^{\frac{1}{2}}(\tau,\tau)f(\tau)d\mu(\tau)$ and it suffices to show that $a(\tau)R_{ee}^{\frac{1}{2}}(\tau,\tau) \in L_2(\mu)$. This follows from $R_{ee}(\tau,\tau) \le 4R_{XX}(\tau,\tau)$ and

$$|a(\tau)|R_{ee}^{\frac{1}{2}}(\tau,\tau)) = \left| \frac{E[X(\tau)Y(\tau)]}{R_{XX}(\tau,\tau)} \right| R_{ee}^{\frac{1}{2}}(\tau,\tau) \le 2R_{YY}(\tau,\tau)$$

since $R_{YY}^{\frac{1}{2}}(\tau,\tau) \in L_2(\mu)$ by (6b). □

It should be remarked that as it becomes clear from the proof of Theorem 6 in Section 5, h in (10) can be replaced by ϕ satisfying (13) and Theorem 1' is still valid.

5. RECONSTRUCTION OF THE INPUT.

In this section we consider the reconstruction of bandlimited processes which have been distorted by a specific nonlinear system followed by various linear systems. The input X is bandlimited to W, $X \in B(W)$, but not necessarily Gaussian, and the nonlinear system is determined by a function A satisfying the following

A is a real valued, monotonically increasing function on the real line with $A(0) = 0$ and satisfying a Lipschitz condition, i.e., for all x and y and some constants $0<m\le M<\infty$,
$m(x-y) \le A(x) - A(y) \le M(x-y)$. (12)

Then the output $Y = AX$ is a mean square continuous process with sample functions in $L_2(\mu)$ with probability one. All the results of this section remain valid for monotonically decreasing functions A satisfying the corresponding Lipschitz condition $-M(x-y) \le A(x) - A(y) \le -m(x-y)$ where $0 < m \le M < \infty$.

When the linear system used is the projection operator P from $L_2(\mu)$ onto $B(W)$, then the sample functions of the input X can be reconstructed from the sample functions of the output $Z = PAX$ by means of the algorithm given in Theorem 3, whose proof is similar to that in [4, p. 101] and is thus omitted. The algorithm is first shown to converge in $L_2(\mu)$ and then Lemma 2(c) implies convergence pointwise on the real line.

<u>THEOREM 3.</u> Let X be a process bandlimited to W $(X \in B(W))$, let A satisfy (12) and let $Z = PAX$. Then with probability one

$$X(t) = \lim_{n\to\infty} X_n(t) \qquad \text{for all } t$$

where $X_0(t) = 0$ and for $n = 0,1,2,\ldots$

$$X_{n+1}(t) = X_n(t) + c\{Z(t) - (PAX_n)(t)\} ,$$

$0 < c < 2/M$, and the rate of convergence is geometric, i.e., $||X-X_n|| \le \theta^n||X||$ where $0 < \theta < 1$.

When there is additive observation noise which is bandlimited to W, then the algorithm of Theorem 3 converges to a process bandlimited to W which may serve as an estimate of the input. Specifically, we have the following.

<u>THEOREM 4.</u> Let X and N be processes bandlimited to W $(X,N \in B(W))$, let A satisfy (12) and let $V = PAX + N$. Then there is a process $\hat{X}$ bandlimited to W such that with probability one

$$\hat{X}(t) = \lim_{n\to\infty} X_n(t) \qquad \text{for all } t$$

where $X_0(t) = 0$ and for $n = 0,1,2,\ldots$

$$X_{n+1}(t) = X_n(t) + c\{V(t) - (PAX_n)(t)\} ,$$

$0 < c < 2/M$, and

$$\frac{1}{M^2}\int_{-\infty}^{\infty}\frac{R_N(t,t)}{1+t^2}\,dt \le E\int_{-\infty}^{\infty}\frac{[X(t)-\hat{X}(t)]^2}{1+t^2}\,dt \le \frac{1}{m^2}\int_{-\infty}^{\infty}\frac{R_N(t,t)}{1+t^2}\,dt$$

where R_N is the correlation function of N.

PROOF. The convergence of the algorithm is shown as in Theorem 3; in fact we also have $X_n(t) \to \hat{X}(t)$ in $L_2(\mu)$ with probability one. Since PAX is a measurable process by Proposition 2, and N is assumed measurable, V is a measurable process and hence so is X_n for all n. It follows by $X_n(t,\omega) \to \hat{X}(t,\omega)$ for all t and almost all ω that $\hat{X}$ is a measurable process. Now with probability one the sample functions of V are in $B(W)$ and the same is true for the sample functions of X_n for all n. Then $X_n(t) \to \hat{X}(t)$ in $L_2(\mu)$ implies that the sample functions of $\hat{X}$ are in $B(W)$ with probability one. Also with probability one we have $V(t) = (PA\hat{X})(t)$ and thus

$$(PA\hat{X})(t) = (PAX)(t) + N(t) .$$

For any $f_1, f_2 \in B(W)$ we have [7]

$$m||f_1-f_2|| \le ||PAf_1 - PAf_2|| \le M||f_1-f_2|| .$$

By letting $f_1 = X$ and $f_2 = \hat{X}$ we have that with probability one

$$m||X-\hat{X}|| \le ||N|| \le M||X-\hat{X}||$$

and by taking expectations we obtain the desired inequality. This inequality also implies that $\int_{-\infty}^{\infty} E[\hat{X}^2(t)]d\mu(t) < \infty$, since X and N satisfy (5). It follows as in the last part of the proof of Proposition 2 that $\hat{X}$ is of second order and hence $\hat{X} \in B(W)$. □

We remark here that if the additive observation noise is not in $B(W)$, as in Theorem 4, but with probability one has sample functions in $L_2(\mu)$, then by projecting the observation on $B(W)$ a similar result to Theorem 4 is obtained as stated in the following

<u>COROLLARY 3</u>. Let X be a process bandlimited to W $(X \in B(W))$ and N a measurable second order process satisfying (5). Let A satisfy (12) and let $V = PAX + N$. Then there is a process $\hat{X}$ bandlimited to W such that with probability one

$$\hat{X}(t) = \lim_{n\to\infty} X_n(t) \qquad \text{for all } t$$

where $X_0(t) = 0$ and for $n = 0,1,2,\ldots$

$$X_{n+1}(t) = X_n(t) + c\{(PV)(t) - (PAX_n)(t)\} ,$$

$0 < c < 2/M$, and

$$\frac{1}{M^2}\int_{-\infty}^{\infty} \frac{R_N(t,t)}{1+t^2}\,dt \le E\int_{-\infty}^{\infty} \frac{[X(t)-X(t)]^2}{1+t^2}\,dt \le \frac{1}{m^2}\int_{-\infty}^{\infty} \frac{R_N(t,t)}{1+t^2}\,dt$$

where R_N is the correlation function of N.

Let us return to the (noise-free) case considered in Theorem 3 and approximate the projection operator P by the linear integral type operator P_N defined in (3) which is easily realized. Then we do not have perfect reconstruction of the input and the approximation error is given in Theorem 5 and decreases to zero as N increases.

THEOREM 5. Let X be a process bandlimited to W ($X \in B(W)$), let A satisfy (12), let P_N be defined by (3), and let $Z_N = P_N AX$. Then there is a process $\hat{X}_N$ bandlimited to W (with almost all sample functions in $\mathcal{M}_N$) such that with probability one

$$\hat{X}_N(t) = \lim_{n\to\infty} X_{N,n}(t) \qquad \text{for all } t$$

where $X_{N,0}(t) = 0$, and for $n = 0,1,2,\ldots$

$$X_{N,n+1}(t) = X_{N,n}(t) + c\{Z_N(t) - (P_N A X_{N,n})(t)\} ,$$

$0 < c < 2/M$, and

$$E\int_{-\infty}^{\infty} \frac{[X(t)-X_N(t)]^2}{1+t^2}\, dt \le \left(1 + \frac{M^2}{m^2}\right)\left\{ \int_{-\infty}^{\infty} \frac{R(t,t)}{1+t^2}\, dt - \iint_{-\infty}^{\infty} \frac{R(t,s)K_N(t,s)}{(1+t^2)(1+s^2)}\, dtds \right\} \underset{N\uparrow\infty}{\downarrow 0}$$

where K_N is defined by (4).

PROOF. Again, the convergence of the algorithm is shown as in Theorem 3, and as in the proof of Theorem 4, $\hat{X}_N$ is a measurable process with almost all sample functions of $\hat{X}_N$ in $\mathcal{M}_N$. We also have

$$(P_N A\hat{X}_N)(t) = Z(t) = (P_N AX)(t) \quad .$$

For any $f_1, f_2 \in \mathcal{M}_N$ we have [7] $m||f_1-f_2|| \le ||P_N Af_1 - P_N Af_2||$ and hence with probability one

$$m||P_N X - \hat{X}_N|| \le ||P_N AP_N X - P_N A\hat{X}_N|| \quad .$$

Also for any $f_1, f_2 \in L_2(\mu)$ we have for A satisfying (12), $||P_N Af_1 - P_N Af_2|| \le M||f_1-f_2||$ and thus with probability one

$$||P_N A P_N X - P_N A X|| \le M||P_N X - X|| \ .$$

It follows that with probability one

$$||P_N X - \hat{X}_N|| \le \frac{M}{m}||P_N X - X||$$

and thus $||X-\hat{X}_N||^2 \le ||X-P_N X||^2 + ||P_N X - \hat{X}_N||^2 \le (1 + \frac{M^2}{m^2})||P_N X-X||^2 =$ $(1 + \frac{M^2}{m^2})\{||X||^2 - ||P_N X||^2\}$. By taking expectations we obtain the desired result, if we note that for every $f \epsilon B(W)$, $||f||^2 - ||P_N f||^2 =$ $\sum_{n=N+1}^{\infty}|\langle f, \phi_n\rangle|^2 \downarrow 0$ as $N \uparrow \infty$. This inequality also implies that $\int_{-\infty}^{\infty} E[\hat{X}_N^2(t)]d\mu(t) < \infty$ and thus, as in the proof of Theorem 4, $\hat{X} \in B(W)$.

□

It can be easily shown that if $\{\phi_n\}_{n=1}^{\infty}$ are the eigenfunctions and $\{\lambda_n\}_{n=1}^{\infty}$ are the corresponding nonzero (hence positive) eigenvalues of the integral type operator in $L_2(\mu)$ with kernel $R(t,s)$ then the approximation error is given by

$$E\int_{-\infty}^{\infty} \frac{E[X(t)-\hat{X}_N(t)]^2}{1+t^2} dt = \left(1 + \frac{M^2}{m^2}\right) \sum_{n=N+1}^{\infty} \lambda_n \quad \underset{N\uparrow\infty}{\downarrow 0}$$

(the eigenfunctions ϕ_n are easily shown to be in $B(W)$). It should also be remarked, as it is clear from the proof of Theorem 5, that if with probability one the sample functions of X belong to M_N for some N, then $\hat{X}_N(t) = X(t)$ and thus we have perfect reconstruction.

In the reconstruction algorithms of Theorems 3 to 5 the linear system L is either the projection P on $B(W)$, or the projection P_N on an N-dimensional subspace (M_N) of $B(W)$. We now take L to be a time invariant linear system with impulse response specified below in a simple manner, and we prove that an appropriately modified algorithm recovers the input. The inpulse response ϕ of L satisfies the following

> $\phi(t)$ is the inverse Fourier transform of a real, symmetric, twice continuously differentiable function $\Phi(\lambda)$ with support $[-W-\delta, W+\delta]$, $\delta > 0$, such that $\Phi(\lambda) = 1$ for $-W \le \lambda \le W$. (13)

Then there is a constant C such that for all t

$$|\phi(t)| \le \frac{C}{\pi(1+t^2)} . \tag{14}$$

This is seen as follows. Integrating by parts $\phi(t) = (1/2\pi)\int_{-(W+\delta)}^{(W+\delta)} \Phi(\lambda)e^{i\lambda t}dt$ we have

$$(-it)^k\phi(t) = \frac{1}{2\pi}\int_{-(W+\delta)}^{W+\delta} \Phi^{(k)}(\lambda)e^{it\lambda}d\lambda \quad , \quad k=1,2$$

and thus

$$(1+t^2)\phi(t) = \frac{1}{2\pi}\int_{-(W+\delta)}^{W+\delta} \{\Phi(\lambda) - \Phi^{(2)}(\lambda)\}e^{i\lambda t}d\lambda .$$

It follows that C can be taken

$$C = \sup_t \frac{1}{2}\left|\int_{-(W+\delta)}^{W+\delta} \{\Phi(\lambda)-\Phi^{(2)}(\lambda)\}e^{it\lambda}d\lambda\right| \le \frac{1}{2}\int_{-(W+\delta)}^{W+\delta} \left|\Phi(\lambda)-\Phi^{(2)}(\lambda)\right|d\lambda . \tag{15}$$

If $y \epsilon L_2(\mu)$ and

$$z(t) = (Ly)(t) = \int_{-\infty}^{\infty} \phi(t-\tau)y(\tau)d\tau$$

then, in view of (14), it follows from Lemma 2(a) that $z \epsilon L_2(\mu)$ and

$$||z|| = ||Ly|| \le C||y|| . \tag{16}$$

Also, if $h'(t) = h(t; W+\delta, \delta')$, $\delta' > 0$ (see (1)), it is obvious from (13) that $\phi = \phi * h'$, and by applying Fubini's theorem we obtain $z = z * h'$ and thus $z \in B(W+\delta)$. It follows that L is a bounded linear operator from $L_2(\mu)$ into $B(W+\delta)$. We now prove the corresponding result to Theorem 3.

THEOREM 6. Let X be a process bandlimited to W ($X \in B(W)$), let A satisfy (12), let L have an impulse response ϕ satisfying (13), and let $Z = LAX$. Then with probability one

$$X(t) = \lim_{n\to\infty} X_n(t) \qquad \text{for all } t$$

where $X_0(t) = 0$ and for $n = 0,1,2,\ldots$

$$X_{n+1}(t) = (LX_n)(t) + c\{Z(t) - (LAX_n)(t)\}$$

for all $\frac{1}{m}(1 - \frac{1}{C}) < c < \frac{1}{M}(1 - \frac{1}{C})$, provided that ϕ and A are such that

$$C < \frac{1 + \frac{m}{M}}{1 - \frac{m}{M}} \quad . \tag{17}$$

PROOF. All statements in this proof are valid with probability one. We first show that $t\phi(t) \in L_1(m)$. Since $\phi(t)$ is continuous, it suffices to prove integrability over $|t| \geq 1$. This follows from the Cauchy-Schwarz inequality and the observation that over $|t| \geq 1$, the functions $\frac{1}{t}$ and $t^2\phi(t)$ are in $L_2(m)$ (the latter since it is the Fourier transform of $\Phi^{(2)} \in L_2(m)$). It is then clear that ϕ satisfies the assumptions of Lemma 2, and since $X \in B(W)$, we have

$$(LX)(t) = (\phi * X)(t) = X(t) \quad .$$

Then $X - X_{n+1} = L[X - X_n - c\{AX - AX_n\}]$, and by (16),

$$||X-X_{n+1}|| \le C||X-X_n-c\{AX-AX_n\}|| .$$

If θ is defined by

$$\theta = \sup_{x,y}\left|1 - c\,\frac{A(x) - A(y)}{x - y}\right|$$

and c is as in the statement of the theorem, it is easily seen that $\alpha = C\theta < 1$ and

$$||X-X_{n+1}|| \le \alpha||X-X_n|| .$$

It follows that $||X-X_{n+1}|| \le \alpha^{n+1}||X|| \to 0$ as $n\to\infty$, and the pointwise convergence follows by Lemma 2(c) for the space $B(W+\delta)$. □

The inequality (17) is a joint constraint on the nonlinear system determined by (12) and the linear system determined by (13). If a linear system satisfying (13) is given then C can be calculated and Theorem 6 applies to the nonlinear systems satisfying (12) and $\frac{m}{M} > \frac{C-1}{C+1}$. However, if a nonlinear system satisfying (13) is given, then it is not known how to construct a linear system satisfying (13) and (17) and moreover, such a linear system may not even exist for some values of $\frac{m}{M}$. Hence the following problem, whose solution is not known to us, is of interest: characterize the set of real numbers C defined by (15) for all Φ satisfying (13).

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