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SOME REMARKS ON THE EQUIVALENCE OF GAUSSIAN PROCESSES

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A. GUALTIEROTTI AND S. CAMBANIS

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1. INTRODUCTION. Let \( X = \{X_t, \ t \in T\} \) and \( Y = \{Y_t, \ t \in T\} \) be real, zero-mean Gaussian processes with respective covariances \( R_X \) and \( R_Y \), defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\), where \( T \) is an arbitrary index set.

Denote by \( P_X \) and \( P_Y \) the probabilities induced on \((R^T, \mathcal{C}(R^T))\) by \( P, X \) and \( Y \) respectively, where \( \mathcal{C}(R^T) \) is the \( \sigma \)-algebra generated by the cylinder sets of the set \( R^T \) of real functions on \( T \). \( L_2(X) \) and \( L_2(Y) \) will denote the subspaces of \( L_2(\Omega, \mathcal{F}, \mathbb{P}) \) generated by \( X \) and \( Y \) respectively, and \( H(X) \) and \( H(Y) \) the corresponding reproducing kernel Hilbert spaces (RKHS's).

The associated canonical isometries will be denoted by \( U_X \) and \( U_Y \) respectively \((U_X t = R_X(t, \cdot), \ t \in T, \) and similarly for \( U_Y \)). We say that the processes \( X \) and \( Y \) are equivalent, or that the probabilities \( P_X \) and \( P_Y \) are equivalent, \( P_X \sim P_Y \), if \( P_X \) and \( P_Y \) are mutually absolutely continuous. The following properties are well known:

(i) \( P_X \sim P_Y \) if and only if \( Y_t = FX_t, \ t \in T \), where \( F \) is an equivalence operator from \( L_2(X) \) to \( L_2(Y) \) (i.e., \( F \) has bounded inverse and \( I-F^*F \) is Hilbert-Schmidt) \([6]\); or equivalently if and only if \( Y_t = X_t - AX_t, \ t \in T \), where \( A \) is a Hilbert-Schmidt operator in \( L_2(X) \) and \( I-A \) has bounded inverse, and the equality is in law, i.e. \( P_Y = P(I-A)X \) \([9]\).

(ii) If \( P_X \sim P_Y \) then \( sH(X) = sH(Y) \), where \( s \) indicates that what follows is considered as a set and not as a space \([7, \ p. 181]\).

The first question considered in this note is the converse of (ii), i.e., if \( X \) and \( Y \) have the same RKHS's, under what additional condition on the RKHS's are they equivalent? The answer is given in Propositions 1 and 2 and results in a characterization of the norms in a RKHS corresponding to equivalent Gaussian processes.

The fact, mentioned in (i), that all Gaussian processes equivalent to \( X_t \)
are of the form $X_t - AX_t$, with $A$ a Hilbert-Schmidt operator in $L_2(X)$, raises the problem of expressing $AX_t$ in a more explicit way in terms of the process $X$. This is done in Propositions 4, 5 and 6.

2. THE RKHS OF EQUIVALENT GAUSSIAN PROCESSES. Here, and in the next section, we adopt the notation of the introduction.

Proposition 1. $P_X \sim P_Y$ if and only if $sh(X) = sh(Y)$ and (a) the identity $J$ on $sh(X) = sh(Y)$ is an equivalence operator from $H(X)$ to $H(Y)$, or (b) for every $f$ in $sh(X)$

$$||f||^2_{H(Y)} = ||f||^2_{H(X)} + \langle f \otimes f, \lambda \rangle_{H(X) \otimes H(X)}$$

for some $\lambda \in H(X) \otimes H(X)$ which is symmetric and such that $-R_X \ll \lambda$.

Proof. Suppose first that $sh(X) = sh(Y)$ and $J$ is an equivalence. For every $\xi \in L_2(Y)$ we have $<\xi, Y_t>_{L_2(Y)} = (U_Y(t)(t)) = (J^{-1}U_Y(t)(t)) =<U_X^* J^{-1} U_Y \xi, X_t>_{L_2(X)}$. Let $F^* = U_X^* J^{-1} U_Y$. Since $J$ is an equivalence, so is $J^{-1}$, and since $U_X^*, U_Y$ are unitary, $F^*$ is an equivalence and so is $F$.

It now follows from $<\xi, Y_t>_{L_2(Y)} = <F^* \xi, X_t>_{L_2(X)} = <\xi, FX_t>_{L_2(Y)}$, for all $\xi$ in $L_2(Y)$, that $Y_t = FX_t$. Thus $P_X \sim P_Y$.

Conversely, suppose that $P_X \sim P_Y$. Then $Y_t = FX_t$ where $F$ is an equivalence operator from $L_2(X)$ to $L_2(Y)$. For every $f$ in $H(Y)$ we have

$f(t) = <U_Y^* f, Y_t>_{L_2(Y)} = <F^* U_Y^* f, X_t>_{L_2(X)} = [U_X^* F^* U_Y^* f](t) = [J U_X^* F^* U_Y^*](t)$. Thus $J U_X^* F^* U_Y^* = I_{H(Y)}$ and $J = U_Y(F^*)^{-1} U_X^*$. Since $F$ is an equivalence, so is $(F^*)^{-1}$ and since $U_X^*, U_Y$ are unitary, $J$ is an equivalence.

Finally, (b) is equivalent to (a) as it follows from Property (i) and the fact that Hilbert-Schmidt operators on RKHS's have kernels in the direct product of the considered RKHS's [1].

The characterization of Proposition 1 is particularly useful if the elements in the common RKHS can be obtained in the way described in the following Proposition.
PROPOSITION 2. Suppose there exists a pair \((H, L)\), where \(H\) is a Hilbert space and \(L\) a unitary map from \(H\) to \(H(X)\). Then \(P_X \sim P_Y\) if and only if \(sh(X) = sh(Y)\) and for all \(h\) in \(H\),

\[
||Lh||^2_{H(Y)} = ||h||^2_H + \langle Kh, h \rangle_H
\]

where \(K\) is a self-adjoint, Hilbert-Schmidt operator on \(H\) such that \(-1 < \sigma(K)\).

REMARK 1. Condition (1) is equivalent to

(a) \(||Lh||^2_{H(Y)} = ||h||^2_H + \langle Kh, h \rangle_H\), or

(b) \(\langle Lh, Lh \rangle_{H(Y)} = \langle Lh, Lh \rangle_{H(X)} + \langle LKh, Lh \rangle_{H(X)} = \langle h, h \rangle_H + \langle Kh, h \rangle_H\).

PROOF OF PROPOSITION 2. Suppose first that \(P_X \sim P_Y\). Then \(J\) is an equivalence and \(J\) can be decomposed as \(J = UW\), with \(W = (J^*J)^{\frac{1}{2}}\) and \(U\) unitary. \((W: H(X) \to H(X)\) and \(U: H(X) \to H(Y)\)). Since \(W\) is onto, for every \(h\) in \(H\) there is a \(g\) in \(H\) such that \(Lh = Wg\). Thus every \(h\) in \(H\) can be obtained as \(L^*Wg\), for some \(g\) in \(H\). Set \(S = L^*WL\). Then \(J = ULSL^*\) and thus for \(h\) in \(H\),

\[
||JLh||^2_{H(Y)} = ||ULSL^*Lh||^2_{H(Y)} = ||LSh||^2_{H(X)} = ||Sh||^2_H.
\]

Now it follows from \(S = L^*U^*JL\) that \(S\) is an equivalence, since it is obtained from an equivalence operator \(J\) by "unitary multiplication." Thus \(I_H - S^*S\) is equal to a self-adjoint, Hilbert-Schmidt operator \(-K\) that does not have \(-1\) among its eigenvalues. Hence, as

\[
||JLh||^2_{H(Y)} = ||Sh||^2_H = \langle (I+K)h, h \rangle_H = ||h||^2_H + \langle Kh, h \rangle_H,
\]

and (1) follows from (a) of Remark 1.

Conversely, suppose that \(sh(X) = sh(Y)\) and (1) holds. Define a unitary operator \(T: H \to H(Y)\) by \(T((I_H + K)^{\frac{1}{2}}h) = JLh\). This definition makes sense since \(T\) is obviously onto and by (1),

\[
||T((I_H + K)^{\frac{1}{2}}h)||^2_{H(Y)} = ||JLh||^2_{H(Y)} = ||(I_H + K)^{\frac{1}{2}}h||^2_H.
\]

But then \(J = T((I_H + K)^{\frac{1}{2}}L^*\), where \(T\) and \(L^*\) are unitary and \((I_H + K)^{\frac{1}{2}}\) is an
equivalence. Hence, \( J \) is an equivalence and \( P_X \sim P_Y \).

REMARK 2. The existence of the assumed pair \((H, L)\) in Proposition 2 is not as restrictive as it appears. In fact, whenever \( H(X) \) (or equivalently \( L_2(X) \)) is separable, this assumption is satisfied and one can take \( H \) to be an \( L_2 \) space. This follows from Theorem 2 of [3]. Indeed, if \( H(X) \) is separable, then, for an arbitrary measure space \((E, \mathcal{E}, \mu)\) such that \( L_2(\mu) \) is separable and infinite dimensional, we have for all \( t \) in \( T \), \( X_t = \int_E \phi_t(u) dZ(u) \), where \( \phi_t \in L_2(\mu) \) and \( Z \) is an orthogonal random measure on \( E \) such that \( L_2(Z) = L_2(X) \). This implies that there is a unitary map \( A: L_2(\mu) \rightarrow L_2(X) \) such that \( A\phi_t = X_t, t \in T \), and clearly \( L: L_2(\mu) \rightarrow H(X) \) defined by \( L = \bigcup_X A \) is unitary. The Hilbert space \( H \) is clearly non-unique. However in some specific cases, \( H \) can be chosen in a natural way. In fact, most of the known RKHS's are obtained in this way. We list some examples below.

EXAMPLES. 1) Let \( X \) have orthogonal increments on \([0,1]\) and start almost surely at the origin, with \( R_X(s,t) = F(s \wedge t), F \) continuous. Then

\[
H(X) = \{ \int_0^t \phi(u) dF(u), \phi \in L_2(dF) \}, < \int_0^t \phi u, \int_0^t \psi dF>_H(X) = <\phi, \psi>_{L_2(dF)},
\]

and \( L: L_2(dF) \rightarrow H(X) \) defined by \( [L\phi](t) = \int_0^t \phi dF \) is an isometry. This example includes the Wiener process \((F(u) = u)\).

2) Let \( X \) have the covariance \( R_X(s,t) = F(s \wedge t) G(s \wedge t) \), where \( F \) and \( G \) are continuous with bounded variation on \([0,1]\), \( F \) is strictly positive, except at the origin, and \( G \) is strictly positive. Suppose further that \( H(u) = F(u)/G(u) \) is strictly increasing. Then \( H(X) = \{ G(t) \int_0^t \phi(u) dH(u), \phi \in L_2(dH) \} \) and

\[
<G(\cdot) \int_0^\cdot \phi dH, G(\cdot) \int_0^\cdot \psi dH>_H(X) = <\phi, \psi>_{L_2[0,1]}.
\]

Thus \( L: L_2(dH) \rightarrow H(X) \) defined by \( [L\phi](t) = G(t) \int_0^t \phi dH \) is unitary.

3) Let \( X \) be a linear operation on a stationary process with spectral measure \( \mu \). Then \( R_X(s,t) = \int_{-\infty}^{\infty} \phi_s(\lambda) \overline{\phi_t(\lambda)} d\mu(\lambda) \) and if \( H \) is the closure in \( L_2(\mu) \)
of the linear span of \( \{ \phi_t, t \in T \} \) then \( H(X) = \{ \int_{-\infty}^{\infty} \phi_t \overline{\phi}_u, \phi \in H \} \) and \( \langle \phi, \psi \rangle_{H(X)} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_t \overline{\phi}_u \, d\mu \). Thus \( L : H \to H(X) \) defined by \( [L\phi](t) = \int_{-\infty}^{\infty} \phi_t \overline{\phi}_u \, d\mu \) is unitary. This includes the case where \( X \) is stationary \( (\phi_t(\lambda) = e^{it\lambda}) \) and then \( H = L_2(\mu) \) if \( T = R \).

4) Let \( X \) be a linear operation on a harmonizable process with two-dimensional spectral measure \( r \). Then \( R_X(s,t) = \int_{-\infty}^{\infty} \phi_s(u) \overline{\phi}_t(v) \, dr(u,v) \) and if \( H \) is the closure in the Hilbert space \( \Lambda_2(r) [5] \) of the linear span of \( \{ \phi_t, t \in T \} \) then
\[
H(X) = \{ \int_{-\infty}^{\infty} \phi_t(u) \overline{\psi}_v \, dr(u,v), \, \phi \in H \}
\]
and
\[
\langle \int_{-\infty}^{\infty} \phi_t \overline{\psi}r, \int_{-\infty}^{\infty} \phi_u \overline{\psi}u \rangle_{H(X)} = \langle \phi, \psi \rangle_{\Lambda_2(r)} = \int_{-\infty}^{\infty} \phi(u) \overline{\psi}(v) \, dr(u,v).
\]
Thus \( L : H \to H(X) \) defined by \( [L\phi](t) = \int_{-\infty}^{\infty} \phi_t(u) \overline{\psi}_v \, dr(u,v) \) is unitary. This includes the case where \( X \) is harmonizable \( (\phi_t(u) = e^{it\mu}) \) and then \( H = \Lambda_2(r) \) if \( T = R \).

REMARK 3. The existence of the pair \((H,L)\) as described in Proposition 2 is easily seen to be equivalent to the existence of a representation of the covariance \( R_X \) of the form \( R_X(s,t) = \langle \phi_s, \phi_t \rangle_X \) where \( \{ \phi_t, t \in T \} \subseteq H \).

Proposition 2 can be also expressed in terms of covariances and it then contains as particular cases the results of [10] and [8].

PROPOSITION 3. Suppose there exists a pair \((H,L)\), where \( H \) is a Hilbert space and \( L \) a unitary map from \( H \) to \( H(X) \). Let \( \phi_t = L^* R_X(t,*) \) or \( R_X(s,t) = \langle \phi_s, \phi_t \rangle_X \) (see Remark 3). Then \( P_X \sim P_Y \) if and only if
\[
(2) \quad R_Y(s,t) = R_X(s,t) - \langle M \phi_s, \phi_t \rangle_X
\]
where \( M \) is self-adjoint, Hilbert-Schmidt and such that \( \sigma(M) < 1 \).

PROOF. Assume first that \( P_X \sim P_Y \). Then \( Y_t = FX_t \) where \( F \) is an equivalence operator from \( L_2(X) \) to \( L_2(Y) \). It follows that \( Y_t = FU_X^t L \phi_t \) and
\[
R_Y(s,t) = \langle L^* U_{X} F^* U_{X}^* L \phi_s, \phi_t \rangle_{H}.
\]
Thus (2) is valid with \( M = L^* U_{X} (I_{L_2(X)} - F^* F) U_{X}^* \). Since \( I_{L_2(X)} - F^* F \) is self-adjoint, Hilbert-Schmidt with \( \sigma(I_{L_2(X)} - F^* F) < 1 \), and \( U_{X}, L \) are unitary, it follows that \( M \) is self-adjoint Hilbert-Schmidt with
σ(M) < 1.

Conversely, assume that (2) is valid. Then \( \phi_t = L^*U_Xx_t \) yields

$$
<\phi_t, \phi_t> = \langle U_X^*LML^*U_Xx_s, x_t \rangle L_2(X) \quad \text{and} \quad (2) \text{ is written } R_Y(s, t) =
$$

$$
= \langle (I_{L_2(X)} - U_X^*LML^*U_X)x_s, x_t \rangle L_2(X). \quad \text{Since } M \text{ is self-adjoint, Hilbert-Schmidt,}
$$

with \( \sigma(M) < 1 \), and \( U_X, L \) are unitary, it follows that \( U_X^*LML^*U_X \) is self-adjoint, Hilbert-Schmidt. Also \( I_{L_2(X)} - U_X^*LML^*U_X = U_X^*L(I_H - M)L^*U_X > 0 \) and hence its square root \( F_0 \) is an equivalence. We then have \( \langle Y_s, Y_t \rangle_{L_2(Y)} = R_Y(s, t) = \quad \quad \) 

$$
= \langle F_0x_s, F_0x_t \rangle_{L_2(X)} \quad \text{which implies that the map } F_0x_t \rightarrow Y_t \text{ extends to a unitary operator } U \text{ from } L_2(X) \text{ to } L_2(Y) \text{ and then } Y_t = FX_t \quad \text{where } F = UF_0 \text{ is an equivalence. Thus } P_X \sim P_Y.
$$

REMARK 4. The relationship between the operators \( F, K \) and \( M \) of Property (i) and Propositions 2 and 3 respectively is as follows

$$
F = U[U_X^*L(I_H + K)^{-1}L^*U_X]^{\frac{1}{2}} = U[U_X^*L(I_H - M)L^*U_X]^{\frac{1}{2}}
$$

$$
K = L^*U_X[(F^*F)^{-1} - I_{L_2(X)}]U_X^*L = (I_H - M)^{-1} - I_H
$$

$$
M = L^*U_X(I_{L_2(X)} - F^*F)U_X^*L = I_H - (I_H + K)^{-1}.
$$

We also have

$$
R_Y(s, t) = <\psi_s, \psi_t>_H \quad \text{where } \psi_t = (I_H - M)\phi_t.
$$

PROOF. The expressions relating \( F \) and \( M \) are derived in the proof of Proposition 3. It then suffices to derive the relationship between \( K \) and \( M \).

Since \( sH(X) = sH(Y) \) we have \( R_Y(t, \cdot) \in H(X) \). Let \( \psi_t = L^*R_Y(t, \cdot) \). Then

$$
R_Y(s, t) = <\psi_s, \psi_t>_H. \quad \text{By Remark 1 we obtain } R_X(s, t) = <R_X(s, \cdot), R_Y(t, \cdot)>_{H(Y)} =
$$

$$
= <J\phi_s, J\phi_t>_Y = <\phi_s, (I_H + K)\phi_t>_Y. \quad \text{Since } R_X(s, t) = <\phi_s, \phi_t>_H \text{ we have}
$$

$$
<\phi_s, \phi_t>_H = <\phi_s, (I_H + K)\phi_t>_H. \quad \text{Since } \{R_X(t, \cdot), t \in T\} \text{ is complete in } H(X) \text{ and } L
$$

is unitary, \( \{\phi_t, t \in T\} \) is complete in \( H \) and hence \( \phi_t = (I_H + K)\psi_t \).

We now have \( R_Y(s, t) = <R_Y(s, \cdot), R_X(t, \cdot)>_{H(X)} = <L\phi_s, L\phi_t>_H(X) = <\psi_s, \phi_t>_H = <(I_H + K)^{-1}\phi_s, \phi_t>_H \) and since \( R_X(s, t) = <\phi_s, \phi_t>_H \) it follows from (2) by inspection
that \( M = I_H - (I_H + K)^{-1} \). Hence \( K = (I_H - M)^{-1} - I_H \) and also \( \psi_t = (I_H + K)^{-1} \phi_t = (I_H - M) \phi_t \).

3. REPRESENTATION OF EQUIVALENT GAUSSIAN PROCESSES. When a pair \((H, L)\) as described in Propositions 2 and 3 exists, then one can obtain explicit representations of the process \( AX_t \) in Property (i). Here it is more appropriate to consider the unitary map \( V: H \to L_2(X) \) related to \( L \) by \( V = U^* L \).

PROPOSITION 4. Suppose that there exists a pair \((H, V)\), where \( H \) is a Hilbert space and \( V \) a unitary map from \( H \) to \( L_2(X) \). If \( A \) is a Hilbert-Schmidt operator in \( L_2(X) \), then \( AX_t = VAh_t \), where \( A \) is a Hilbert-Schmidt operator in \( H \) and \( h_t = V^* X_t \).

PROOF. If \( \bar{A} = V^* AV \), then \( \bar{A} \) is Hilbert-Schmidt in \( H \) and \( AX_t = AVh_t = VV^* AVh_t = V\bar{A}h_t \).

EXAMPLES. 5) Let \( X \) be as in Example 1. Then \( V: L_2(dF) \to L_2(X) \) is defined by \( V\phi = \int_0^1 \phi(u) dX_u \). Consequently \( h_t = I_t \), the indicator function of \([0, t]\) and \( AX_t = \int_0^1 \bar{A}h_t(u) dX_u \). Since a Hilbert-Schmidt operator in \( L_2(dF) \) is of integral type with kernel \( \alpha(u, v) \) in \( L_2(dF \times dF) \) we finally have

\[
AX_t = \int_0^1 \int_0^t \alpha(u, v) dF(v) dX_u.
\]

This result is obtained for the Wiener process in [9].

6) Consider the case where the covariance of \( X \) has the representation \( R_X(s, t) = \int_E \phi_s(u) \phi_t(u) d\mu(u) \) with \((E, E, \mu)\) a finite (for simplicity) measure space and \( \phi_t \in L_2(\mu) \). This includes both Examples 2 and 3. Then there is an orthogonal random measure \( Z \) on \( E \) such that \( X_t = \int_E \phi_t(u) dZ(u) \) [5]. Let \( H \) be the closure in \( L_2(\mu) \) of the linear span of \( \{\phi_t, t \in T\} \). Then \( V: H \to L_2(X) \) is defined by \( V\phi = \int_E \phi(u) dZ(u) \). Consequently \( h_t = \phi_t \) and \( AX_t = \int_E \bar{A}\phi_t(u) dZ(u) \). As in Example 5, \( \bar{A} \) is of integral type with kernel \( \alpha(u, v) \) in \( L_2(\mu \times \mu) \) and finally we have

\[
AX_t = \int_{EE} \alpha(u, v) \bar{\phi}_t(v) d\mu(v) dZ(u).
\]

In the case of Example 2, \( H = L_2(dH) \) and \( AX_t = G(t) \int_0^t \alpha(u, v) dH(v) dZ(u) \).
7) Consider the case where the covariance of $X$ has the representation
\[ R_X(s,t) = \int \int \phi_s(u) \overline{\phi}_t(v) \text{d}r(u,v) \] with $(E,E)$ a measurable space, $r$ a two-dimensional spectral measure on $E \times E$ and $\phi_t$ in $\Lambda_2(r)$. Then there is a random measure $Z$ on $E$ such that $X_t = \int \phi_t(u) dZ(u)$ [5]. Let $H$ be the closure in $\Lambda_2(r)$ of the linear span of $\{\phi_t, t \in T\}$. Then $V: H \rightarrow L_2(X)$ is defined by $V\phi = \int \phi(u) dZ(u)$. Consequently $h_t = \phi_t$ and $AX_t = \int \overline{A\phi_t}(u) dZ(u)$, where $\overline{A}$ is a Hilbert-Schmidt operator in $\Lambda_2(r)$. However, no kernel representation of $\overline{A}$ seems to be available because $\Lambda_2$ is a more complicated space than $L_2$. Nevertheless, as follows from (i) of the Lemma at the end of this section, if $E$ is an interval, $\overline{A}$ is the limit in the operator norm of a sequence of Hilbert-Schmidt operators $\{A_n\}_{n=1}^{\infty}$ in $L_2(u)$ with kernels $\{a_n\}_{n=1}^{\infty}$ and we thus have
\[ AX_t = \lim_{n \rightarrow \infty} \int \int \phi_t(v) \overline{a_n}(w,u) d\nu(v,w) dZ(u) \]
where the limit is in $L_2(P)$. Consider now the important particular case where there is a measurable process $\{Z_u, u \in E\}$ and a measure $\mu$ on $E$ equivalent to the Lebesgue measure such that $\int Z(u,u) d\mu(u) < \infty$ and for every $B \subset E$,
\[ Z(B) = \int_B Z_u d\mu(u). \]
Then $X_t = \int \phi_t(u) Z_u d\mu(u)$ and
\[ AX_t = \lim_{n \rightarrow \infty} \int g^{(n)}_t(u) Z_u d\mu(u) \]
where $g^{(n)}_t(u) = \int \phi_t(v) \overline{a_n}(w,u) R_Z(v,w) d\nu(v,w) du(w)$ is in $L_2(u)$, the integral is defined almost surely, i.e., on the paths of $Z$, and the limit is in $L_2(P)$. As particular cases of this example we obtain the following representations.

PROPOSITION 5. Let $X$ be mean square continuous, $T$ an interval and $A$ a Hilbert-Schmidt operator on $L_2(X)$. Then
\[ AX_t = \int g_t(u) X_u d\mu(u) = \lim_{n \rightarrow \infty} \int g^{(n)}_t(u) X_u d\mu(u) \]
where the measure $\mu$ on the Borel sets of $T$ satisfies (ii) of the Lemma, $g_t \in \Lambda_2(R_x \times u \times \mu)$, the first integral is defined in quadratic mean, $g^{(n)}_t \in L_2(u)$,
the second integral is defined almost surely, i.e., on the paths of $X$, and
the limit is in $L_2(P)$.

PROOF. $X$ has a measurable modification which is henceforth considered.
There exist finite measures $\mu$, equivalent to the Lebesgue measure, and such
that $\int_T R_X(t,t) d\mu(t) < \infty$ [2]. Let $dr(u,v) = R_X(u,v) d\mu(u) d\mu(v)$. Since $X$ is
mean square continuous, it has a representation $X_t = \int_T \phi_t(u) X_u d\mu(u)$ with
$\{\phi_t, t \in T\} \subseteq \Lambda_2(\mathbb{R})$ [2]. The result then follows from the last case considered in
Example 7.

PROPOSITION 6. Let $X$ be mean square continuous on $[0,1]$, $X_0 = 0$ a.s., and
$R_X$ of bounded variation on $[0,1] \times [0,1]$. Let $A$ be a Hilbert-Schmidt operator on
$L_2(X)$. Then

$$AX_t = \int_0^1 g_t(u) dX_u = \lim_{n \to \infty} \int g_t^{(n)}(u) dX_u$$

where $g_t \in \Lambda_2(d^2 R_X)$, $\mu$ corresponds to $d^2 R_X$ as in (i) of Lemma, $g_t^{(n)} \in L_2(\mu)$
are of the form $g_t^{(n)}(u) = \int_0^t \int_0^1 a_n(w,u) d^2 R_X(v,w) \, dv$, and $a_n \in L_2(\mu \times \mu)$.

PROOF. The proof is obvious from Example 7 and the observation that

$$X_t = \int_0^1 I_t(u) dX_u, \ I_t \ \text{the characteristic function of the interval } [0,t], \ i.e.,$$

$$\phi_t = I_t.$$

LEMMA. Let $E$ be an interval, $E$ its Borel sets, $r$ a finite, two-dimensional
spectral measure on $E \times E$, and $K$ a Hilbert-Schmidt operator on $\Lambda_2(\mathbb{R})$. If

(i) $\mu$ is the finite measure defined on $E$ by $\mu(B) = |r|(E \times B)$ for all $B \in E$, or if

(ii) $dr(u,v) = R(u,v) d\mu(u) d\mu(v)$, where $R$ is a covariance and $\mu$ a finite measure

on $E$, equivalent to the Lebesgue measure and such that $\int_E R(u,u) d\mu(u) < \infty$,

then $L_2(\mu) \subseteq \Lambda_2(\mathbb{R})$, and there is a sequence of Hilbert-Schmidt operators $\{K_n\}_{n=1}^\infty$
on $L_2(\mu)$ with kernels $\{k_n\}_{n=1}^\infty$, that are defined from $\Lambda_2(\mathbb{R})$ to $L_2(\mu)$ by

$$[K_n f](u) = \langle f(\cdot), k_n (\cdot, u) \rangle_{\Lambda_2(\mathbb{R})}$$

and are such that $K_n \to K$ in the operator norm in

$\Lambda_2(\mathbb{R})$.

PROOF. Both (i) and (ii) imply that $L_2(\mu) \subseteq \Lambda_2(\mathbb{R})$ and that there is a sequence


\( \{f_n\}_{n=1}^{\infty} \) in \( L_2(\mu) \) which is orthonormal and complete in \( \Lambda_2(\mathfrak{r}) \). For (ii) this is shown in [2] and for (i) it is shown as Theorem 2 of [4].

In the sequel \( \langle \cdot, \cdot \rangle \) and \( ||\cdot|| \) denote inner product and norm in \( \Lambda_2(\mathfrak{r}) \). Since \( K \) is Hilbert-Schmidt we have 
\[
\sum_{n,m=1}^{\infty} |\langle Kf_n, f_m \rangle|^2 < \infty.
\]
For every \( f \in \Lambda_2(\mathfrak{r}) \) we have 
\[
f = \sum_{n=1}^{\infty} \langle f, f_n \rangle f_n \quad \text{and thus} \quad Kf = \sum_{n=1}^{\infty} \langle f, f_n \rangle Kf_n = \sum_{m=1}^{\infty} \{ \sum_{n=1}^{\infty} \langle Kf_n, f_m \rangle \langle f, f_n \rangle \} f_m.
\]
Define 
\[
k_N(u,w) = \sum_{n,m=1}^{N} \langle Kf_n, f_m \rangle f_n(u) f_m(w).
\]
Since \( k_N \) is in \( L_2(\mu \times \mu) \), it defines a (finite rank) Hilbert-Schmidt operator \( K_N \) on \( L_2(\mu) \). \( K_N \) is also defined from \( \Lambda_2(\mathfrak{r}) \) to \( L_2(\mu) \) by 
\[
[K_N f](u) = \langle f(\cdot), k_N(\cdot, u) \rangle = \sum_{n,m=1}^{N} \langle Kf_n, f_m \rangle \langle f, f_n \rangle f_m.
\]
Then 
\[
||Kf - K_N f||^2 \leq ||f||^2 \sum_{n,m=N+1}^{\infty} |\langle Kf_n, f_m \rangle|^2
\]
which implies that \( K_N \rightarrow K \) in the operator norm in \( \Lambda_2(\mathfrak{r}) \).

REFERENCES


