WEAK CONVERGENCE OF RAO-BLACKWELL ESTIMATOR OF DISTRIBUTION FUNCTION

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ABSTRACT

Under the condition that the minimal sufficient statistics are transitive, the sequence of Rao-Blackwell estimators of distribution function has been shown to form a reverse martingale sequence. Weak convergence of the corresponding empirical process to a Gaussian process has been established by assuming that the sufficient statistics are U-statistics and utilizing certain results on the convergence of conditional expectations of functions of U-statistics along with the functional central limit theorems for (reverse) martingales by Loyns(1970) and Brown(1971).

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Key Words and Phrases: Gaussian process, Rao-Blackwell estimator, Reverse martingale, Transitive sufficiency, U-statistics, weak convergence.

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1. Introduction. Let $\{\underset{\sim}{X}_i,\ i \geq 1\}$ be a sequence of independent and identically distributed random vectors (iidrv) defined on a probability space $(\Omega, \mathcal{G}, P)$ with each $\underset{\sim}{X}_i$ having a continuous distribution function (df) $F(\underset{\sim}{x})$, $\underset{\sim}{x} \in R^p$, the $p(\geq 1)$ dimensional Euclidean space. For every $n(\geq 1)$, the <u>empirical df</u> F_n is defined by

$$(1.1) \qquad F_n(\underset{\sim}{x}) = n^{-1} \Sigma_{i=1}^{n} c(\underset{\sim}{x} - \underset{\sim}{X}_i), \ \underset{\sim}{x} \in R^p,$$

where for a p-vector $\underset{\sim}{u} = (u_1, \ldots, u_p)'$,

$$(1.2) \qquad c(\underset{\sim}{u}) = 1 \text{ if } \underset{\sim}{u} \geq \underset{\sim}{0} \text{ and is 0 otherwise,}$$

and by $\underset{\sim}{x} \geq \underset{\sim}{y}$ we denote the coordinatewise inequalities $x_j \geq y_j$, $1 \leq j \leq p$. We assume that the df F admits of the existence of a complete sufficient statistic (vector) $\underset{\sim}{T}_n$, where

$$(1.3) \qquad \underset{\sim}{T}_n' = (T_{n1}, \ldots, T_{nq}), \text{ for some } q \geq 1.$$

[When we encounter the asymptotic situation where $n \to \infty$, we assume that q remains fixed. Note that the sample order statistics (for $p=1$) or the collection matrix (for $p \geq 1$) constitute a sufficient statistic where $q=n$, and in that case, our study is of no real interest.]

Let $\mathcal{B}_n = \mathcal{B}(\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_n)$ be the σ-field generated by $\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_n$, and let $\mathcal{B}_n^{(s)} = \mathcal{B}(\underset{\sim}{T}_n)$ be the σ-field generated by $\underset{\sim}{T}_n$, so that $\mathcal{B}_n^{(s)} \subset \mathcal{B}_n$. Also, let $\mathcal{F}_n = \mathcal{F}(\underset{\sim}{T}_n, \underset{\sim}{T}_{n+1}, \ldots)$ be the σ-field generated by $\{\underset{\sim}{T}_k,\ k \geq n\}$, $n \geq 1$, so that $\mathcal{F}_n$ is a monotone non-increasing σ-field. Finally, let $\mathcal{C}_n$ be the σ-field generated by the unordered collection $\{\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_n\}$ and $\underset{\sim}{X}_{n+1}, \underset{\sim}{X}_{n+2}, \ldots$, $n \geq 1$, so that $\mathcal{C}_n$ is monotone non-increasing and $\mathcal{F}_n \subset \mathcal{C}_n$.

Consider the usual <u>empirical process</u>

$$(1.4) \qquad V_n(\underset{\sim}{x}) = n^{1/2}[F_n(\underset{\sim}{x}) - F(\underset{\sim}{x})], \ \underset{\sim}{x} \in R^p, \ n \geq 1.$$

Then, the Rao-Blackwell empirical process is defined by

$$(1.5) \qquad V_n^*(\underset{\sim}{x}) = E[V_n(\underset{\sim}{x}) \mid \mathcal{B}_n^{(s)}] = n^{1/2}[\Psi_n(\underset{\sim}{x}) - F(\underset{\sim}{x})], \ \underset{\sim}{x} \in R^p,$$

where for every $\underset{\sim}{x} \in R^p$,

$$(1.6) \qquad \Psi_n(\underset{\sim}{x}) = E[F_n(\underset{\sim}{x}) \mid \mathcal{B}_n^{(s)}] = E[c(\underset{\sim}{x} - \underset{\sim}{X}_1) \mid \mathcal{B}_n^{(s)}]$$

is the Rao-Blackwell empirical df. Note that when $T_{\tilde{n}}$ is a complete sufficient statistic, Ψ_n is the unique minimum variance unbiased estimator (UMVU) of F. Specific expressions for Ψ_n for various F (mostly belonging to the exponential family) have been worked out by various workers; we may refer to Tate(1959), Ghurye and Olkin(1969) where additional references are cited. In the context of the UMVU estimation of the density function $f(\tilde{x})$ and its various functions, the recent works of Seheult and Quesenberry(1971) and O'Reilly and Quesenberry(1972) deserve mention.

In view of the fact that the empirical process in (1.4) weakly converges to an appropriate multiparameter Gaussian process [viz., Neuhaus(1971) for $p \geq 1$ and Billingsley(1968) for p=1], our interest centers here on deriving similar weak convergence results for the process V_n^* in (1.5). That, in general, the weak convergence of V_n does not necessarily imply the same for V_n^* can easily be verified with the simple example of the uniform $[0,\theta]$ df, where $\Psi_n(x) = (n-1)x/nX_{(n)}$, if $0 \leq x < X_{(n)}$ and is 1 for $x \geq X_{(n)}$; $X_{(n)} = \max\{ X_i: 1 \leq i \leq n\}$, and the distribution of $\sup\{ |\Psi_n(x) - F(x)|: 0 \leq x \leq \theta \}$ is independent of θ . Hence, assuming $\theta = 1$, $n^{1/2}\{\sup_{0 \leq x \leq 1} |\Psi_n(x) - F(x)|\} \leq \max\{ n^{1/2}|(n-1)/nX_{(n)} - 1|, n^{1/2}|1-X_{(n)}| \} \to 0$, in probability, as $n \to \infty$. We also note that by definition in (1.6), $\Psi_n(\tilde{x})$ ceases to be an average of iidrv and $(n+1)\Psi_{n+1}(\tilde{x}) - n\Psi_n(\tilde{x})$ is no longer stochastically independent of $n\Psi_n(\tilde{x})$. Thus, the basic technique of deriving the functional central limit theorems, displayed in detail in Billingsley(1968) and extended to the multiparameter case by others, is not readily applicable. our task is accomplished by showing that under the additional assumption of the transitivity of $T_{\tilde{n}}$, for every $\tilde{x} \in R^p$, $\{ \Psi_n(\tilde{x}), \mathfrak{F}_n; n \geq 1\}$ is a reverse martingale on which the central limits theorems of Loyns(1970) and Brown(1971) can be applied.

The basic results on Ψ_n and some properties of U-statistics are studied in section 2. The main theorem is stated and proved in section 3. The last section includes (by way of concluding remarks) certain additional results.

2. Some basic lemmas. Let us denote by

(2.1) $\quad B(\underset{\sim}{a},\underset{\sim}{b}) = \{ \underset{\sim}{x} : \underset{\sim}{a} \le \underset{\sim}{x} \le \underset{\sim}{b} \}$ where $\underset{\sim}{a} \le \underset{\sim}{b}$;

(2.2) $\quad P_F(\underset{\sim}{a},\underset{\sim}{b}) = P\{ \underset{\sim}{X}_1 \in B(\underset{\sim}{a},\underset{\sim}{b}) \} = P\{ \underset{\sim}{a} \le \underset{\sim}{X}_1 \le \underset{\sim}{b} \}.$

Also, for every $n \ge 1$, let

(2.3)
$$Z_n^*(\underset{\sim}{a},\underset{\sim}{b}) = P\{ \underset{\sim}{X}_1 \in B(\underset{\sim}{a},\underset{\sim}{b}) | \mathcal{B}_n^{(s)} \} = \Sigma^* (-1)^{\underset{\sim}{j}'\underset{\sim}{1}} \Psi_n(j_i a_i + (1-j_i)b_i, 1 \le i \le p),$$

where the summation Σ^* extends over all $\underset{\sim}{j}' = (j_1,\ldots,j_p)$, $j_i = 0,1, i=1,\ldots,p$.

Lemma 2.1. Let C and h be some positive numbers such that

(2.4) $\quad 1 \le nh \le n$ and $P_F(\underset{\sim}{a},\underset{\sim}{b}) \le Ch.$

Then, for every positive integer s , there exists a constant K_s (independent of h, n and $P_F(\underset{\sim}{a},\underset{\sim}{b})$), such that

(2.5) $\quad n^s E| Z_n^*(a,b) - P_F(a,b) |^{2s} \le K_s h^s.$

Proof. Let us define

(2.6) $\quad Z_n(\underset{\sim}{a},\underset{\sim}{b}) = \Sigma^*(-1)^{\underset{\sim}{j}'\underset{\sim}{1}} F_n(j_i a_i + (1-j_i)b_i, 1 \le i \le p),$

so that by (1.6) and (2.3),

(2.7) $\quad Z_n^*(\underset{\sim}{a},\underset{\sim}{b}) = E[Z_n(\underset{\sim}{a},\underset{\sim}{b}) | \mathcal{B}_n^{(s)}].$

Therefore, by the Jensen inequality,

(2.8)
$$E|Z_n^*(\underset{\sim}{a},\underset{\sim}{b}) - P_F(\underset{\sim}{a},\underset{\sim}{b})|^{2s} = E|E\{Z_n(\underset{\sim}{a},\underset{\sim}{b}) - P_F(\underset{\sim}{a},\underset{\sim}{b}) | \mathcal{B}_n^{(s)}\}|^{2s} \le E|Z_n(\underset{\sim}{a},\underset{\sim}{b}) - P_F(\underset{\sim}{a},\underset{\sim}{b})|^{2s}.$$

Now, $nZ_n(\underset{\sim}{a},\underset{\sim}{b})$ has the binomial distribution with the parameters $(n, P_F(\underset{\sim}{a},\underset{\sim}{b}))$, so that by Lemma 5.2 of Neuhaus(1971), under (2.4),

(2.9) $\quad E| Z_n(\underset{\sim}{a},\underset{\sim}{b}) - P_F(\underset{\sim}{a},\underset{\sim}{b}) |^{2s} \le K_s n^{-s} h^s.$

The lemma follows directly from (2.8) and (2.9).

Lemma 2.2. If $\{ \underset{\sim}{T}_n, n \ge 1 \}$ is transitive sufficient, then for every $\underset{\sim}{x} \in R^p$, $\{\Psi_n(\underset{\sim}{x}), \mathcal{F}_n; n \ge 1 \}$ is a reverse martingale.

Proof. By definition in (1.6),

(2.10) $\Psi_{n+1}(\underset{\sim}{x}) = E\{c(\underset{\sim}{x}-\underset{\sim}{X}_1) | \mathcal{B}_{n+1}^{(s)}\} = E\{E[c(\underset{\sim}{x}-\underset{\sim}{X}_1) | \sigma(\mathcal{B}_n^{(s)}, \mathcal{B}_{n+1}^{(s)})] | \mathcal{B}_{n+1}^{(s)}\}$.

Since $\{\underset{\sim}{T}_n\}$ is transitive, by the Wijsman theorem [viz., Zacks(1971,p.84)], $\mathcal{B}_n$ and $\mathcal{B}_{n+1}^{(s)}$ are conditionally independent given $\mathcal{B}_n^{(s)}$. Therefore,

(2.11) $E[c(\underset{\sim}{x}-\underset{\sim}{X}_1) | \sigma(\mathcal{B}_n^{(s)}, \mathcal{B}_{n+1}^{(s)})] = E[c(\underset{\sim}{x}-\underset{\sim}{X}_1) | \mathcal{B}_n^{(s)}] = \Psi_n(\underset{\sim}{x})$ a.s.

Hence, from (2.10) and (2.11), we have for every $n \geq 1, \underset{\sim}{x} \in R^p$,

(2.12) $\Psi_{n+1}(\underset{\sim}{x}) = E\{\Psi_n(\underset{\sim}{x}) | \mathcal{B}_{n+1}^{(s)}\} = E\{\Psi_n(x) | \mathcal{F}_{n+1}\}$ a.s.,

and the lemma follows.

For a large class of df's (including the exponential family), $\underset{\sim}{T}_n$ can be equivalently expressed in terms of a set of U-statistics. Thus, we write

(2.13) $\underset{\sim}{T}_n' = [U_n^{(1)}, \ldots, U_n^{(q)}]$, for $n \geq m \geq 1$,

where m is the maximum of the individual degrees of the kernels for the $U_n^{(j)}$. From the results of Hoeffding(1948), it follows that if the kernels are all square integrable, then

(2.14) $E\{(\underset{\sim}{T}_n - E\underset{\sim}{T}_n)'(\underset{\sim}{T}_n - E\underset{\sim}{T}_n)\} = n^{-1}\underset{\sim}{A}_1 + n^{-2}\underset{\sim}{A}_2 + \ldots,$

where the $\underset{\sim}{A}_k$, $k \geq 1$ are positive semi-definite matrices. For proving a few results on U-statistics, to follow in the lemmas 2.3-2.6, we assume for simplificity that q=1, and let $\theta = E\phi(\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_m)$ where ϕ is the symmetric kernel for U_n. Let then

(2.15) $\phi_h(\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_h) = E\phi(\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_h, \underset{\sim}{X}_{h+1}, \ldots, \underset{\sim}{X}_m)$,

(2.16) $\zeta_h = E\{\phi_h^2(\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_h) - \theta^2\}$, $h=0,\ldots,m$,

so that $\phi_o = 0$, $\zeta_o = 0$ and for q=1, $A_1 = m^2 \zeta_1$.

<u>Lemma 2.3</u>. <u>If</u> $E\{\phi^2(\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_m)\} < \infty$, <u>then</u>

(2.17) $n(n+1)E\{(T_n - T_{n+1})^2 | \mathcal{C}_{n+1}\} \to m^2 \zeta_1$ a.s., <u>as</u> $n \to \infty$.

Proof. Following Miller and Sen(1972), for all $n \geq m$,

(2.18) $$T_n = \binom{n}{m}^{-1} \Sigma_{C_{n,m}} \int \cdots \int_{R^{pm}} \phi(\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_m) \prod_{j=1}^{m} d[c(\underset{\sim}{x}_j - \underset{\sim}{X}_{i_j})]$$

$$= \theta + \Sigma_{h=1}^{m} \binom{m}{h} T_{n,h} ,$$

where

(2.19) $T_{n,1} = n^{-1} \Sigma_{i=1}^{n}[\ \phi_1(\underset{\sim}{X}_i) - \theta\ ],$

(2.20) $T_{n,h} = \binom{n}{h}^{-1} \Sigma_{C_{n,h}}\ \phi_h^*(\underset{\sim}{X}_{i_1},\ldots,\underset{\sim}{X}_{i_h});$

(2.21) $\phi_h^*(\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_h) = \phi_h(\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_h) - \Sigma_{j=1}^{h}\phi_{h-1}(\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_{j-1},\underset{\sim}{x}_{j+1},\ldots,\underset{\sim}{x}_h)$

$+\ \Sigma_{1\le j<k\le h}\ \phi_{h-2}(\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_{j-1},\underset{\sim}{x}_{j+1},\ldots,\underset{\sim}{x}_{k-1},\underset{\sim}{x}_{k+1},\ldots,\underset{\sim}{x}_h) -$

$\ldots + (-1)^k\theta$, for $h=2,\ldots,m$.

To avoid notational complexities, we shall take the case of $m=2$; the extension to the case of $m \geq 3$ can be made ina similar but more laborious way. For $m=2$,

(2.22) $T_n - T_{n+1} = 2(T_{n,1}-T_{n+1,1}) + (T_{n,2} - T_{n+1,2}).$

Hence,

(2.23) $n(n+1)(T_n - T_{n+1})^2 = 4n(n+1)(T_{n,1} - T_{n+1,1})^2 +$

$n(n+1)(T_{n,2}-T_{n+1,2})^2 + 4n(n+1)(T_{n,1}-T_{n+1,1})(T_{n,2}-T_{n+1,2}).$

Now,

(2.24) $T_{n,1} - T_{n+1,1} = n^{-1}[\ T_{n+1,1} - \phi_1(\underset{\sim}{X}_{n+1}) + \theta\ ],$

so that

(2.25) $n(n+1)E\{\ (T_{n,1} - T_{n+1,1})^2\ |\ \mathcal{C}_{n+1}\ \}$

$= n^{-1}(n+1)[\ (n+1)^{-1}\ \Sigma_{i=1}^{n+1}\{\ \phi_1(\underset{\sim}{X}_i) - \theta\}^2 - T_{n+1,1}^2].$

By the Kintchine strong law of large numbers, $(n+1)^{-1}\ \Sigma_{i=1}^{n+1}\{\ \phi_1(\underset{\sim}{X}_i) - \theta\ \}^2 \ \rightarrow\ \zeta_1$ a.s., as $n \rightarrow \infty$, whereas $\{T_{n+1,1}, \mathcal{C}_{n+1}\}$ is a reverse martingale sequence which converges a.s. to its expectation 0, and hence, $T_{n+1,1}^2$ also a.s. converges to 0 as $n \rightarrow \infty$. Again

(2.26) $T_{n,2} - T_{n+1,2} = \frac{2}{n-1}T_{n+1,2} - \binom{n}{2}^{-1}\Sigma_{i=1}^{n}\ \phi_2^*(\underset{\sim}{X}_i,\underset{\sim}{X}_{n+1}).$

Hence,

(2.27) $n(n+1)(T_{n,2} - T_{n+1,2})^2 \leq (n-1)^{-2}8n(n+1)T_{n+1,2}^2 +$

$n(n+1)2\binom{n}{2}^{-2}\ \{\Sigma_{i=1}^{n}\ \phi_2^*(\underset{\sim}{X}_i,\underset{\sim}{X}_{n+1})\ \}^2.$

By the Berk (1966) reverse martingale property of U-statistics, $\{T_{n+1,2}, \mathcal{C}_{n+1}\}$ is a reverse martingale with expectation 0, so that $E(T_{n+1,2}^2 |\ \mathcal{C}_{n+1}) = T_{n+1,2}^2$ a.s. converges to 0 as $n \rightarrow \infty$. Whereas

$$E\{(\Sigma_{i=1}^{n} \phi_2^*(\underset{\sim}{X}_i,\underset{\sim}{X}_{n+1}))^2|\mathcal{C}_{n+1}\}$$

(2.28) $$= n\binom{n+1}{2}^{-1} \sum_{1\leq i<j\leq n+1} \phi_2^{*2}(\underset{\sim}{X}_i,\underset{\sim}{X}_j) + n(n-1)\binom{n+1}{3}^{-1} \sum_{1\leq i<j<k\leq n+1} \phi_2^*(\underset{\sim}{X}_i,\underset{\sim}{X}_j)\phi_2^*(\underset{\sim}{X}_i,\underset{\sim}{X}_k).$$

By (2.28), the 2nd term on the rhs of (2.27) is given by:

(2.29) $$\frac{8(n+1)}{(n-1)}\{\frac{1}{n-1}\binom{n+1}{2}^{-1} \sum_{1\leq i<j\leq n+1} \phi_2^{*2}(\underset{\sim}{X}_i,\underset{\sim}{X}_j) + \binom{n+1}{3}^{-1} \sum_{1\leq i<j<k\leq n+1} \phi_2^*(\underset{\sim}{X}_i,\underset{\sim}{X}_j)\phi_2^*(\underset{\sim}{X}_i,\underset{\sim}{X}_k)\}$$
$$= 8(n-1)^{-1}(n+1)[\ (n-1)^{-1}\tilde{U}_n^{(1)} + \tilde{U}_n^{(2)}\ ]\text{, say.}$$

Now, $\tilde{U}_n^{(1)}$ is a U-statistic, and hence, converges a.s. to its expectation which is finite, while $\tilde{U}_n^{(2)}$ is also a U-statistic with expectation $E\phi_2^*(\underset{\sim}{X}_1,\underset{\sim}{X}_2)\phi_2^*(\underset{\sim}{X}_1,\underset{\sim}{X}_3) = 0$, and hence, it a.s. converges to 0 as $n \to \infty$. Thus, (2.29) a.s. converges to 0 as $n \to \infty$. Finally, $[E\{n(n+1)(T_{n,1} - T_{n+1,1})(T_{n,2} - T_{n+1,2})|\mathcal{C}_{n+1}\}]^2$ converges a.s. to 0 [by the Schwarz inequality and (2.25)- (2.29)]. Hence, the proof is complete.

<u>Lemma 2.4</u>. <u>Under the assumptions of the previous lemma,</u>

(2.30) $$n(n+1)E\{(T_n - T_{n+1})^2|\mathcal{F}_{n+1}\} \xrightarrow[L_1]{P} m^2\zeta_1 \text{ a.s., as } n \to \infty .$$

<u>Proof</u>. Since $\mathcal{F}_{n+1} \subset \mathcal{C}_{n+1}$,

(2.31) $$n(n+1)E\{(T_n - T_{n+1})^2|\mathcal{F}_{n+1}\} = n(n+1)E[E\{(T_n - T_{n+1})^2|\mathcal{C}_{n+1}\}|\mathcal{F}_{n+1}].$$

Now, by Lemma 2.3, $E\{n(n+1)(T_n - T_{n+1})^2|\mathcal{C}_{n+1}\} \to m^2\zeta_1$ a.s., as $n \to \infty$, $\mathcal{F}_n$ is $\downarrow$ in n and $n(n+1)E\{(T_n - T_{n+1})^2|\mathcal{C}_{n+1}\} \in L_1$. Hence, by Loeve(1963,p.409), the result follows.

<u>Lemma 2.5</u>. <u>If</u> $\phi^4\{\underset{\sim}{X}_1, \underset{\sim}{X}_2, \ldots, \underset{\sim}{X}_m\}$, <u>then</u> <u>for every</u> $k \geq m$,

(2.32) $$\sup_{n\geq k} n(n+1)E\{(T_n - T_{n+1})^2|\mathcal{C}_{n+1}\} \in L_1.$$

<u>Proof</u>. Here also, for simplicity, we take m = 2. It suffices to show that for each $j(=1,2)$, $\sup_{n\geq k} n(n+1)E\{(T_{n,j} - T_{n+1,j})^2|\mathcal{C}_{n+1}\}$ is integrable. Note that by (2.24),

(2.33) $$\sup_{n\geq k} n(n+1)E\{(T_{n,1} - T_{n+1,1})^2|\mathcal{C}_{n+1}\}$$
$$= \sup_{n\geq k}[n^{-1}\Sigma_{i=1}^{n+1}\{\phi_1(\underset{\sim}{X}_i) - T_{n+1,1}\}^2] = \sup_{n\geq k} M_{n+1}\text{, say.}$$

Now, by definition, $\{ M_n, \mathcal{C}_n ; n \geq 2\}$ is a reverse martingale, so that by the Kolmogorov inequality, for every $t \geq 0$,

(2.34) $$P\{ \sup_{n \geq k} M_n > t \} \leq t^{-2}E[M_k^2] = t^{-2}\zeta_1 ,$$

and this implies the integrability of $\sup_{n \geq k} M_n$, for every $k \geq 1$. A similar but more lengthy proof applies for $j = 2$. Q.E.D.

Lemma 2.6. Under the conditions of Lemma 2.5,

(2.35) $$n(n+1)E\{(T_n - T_{n+1})^2 \mid \mathfrak{F}_{n+1}\} \xrightarrow{a.s.} m^2 \zeta_1 , \text{ as } n \to \infty.$$

Proof. Since $E\{(T_n - T_{n+1})^2 \mid \mathfrak{F}_{n+1}\} = E[E\{(T_n - T_{n+1})^2 \mid \mathcal{C}_{n+1}\} \mid \mathfrak{F}_{n+1}]$ and since $\mathfrak{F}_n$ is $\downarrow$ in n, the lemma follows directly from Lemma 2.5 and Loeve(1963,p.409). Q.E.D.

3. The main results. To motivate the main theorem, we require to introduce certain assumptions. Note that by the Rao-Blackwellization, for every $n \geq 1$,

(3.1) $$nE[\Psi_n(\tilde{x}) - F(\tilde{x})]^2 \leq nE[F_n(\tilde{x})-F(\tilde{x})]^2 = F(\tilde{x})[1-F(\tilde{x})], \ \forall \ \tilde{x} \in R^p.$$

But, as in the example of the uniform df, the lhs of (3.1) may tend to 0 as $n \to \infty$. So, in order to avoid this degeneracy, we assume that for every $\tilde{x}, \tilde{y} \in R^p$,

(3.2) $$\lim_{n \to \infty} nE[\{\Psi_n(\tilde{x}) - F(\tilde{x})\}\{ \Psi_n(\tilde{y}) - F(\tilde{y})\}] = h(\tilde{x},\tilde{y}),$$

where there exists a subset $\mathfrak{X} \subset R^p$ such that for every $x_i \in \mathfrak{X}$, $i=1,\ldots,m$, the matrix $(h(\tilde{x}_i,\tilde{x}_j))$ is positive definite for every $m \geq 1$. We also assume that

(3.3) $$E\{ n(n+1)[\Psi_n(\tilde{x}) - \Psi_{n+1}(\tilde{x})]^2 I(n|\Psi_n(\tilde{x})- \Psi_{n+1}(\tilde{x})|> \lambda) \mid \mathfrak{F}_{n+1} \}$$

converges to 0 as $\lambda \to \infty$, uniformly in $n(\geq n_0)$, and

(3.4) $$n(n+1)E[\{\Psi_n(\tilde{x}_1)- \Psi_{n+1}(\tilde{x}_1)\}\{ \Psi_n(\tilde{x}_2) - \Psi_{n+1}(\tilde{x}_2)\} \mid \mathfrak{F}_{n+1}\} \to h(\tilde{x}_1,\tilde{x}_2) \text{ a.s.},$$

for every $\tilde{x}_1, \tilde{x}_2 \in R^p$, where I(A) stands for the indicator function of a set A. We may note that (3,4) implies that

(3.5) $$n\Sigma_{k=n}^{\infty}E[\{ \Psi_k(\tilde{x}_1) - \Psi_{k+1}(\tilde{x}_1)\}\{ \Psi_k(\tilde{x}_2) - \Psi_{k+1}(\tilde{x}_2)\} \mid \mathfrak{F}_{k+1}\} \to h(\tilde{x}_1,\tilde{x}_2) \text{ a.s.}$$

whereas the uniform integrability condition in (3.3) implies that whenever $h(\tilde{x},\tilde{x})> 0$, for every $\varepsilon > 0$,

$$[\Sigma_{k=n}^{\infty} E[\Psi_k(\underset{\sim}{x}) - \Psi_{k+1}(\underset{\sim}{x})]^2]^{-1}[\{\Sigma_{k=n}^{\infty}\{E[(\Psi_k(\underset{\sim}{x}) - \Psi_{k+1}(\underset{\sim}{x}))^2 . I(n^{1/2}|\Psi_k(\underset{\sim}{x}) - \Psi_{k+1}(\underset{\sim}{x})| > \varepsilon h^{1/2}(\underset{\sim}{x},\underset{\sim}{x}) \mid \mathfrak{F}_{k+1})]\}] \to 0 \text{ a.s., as } n \to \infty. \tag{3.6}$$

First, we present the main theorem. Later on we shall show that (3.3) and (3.4) hold under general conditions on Ψ_n .

Theorem 3.1. If $\{T_n\}$ is transitively sufficient, then under (3.2), (3.3) and (3.4), $V_n^* = [V_n^*(\underset{\sim}{x}), \underset{\sim}{x} \in R^p]$ converges weakly to a multiparameter Gaussian function $W = [W(\underset{\sim}{x}), \underset{\sim}{x} \in R^p]$ whose mean (function) is 0 and covariance function is $h(\underset{\sim}{x},\underset{\sim}{y})$.

Proof. Let $F_{[j]}$ be the marginal df of the jth compon nt of X_1, j=1,...,p. Let then $Y_{ij} = F_{[j]}(X_{ij})$, $1 \le j \le p$, $i \ge 1$, and let $\underset{\sim}{Y}_i = (Y_{i1},\dots,Y_{ip})$,

$$G_n(\underset{\sim}{t}) = n^{-1}\Sigma_{i=1}^{n} c(\underset{\sim}{t} - \underset{\sim}{Y}_i), \quad \underset{\sim}{t} \in E^p, \; n \ge 1, \tag{3.7}$$

where E^p denotes the unit p-cube $\{\underset{\sim}{t}: \underset{\sim}{0} < \underset{\sim}{t} < \underset{\sim}{1}\}$. Also, let

$$G(\underset{\sim}{t}) = P\{\underset{\sim}{Y}_i \le \underset{\sim}{t}\}, \quad \underset{\sim}{t} \in E^p; \tag{3.8}$$

$$\tilde{V}_n(\underset{\sim}{t}) = E\{n^{1/2}[G_n(\underset{\sim}{t}) - G(\underset{\sim}{t})] \mid \mathcal{B}_n^{(s)}\}, \quad \underset{\sim}{t} \in E^p. \tag{3.9}$$

Then, it suffices to show that $\tilde{V}_n$ converges weakly in the Skorokhod J_1-topology on $D^p[0,1]$ to a Gaussian function $\tilde{W} = [\tilde{W}(\underset{\sim}{t}), \underset{\sim}{t} \in E^p]$. For this, we require to show (a) that the process $\tilde{V}_n$ is tight and (b) that the finite dimensional distributions (f.d.d.) of $\tilde{V}_n$ converge to the corresponding ones of $\tilde{W}$, as $n \to \infty$. For the proof of (a), we principally follow the treatment of Section 5 of Neuhaus(1971) where the original empirical process has been considered. We only need to replace his Lemma 5.2 by our Lemma 2.1, and the rest follows on the same line.

The proof of (b) is a little more complicated. By virtue of our Lemma 2.2, for every $\underset{\sim}{t} \in E^p$, $\{\tilde{V}_n(\underset{\sim}{t}), \mathfrak{F}_n, n \ge 1\}$ is a reverse martingale. As such, for every $m \ge 1$, $\underset{\sim}{0} \le \underset{\sim}{t}_1 \ne \dots \ne \underset{\sim}{t}_m \le 1$ and $\underset{\sim}{\lambda} = (\lambda_1,\dots,\lambda_m) \ne \underset{\sim}{0}$, $\{\Sigma_{j=1}^{m}\lambda_j\tilde{V}_n(\underset{\sim}{t}_j), \mathfrak{F}_n, n \ge 1\}$ is a reverse martingale. This tempts us to use the recent functional central limit theorems of Loynes(1970) and Brown(1971). For this, we let $s_n^2 = E[\Sigma_{k=n}^{\infty} Z_k]^2$, where

$$Z_k = \Sigma_{j=1}^{m}\lambda_j\{E[G_k(\underset{\sim}{t}_j) \mid \mathcal{B}_k^{(s)}] - E[G_{k+1}(\underset{\sim}{t}_j) \mid \mathcal{B}_{k+1}^{(s)}]\}, \quad k \ge 1. \tag{3.10}$$

Then, by the Brown(1971) modifications of Loynes'(1970) results, we are only to show that as $n \to \infty$,

(3.11) $\quad [\,\Sigma_{k=n}^{\infty} E(Z_k^2 \mid \mathcal{F}_{k+1})]/s_n^2 \to 1$, in probability,

(3.12) $\quad s_n^{-2}[\,\Sigma_{k=n}^{\infty} E\{Z_k^2 I(|Z_k| > \varepsilon s_n) \mid \mathcal{F}_{k+1}\}] \to 0$, in probability,

for every $\varepsilon > 0$. Now, (3.11) follows from (3.4) and (3.5) while (3.12) follows from (3.3), (3.6) and the inequality

(3.13) $\quad (\Sigma_{i=1}^{m} u_i)^2 I(|\Sigma_{i=1}^{m} u_i| > m\varepsilon)\} \leq m2^{m-1}\{\Sigma_{i=1}^{m} u_i^2 I(|u_i| > \varepsilon)\}$.

Let us now examine the conditions (3.3) and (3.4). Note that $\Psi_n(\underset{\sim}{x}) = \Psi_n(\underset{\sim}{x}, \underset{\sim}{T}_n)$ is a bounded function and for a broad class of df's (including the exponential family having a se of minimal complete sufficient statistics), we may by the usual Taylor's expansion write

(3.14)
$$\Psi_n(\underset{\sim}{x}) = \Psi_n(\underset{\sim}{x} + \underset{\sim}{\theta} + (\underset{\sim}{T}_n - \underset{\sim}{\theta})) = F(\underset{\sim}{x}) + n^{-1}v_o(\underset{\sim}{x}) + \Sigma_{j=1}^{q} v_j(\underset{\sim}{x},\underset{\sim}{\theta})[T_{nj} - \theta_j]$$
$$+ \Sigma_{j=1}^{q}\Sigma_{j'=1}^{q} v_{jj'}(\underset{\sim}{x},\underset{\sim}{\theta})[T_{nj} - \theta_j][T_{nj'} - \theta_{j'}] + R_n$$

where $|R_n| < cn^{-2}$, $0<c<\infty$ and v_j, $j=0,1,\ldots,q$, $v_{jj'}$, $j,j'=1,\ldots,q$ are all bounded and continuous functions of $\underset{\sim}{x}$ and $\underset{\sim}{\theta}$ .For example, consider the case of the univariate normal df with mean μ and variance σ^2 ; the corresponding $\Psi_n(x)$ [cf. Lieberman and Resnikoff(1955)] for $n > 2$ is given by

(3.15)
$$\psi_n(x) = \begin{cases} 0, \text{ if } x < \bar{X}_n - (n-1)^{1/2}S_n, \\ 1, \text{ if } x > \bar{X}_n + (n-1)^{1/2}S_n, \\ G_{n-2}\{(n-2)^{1/2}(x-\bar{X}_n)/[(n-1)S_n^2 - (x-\bar{X}_n)^2]^{1/2}\}, \text{ if } |x-\bar{X}_n| < (n-1)^{1/2}S_n, \end{cases}$$

where $\underset{\sim}{T}_n = (\bar{X}_n, S_n^2) = (n^{-1}\Sigma_{i=1}^{n} X_i,\ n^{-1}\Sigma_{i=1}^{n}(X_i - \bar{X}_n)^2)$ and G_{n-2} is the Student t distribution with n-2 degrees of freedom. Now , from Johnson and Kotz(1970), we have

(3.16) $\quad G_n(t) = \Phi(t) - Z(t)[\frac{1}{4}t(t^2+1)n^{-1} + O(n^{-2})]$,

so that (3.14) is valid in this case.

From (3.14), we have

$$\Psi_n(\underset{\sim}{x}) - \Psi_{n+1}(\underset{\sim}{x}) = v_o(\underset{\sim}{x})/n(n+1) + \Sigma_{j=1}^{q} v_j(\underset{\sim}{x},\underset{\sim}{\theta})(T_{nj} - T_{n+1j}) +$$

(3.17) $$\Sigma_{j=1}^{q} v_{jj}(\underset{\sim}{x},\underset{\sim}{\theta})(T_{nj} - T_{n+1j})(T_{nj} + T_{n+1j} - 2\,\theta_j) +$$

$$\Sigma_{1\le j\ne j'\le q}\, v_{jj'}(\underset{\sim}{x},\underset{\sim}{\theta})[(T_{nj}-T_{n+1j})(T_{nj'}-\theta_{j'}) + (T_{n+1j}-\theta_j)(T_{nj'}-T_{n+1j'})] +$$

$$+ O(n^{-2}), \quad \text{a.s.}$$

Thus, if the $\{\underset{\sim}{T}_n\}$ form a U-statistics sequence, by the a.s. convergence results of Berk(1966), $\underset{\sim}{T}_n \to \underset{\sim}{\theta}$ a.s., as $n \to \infty$, and hence, by (3.17), we have

(3.18) $$\Psi_n(\underset{\sim}{x}) - \Psi_{n+1}(\underset{\sim}{x}) = \Sigma_{j=1}^{q}(T_{nj} - T_{n+1j})[v_j(\underset{\sim}{x},\underset{\sim}{\theta}) + 2\Sigma_{j'=1}^{q} v_{jj'}(\underset{\sim}{x},\underset{\sim}{\theta})\{o(1)\}] + O(n^{-2})$$

with probability one, as $n \to \infty$. Consequently, (3.3) and (3.4) follows from (3.18), Lemma 2.5 and Lemma 2.6.

<u>Remarks</u>. Srinivasan(1970), Lilliefors(1967) and others have used the studentized Kolmogorov-Smirnov statistic (in the univariate case)

(3.19) $$n^{1/2}\{\sup_x |\ \Psi_n(x) - F_n(x)|\}$$

as a goodness of fit statistic when F is an exponential or normal df, and in those cases, the distribution of (3.19) is known to be independent of the population parameters. Condition for this statistic to be independent of nuisance parameters in the general case has been given by O'Reilly(1971). We should note that if $\Psi_n(x,\underset{\sim}{T}_n)$ is absolutely continuous then $\Psi_n(X_i,\underset{\sim}{T}_n)$, i=1,...,n are ancilliary statistics, each having marginally the uniform(0,1) df[viz., O'Reilly and Quesenberry(1973)]. Let us now assume that the probability structure of the problem is such that Stein's Theorem [cf. Zacks(1971,p.79)] is applicable, and let $\mathcal{G}$ denote the group of transformations operting on the sample space. Then, if $g \in \mathcal{G}$ is a monotone increasing transformation, it is easy to show that

(3.20) $$\Psi_n(gX_i,\ \underset{\sim}{T}_n(g\underset{\sim}{X})) = \Psi_n(X_i,\underset{\sim}{T}_n(\underset{\sim}{X}))\ ;\ \underset{\sim}{X} = (X_1,\ldots,X_n)\ ;$$

(3.21) $$\Psi_n(gX_i,\underset{\sim}{T}_n(g\underset{\sim}{X})) - F_n(gX_i) = \Psi_n(X_i,\underset{\sim}{T}_n(\underset{\sim}{X})) - F_n(X_i).$$

Hence, noting that $\sup_x |\Psi_n(x,\underset{\sim}{T}_n) - F_n(x)|$ occurs at one of the jump points of $F_n(x)$ if Ψ_n is a continuous function, and by (3.21) we conclude that (3.19) is an invariant statistic and that the distribution of (3.19) depends on the maximal invariant function in the parameter space. Thus, if the maximal invariant function is constant, then (3.19) has a distribution independent of the parameters. Even when this distribution is not free of nuisance parameters, under fairly general conditions, $n^{1/2}[\Psi_n(x,\tilde{T}_n) - F_n(x)]$, $x \in R$ will converge to a Gaussian process and the percentage points of the original Kolmogorov-Smirnov statistic[viz., $\sup_x |F_n(x) - F(x)|$] will give a crude but distribution free upper bound for the corresponding percentage points of the asymptotic distribution of (3.19).

4. Some concluding remarks. First, let us consider a real valued $g(\tilde{x})$, $\tilde{x} \in R^p$, such that $\int g(\tilde{x})dF(\tilde{x}) = \mu$ and $\int g^2(\tilde{x})dF(\tilde{x}) < \infty$. Let then

(4.1) $W_n = \int g(\tilde{x})d\,\Psi_n(\tilde{x})$ so that $W_n = E[\, g(\tilde{X}_1) \mid \mathcal{B}_n^{(s)} \}$, $n \geq 1$.

By Lemma 2.2, $\{ W_n, \mathcal{F}_n, n \geq 1\}$ is a reverse martingale, and hence, under the conditions of Brown(1971) or Loynes(1970), $n^{1/2}(W_n - \mu)$ is asymptotically normal. Consider then the general U-statistic

(4.2) $U_n = \binom{n}{m}^{-1} \sum_{C_{n,m}} g(\tilde{X}_{i_1},\ldots,\tilde{X}_{i_m})$, $n \geq m$.

By Berk(1966), $\{ U_n \}$ forms a reverse martingale sequence. Loynes(1970) cites this and conjectured that the functional central limit theorem is applicable to the U-statistics . He, however, **fails** to provide the necessary verification of the underlying conditions needed to apply his main theorem.By virtue of our Lemmas 2.2-2.6, we immediately conclude that the assumptions of Loynes' main theorem are all met for U_n , and hence, the results hold good. In fact, we may proceed one step further and define a Rao-Blackwell U-statistic

(4.3) $U_n^* = E[\, U_n \mid \mathcal{B}_n^{(s)}] = E[g(\tilde{X}_1,\ldots,\tilde{X}_m) \mid \mathcal{B}_n^{(s)}]$, $n \geq m$.

Under the assumption of transitivity of $\{T_n\}$, $\{U_n^*$, $\mathcal{F}_n$; $n \geq m\}$ is a reverse martingale, and by using the results of Miller and Sen(1972), the asymptotic normality and tightness of the process based on $\{U_n^*\}$ can be established. Similar results hold for the von Mises differentiable statistical functions.

If for $n \geq n_o$, $\Psi_n(x,\underset{\sim}{T}_n)$ has a continuous derivative $\Psi_n'(x,\underset{\sim}{T}_n)$ such that $\Psi_n'(x,\underset{\sim}{T}_n) \leq h_n(\underset{\sim}{T}_n)$ is integrable, then under the assumption of transitivity of $\{T_n\}$, $\{\Psi_n'(x,\underset{\sim}{T}_n), \mathcal{F}_n, n \geq n_o\}$ is a reverse martingale. Hence, under conditions essentially similar to those in Theorem 3.1, asymptotic normality of $n^{1/2}\{\Psi_n'(x,\underset{\sim}{T}_n) - f(x)\}$ may be established.

It has been shown in Sen, Bhattacharyya and Suh(1973) that the bundle strength of filaments under suitable conditions may be represented as $Z_n = \sup\{x[1-F_n(x)]: x \geq 0\}$. In view of the Rao-Blackwell estimator $\Psi_n(x)$ of $F(x)$, we consider

(4.4) $$Z_n^* = \sup\{x[1-\Psi_n(x)] : x \geq 0\}.$$

Then, it is easily seen that $\{Z_n^*\}$ is a reverse submartingale sequence. Also,

(4.5) $$P\{\sup_x n^{1/2}|\Psi_n(x) - F(x)| > K_\varepsilon\} < \varepsilon, \quad K_\varepsilon < \infty.$$

Hence on using our Theorem 3.1 and then proceeding as in the proof of the main theorem in Sen, Bhattacharyya and Suh(1973), it follows that the asymptotic normality results apply to $\{Z_n^*\}$ as well.

Finally, the weak convergence of V_n^* for random sample sizes can also be established by using our lemmas 2.1 - 2.6 along with the results of Sen(1973) on the weak convergence of V_n for random sample sizes.

REFERENCES

[1] Berk, R. H. (1966). Limiting behavior of posterior distribution when the model is incorrect. Ann. Math. Statist. 37 51-58.

[2] Billingsley, P. (1968). Convergence of probability measures. John-Wiley, New York.

[3] Brown, B. M. (1971). Martingale central limit theorem. Ann. Math. Statist. 42 59-66.

[4] Ghurye, S. G. and Olkin, I. (1969). Unbiased estimation of some multivariate probability densities and related functions. Ann. Math. Statist. 40 1261-1271.

[5] Johnson, N. L. and Kotz, S. (1970). Continuous univariate distributions -2. Houghton Mifflin Company, Boston.

[6] Lieberman, G. J. and Resnikoff, G. J. (1955). Sampling plans for inspection by variables. J. Amer. Statist. Assoc 50 457-516.

[7] Lilliefors, H. W. (1967). On the Kolmogorov-Smirnov test for normality with mean and variance unknown. J. Amer. Statist. Assoc. 62 399-402.

[8] Loeve, M. (1963). Probability theory. 3rd. Ed. van Nostrand, New York.

[9] Loynes, R. M. (1970). An invariance principle for reversed martingales. Proc. Amer. Math. Soc. 25 56-64.

[10] Miller, R. G., Jr. and Sen, P. K. (1972). Weak convergence of u-statistics and von Mise's differentiable statistical functions. Ann. Math. Statist. 43 31-41.

[11] O'Reilly, F. J. (1971) On goodness of fit tests based on Rao-Blackwell distribution function estimators. Ph.D. Thesis. N. C. State University.

[12] O'Reilly, F. J. and Quesenberry, C. P. (1973). The probability integral transformation. Ann. Statist. 1 74-83.

[13] Sen, P. K. (1973). On weak convergence of empirical processes for random number of independent stochastic vectors. Proc. Cambridge Phil. Soc. 73 139-144 .

[14] Sen, P. K., Bhattacharyya, B. B. and Suh, M. W. (1973). Limiting behavior of the extremum of certain sample functions. Ann. Statist. 1 297-311.

[15] Seheult, A. H. and Quesenberry, C. P. (1971). On unbiased estimation of density functions. Ann. Math. Statist. 42 1434-1438.

[16] Srinivasan, R. (1970). An approach to testing the goodness of fit of incompletely specified distributions. Biometrika 57 605-611.

[17] Tate, R. F. (1959). Unbiased estimation: Function of location and scale parameters. Ann. Math. Statist. 30 341-366.

[18] Zacks, S. (1971). The theory of statistical inference. John Wiley, New York.

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