ASYMPTOTICALLY EFFICIENT TESTS
BY THE METHOD OF n RANKINGS

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Pranab Kumar Sen

University of North Carolina, Chapel Hill
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1. **Summary and Introduction.** When the block-effects are not additive or the method of ranking after alignment is not practicable, the method of n rankings [cf. Friedman (1937), Kendall (1955)] is quite useful for the analysis of two way layouts. Consider n observers, each of which ranks (independently) p objects $0_1, \ldots, 0_p$; the ranks of $0_1, \ldots, 0_p$ by the $i$th observer are denoted by $r_{i1}, \ldots, r_{ip}$, respectively, for $i=1, \ldots, n$. Well-known tests for the hypothesis of no difference among the p objects are due to Friedman (1937) and Brown and Mood (1951), and are respectively based on the test statistics

$$\chi^2_r = \left[12/n p(p+1)\right] \sum_{j=1}^{p} \left(R_j - n(p+1)/2\right)^2; \quad R_j = \sum_{i=1}^{n} r_{ij}, \quad j=1, \ldots, p;$$

and

$$M_r = \frac{p(p-1)}{n a(p-a)} \sum_{j=1}^{p} \left(M_j - na/p\right)^2; \quad M_j = \sum_{i=1}^{n} m_{ij}, \quad j=1, \ldots, p,$$

where $m_{ij}$ is 1 or 0 according as $r_{ij}$ is $\leq$ or $> a$: $1 \leq a < p$; usually a is taken to be the largest integer contained in $(p+1)/2$. Generalizations of the $\chi^2_r$-test to non-orthogonal designs are due to Durbin (1951) and Benard and Elteren (1953) and their asymptotic power-efficiencies are studied by Elteren and Noether (1959). Similar works on the other test are due to Bhapkar (1961, 1963). The present investigation is concerned with (i) the derivation of a class of asymptotically efficient test for the same problem and (ii) the characterizations of the optimality of the tests based on (1.1) and (1.2).

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2. Preliminary notions. We conceive of some stochastic variables \((X_{1i}, \ldots, X_{ip})\) (may or may not be observable) underlying the ranks \((r_{1i}, \ldots, r_{ip})\), for \(i=1, \ldots, n\). It is then assumed that \(X_{1i}, \ldots, X_{ip}\) are independently distributed according to continuous cumulative distribution functions (cdfs) \(F_{1i}(x), \ldots, F_{ip}(x)\), respectively, for \(i=1, \ldots, n\). The null hypothesis states that

\[
H_0: \quad F_{11} = \cdots = F_{1n} = F_1, \quad \text{for all } i=1, \ldots, n,
\]

whatever be \(F_1, \ldots, F_n\). We are interested in translation alternatives viz.,

\[
F_{ij}(x) = F_1(x-t_j), \quad j=1, \ldots, p, \quad i=1, \ldots, n; \quad \sum_{j=1}^p t_j = 0,
\]

\(t = (t_1, \ldots, t_p)\) being a real \(p\)-vector. For our study, we shall assume that

(i) \(n\) (the number of observers or blocks) is large;

(ii) \(F_i(x)\) is absolutely continuous having a continuous density function \(f_i(x)\) where

\[
\int_{-\infty}^{\infty} f_i^2(x) dx < \infty \quad \text{for all } i=1, \ldots, n;
\]

(iii) \(t = t_n = n^{-1/2}\theta; \quad \theta = (\theta_1, \ldots, \theta_p)\) has real and finite elements.

Also, we shall confine ourselves to the following class of rank tests. Let \(\{J(r, p), r=1, \ldots, p\}\) be \(p\) real valued functions, where \(J(r, p)\) is a function of \(r\) and \(p, \quad r=1, \ldots, p\). We define

\[
\bar{J} = (1/p) \sum_{r=1}^p J(r, p) \quad \text{and} \quad A^2(J) = \frac{1}{p-1} \sum_{r=1}^p (J(r, p) - \bar{J})^2.
\]

Concerning \(\{J(r, p)\}\) we assume that for any finite \(p\), they are all finite and are not all equal. Under these assumptions both \(\bar{J}\) and \(A^2(J)\) are finite and \(A^2(J)\) is strictly positive. We then define a rank statistic (vector) \(T_n = (T_{n,1}, \ldots, T_{n,p})\), where
(2.6) \[ T_{n,j} = \frac{1}{n} \sum_{i=1}^{n} J(r_{ij}, p), \text{ for } j = 1, \ldots, p. \]

The tests to be considered here are based on the following type of statistics

(2.7) \[ S_n = n A^{-2}(J) \sum_{j=1}^{p} [T_{n,j} - \bar{J}]^2. \]

It is easily verified that both (1.1) and (1.2) are particular cases of (2.7).

Our main contention is to select \{J(r, p)\} in such a manner that the corresponding \( S_n \) leads to asymptotically efficient test.

3. Asymptotic distribution of \( S_n \). This is presented briefly, as it will be required in the sequel. Let us define for each \( i(=1, \ldots, n) \)

(3.1) \[ p_{i,r}^{(j)} = P(r_{ij} = r), \quad p_{i,rs}^{(j,j')} = P(r_{ij} = r, r_{ij'} = s), \quad r, s = 1, \ldots, p, \quad j \neq j' = 1, \ldots, p. \]

Conventionally, we let

(3.2) \[ p_{i,rs}^{(j,j')} = p_{i,r}^{(j)} \cdot \delta_{rs}, \quad p_{i,rr}^{(j,j')} = \delta_{jj'} p_{i,r}^{(s)}, \quad r, s, j, j' = 1, \ldots, p, \]

where \( \delta_{jj'} \) and \( \delta_{rs} \) are the Kronecker deltas. Let then

(3.3) \[ \mu_{n,j} = \frac{1}{n} \sum_{i=1}^{n} \sum_{r=1}^{P} J(r, p) p_{i,r}^{(j)}, \quad j = 1, \ldots, p; \quad \mu_n = (\mu_{n,1}, \ldots, \mu_{n,p}); \]

(3.4) \[ \sigma_{n,jj'} = \frac{1}{n} \sum_{i=1}^{n} \sum_{r=1}^{P} \sum_{s=1}^{P} J(r, p) J(s, p) p_{i,rs}^{(j,j')} - \mu_{n,j} \mu_{n,j'}, \quad j, j' = 1, \ldots, p; \]

(3.5) \[ \Sigma_n = \{ (\sigma_{n,jj'})_{j,j' = 1, \ldots, p} \}. \]

THEOREM 3.1. \( n^2 (T_n - \mu_n) \) has asymptotically a normal distribution with a null mean vector and dispersion matrix \( \Sigma_n \).
OUTLINE OF THE PROOF. It suffices to show that for any real and non-null \( a = (a_1, \ldots, a_p) \) (other than \( a(1, \ldots, 1) \)) \( Z_n = n \frac{Z_n}{Z_n} \converges \) in law to a normal distribution. By using (2.6), we may rewrite \( Z_n \) as

\[
Z_n = n^{-\frac{1}{2}} \sum_{i=1}^{n} U_i, \quad \text{where} \quad U_i = \sum_{j=1}^{p} a_j J(r_{ij}, p), \quad i=1, \ldots, n.
\]

As \( J(r, p), r=1, \ldots, p \) are all finite, \( U_1, \ldots, U_n \) are all independent and finite valued random variables. The rest of the proof follows by an application of the classical central limit theorem [cf. Loeve (1963, p. 277)] and some routine analysis. Hence, the theorem.

Under the null hypothesis (2.1), it is easily seen that \( U_n = \frac{J_n - \frac{1}{2}}{\frac{1}{2} J_n} = (1, \ldots, 1) \), and \( \sum_{n} = A^2(J) \left[I_p - \frac{1}{p} \cdot \frac{1}{p} J_n \right] \), where \( I_p \) is the identity matrix of order \( p \). In this case, using (3.6), the fact that \( U_1, \ldots, U_n \) are identically distributed and the Berry-Esseen theorem [cf. Loeve (1962, p. 288)], it can be shown that the convergence of \( n^{-\frac{1}{2}}(\sqrt{\sum_{n}} - \mu) \) to a multinormal variable is uniform in \( (F_1, \ldots, F_p) \). Further, considering the random variables \( n^{-\frac{1}{2}}(\sqrt{T_n} - \mu) \), \( j=1, \ldots, p-1 \), taking the reciprocal of their covariance matrix as a suitable discriminant of their quadratic form, symmetrizing it and using some well known results on the asymptotic distribution of quadratic forms associated with asymptotically multinormal distributions, we arrive at the following.

**Corollary 3.1.** Under \( H_0 \) in (2.1), \( S_n \), defined by (2.7), converges in law to a chi-square distribution with \( (p-1) \) degrees of freedom.

We shall now consider the asymptotic distribution of \( S_n \) under the sequence of alternatives in (2.2) and (2.4). For this, we define by

\[
\beta_{s,p-2}^{(i)} = (p-2)^{-s} \int_{-\infty}^{\infty} [F_i(x)]^s [1-F_i(x)]^{p-2-s} f_i^2(x) dx, \quad s=0, \ldots, p-2
\]

and conventionally, we let \( \beta_{-1,p-2}^{(i)} = \beta_{p-1,p-2}^{(i)} = 0 \), for all \( i=1, \ldots, n \); (2.3) ensures
the finiteness of (3.7). Let then

\[ \beta_{s, p-2}^{(n)} = (1/n) \sum_{i=1}^{n} \beta_{s, p-2}^{(i)}, \text{ for } s=-1, 0, \ldots, p-1; \]

\[ \lambda_{r, n} = \beta_{r-1, p-2}^{(i)} - \beta_{r-2, p-2}^{(i)}, \text{ for } r=1, \ldots, p. \]

**THEOREM 3.2.** Under (2.2) through (2.5), \( S_n \), defined by (2.7), has asymptotically a non-central chi-square distribution with \( p-1 \) degrees of freedom and the non-centrality parameter

\[ \Delta_n(J) = p^2 A^{-2}(J)(\sum_{j=1}^{p} \theta_j^2)(\sum_{r=1}^{p} J(r, p) \lambda_{r, n})^2. \]

**PROOF.** By virtue of the result of theorem 3.1, it is sufficient to show that under (2.2) through (2.5)

\[ n^{\frac{1}{2}}(u_{n, j} - \bar{j}) = -p \theta_j \sum_{r=1}^{p} J(r, p) \lambda_{r, n} + o(1), \; j=1, \ldots, p; \]

\[ \sum_{n=0}^{\infty} A^2(J) \{ l \frac{1}{p} \sum_{j=1}^{p} \ell \frac{j}{p} \} \] converges to a null matrix as \( n \to \infty \).

Now, by definition in (3.1)

\[ p_{i, r}^{(j)} = \sum_{s_j} \int_{-\infty}^{\infty} F_{i, s_{r+1}}(x) \cdots F_{i, s_{r}}(x) [1 - F_{i, s_{r+1}}(x)] \cdots [1 - F_{i, s_{r+1}}(x)] dF_{i, j}(x), \]

where the summation \( s_j \) extends over all possible choice of \((s_1, \ldots, s_{r-1})\) from \((1, \ldots, j-1, j+1, \ldots, p)\), \((s_{r+1}, \ldots, s_p)\) being the complementary set. Now, by (2.2) and (2.4), we have

\[ F_{i, j}(x) = F_{i, j}(x - n^{-\frac{1}{2}}[\theta_{j} - \theta]) \] for all \( j, \ell = 1, \ldots, p; \sum_{r=1}^{p} \theta_r = 0. \]

Using (3.13), (3.14) and some routine analysis, it follows that under (2.3),
(3.15) \[ p_{i,r}^{(j)} = (1/p) + n^{-2} [p \theta_j \beta_{r-2}^{(i)} - \beta_{r-1}^{(i)}] + o(n^{-2}), \]

for all \( r,j=1,\ldots,p, i=1,\ldots,n \). (3.3) and (3.15) ascertain (3.11). Proceeding in a similar manner, it can be shown that under (2.2) through (2.4)

(3.16) \[ p_{i,rs}^{(j,j') = 1/p(p-1) + o(n^{-2}) \text{ for all } j\neq j', r\neq s=1,\ldots,p, i=1,\ldots,n.} \]

Hence, from (3.2), (3.4), (3.5), (3.15) and (3.16), we obtain (3.12). Hence the theorem.

It may be noted that if \( F_i \) possesses a finite variance \( \sigma_i^2 \) for all \( i=1,\ldots,n \), then \( (p-1)F(p-1),(n-1)(p-1) \) (where \( F(p-1),(n-1)(p-1) \) is the classical analysis of variance test statistic,) has asymptotically (under (2.2) and (2.4)) a non-central chi-square distribution with \( (p-1) \) degrees of freedom and the non-centrality parameter

(3.17) \[ \Delta^* = (\sum_{j=1}^{p} \sigma_j^2) / \sigma_i^2; \quad \sigma_i^2 = (1/n) \sum_{i=1}^{n} \sigma_i^2. \]

(3.10) and (3.17) will be used in the derivation of our main results.

4. **Intrinsic asymptotic efficiency** and the **efficient \( S_n \)-test.** From theorem 3.2 and (3.17) the asymptotic relative efficiency (A.R.E.) of the test based on \( S_n \) with respect to the classical analysis of variance test is deduced to be equal to

(4.1) \[ e_n(J) = (p-1) \left( \sum_{r=1}^{p} [J(r,p) - \bar{J}]^2 \right) / \left( \sum_{r=1}^{p} \lambda_{r,n} \right)^2 \]

As from (3.7), (3.8) and (3.9), we have \( \sum_{r=1}^{p} \lambda_{r,n} = 0 \), (4.1) may also be written as

(4.2) \[ [p^2 (p-1) \sum_{r=1}^{p} \lambda_{r,n}^2] \left( \sum_{r=1}^{p} [J(r,p) - \bar{J}]^2 \right) / \left( \sum_{r=1}^{p} \lambda_{r,n} \right)^2 \left( \sum_{r=1}^{p} \lambda_{r,n}^2 \right) \left( \sum_{r=1}^{p} [J(r,p) - \bar{J}]^2 \right), \]

where the second factor of (4.2) is uniformly bounded by 1, and the first factor is independent of \( \{J(r,p), r=1,\ldots,p\} \). We now define
as the intrinsic A.R.E. of the method of n rankings as it is the maximum possible
value of $e_n(J)$. Further, a particular $S_n$-test is said to be asymptotically efficient
for $(F_1, \ldots, F_n)$ if $e_n(J) = e_k^*$ for that $(F_1, \ldots, F_n)$. It follows from (4.1), (4.2)
and (4.3) that $e_n(J) = e_k^*$, if and only if,

$$J(r,p) - J = k \lambda_{r,n}$$

for all $r=1, \ldots, p$, $k$ being a constant.

For $p=2$, we have $\lambda_{1,n} = -\lambda_{2,n}$ and also whatever be $\{J(r,p), r=1,2\}$, $J(1,2) - J =
J(2,2) - J$. Hence, (4.4) holds. So any $S_n$-test will be asymptotically efficient
and will attain the intrinsic A.R.E. If $p=3$ and $F_1, \ldots, F_n$ are all symmetric about
zero (so that $\beta_{0,1}^{(1)} = \beta_{1,1}^{(1)}$ for all $i=1, \ldots, n$), we have from (3.9), $\lambda_{1,n} = -\lambda_{3,n}$ and
$\lambda_{2,n} = 0$. As such, the optimum $S_n$ reduces to $\chi^2_r$, defined by (1.1). However, if
$F_1, \ldots, F_n$ are not all symmetric, this property of $\chi^2_r$ will not hold, in general. (It
may be noted that $S_n$ corresponding to $\{J(r,p), r=1, \ldots, p\}$ in (4.4) also corresponds
to the likelihood ratio criterion (for the model (2.2) and (2.4) with any given $\theta$),
and the intrinsic A.R.E. $e_k^*$ is the A.R.E. of the likelihood ratio test for the same
problem. For intended brevity the proof of this statement is omitted.)

For $p \geq 4$, the choice of the optimum $S_n$ (in accordance with (4.4)) in general,
depends on $(F_1, \ldots, F_n)$ through $\lambda_{r,n}$, $r=1, \ldots, n$. If, however, $F_1 = \cdots = F_n = F$,
then in most of the cases, $\lambda_{r,n}$, $r=1, \ldots, p$, are quite simple and the corresponding
$S_n$-statistics are not difficult to compute. We shall consider the following
illustrations.

(I) Uniform distribution: $f(x) = F'(x) = dx$: $0 \leq x \leq 1$. Here, we have

$$\lambda_{r,n} = J(r,p) = \begin{cases} 
1, & r=1 \\
0, & 2 \leq r \leq p-1 \\
-1, & r=p 
\end{cases}$$
Thus, for \( p \geq 4 \), the optimum \( S_n \) is different from \( \chi^2_r \) in (1.1).

(II) **Exponential distribution:** \( f(x) = e^{-x}, \ 0 \leq x < \infty \). Here

\[
\lambda_{r,n} = \begin{cases} 
1/p, & \text{if } r=1 \\
-1/p(p-1), & \text{if } r>1 
\end{cases} \Rightarrow J(r,p) = \begin{cases} 
1, & \text{if } r=1 \\
0, & \text{if } r=2,\ldots,p.
\end{cases}
\]

The corresponding \( S_n \) is a particular case of (1.2) (with \( \alpha = 1 \)).

(III) **Double exponential distribution:** \( f(x) = e^{-|x|}, \ -\infty < x < \infty \). By (3.7), we have

\[
\beta_{s-1,p-2} = \frac{1}{p(p-1)} \left\{ p^x \sum_{r=s}^{p} \left( \frac{p}{r} \right)^{2-p} - (p-s) \sum_{r=s}^{p} \left( \frac{p}{r} \right)^{2-p} \right\}, \ s=1,\ldots,p-1.
\]

Hence, it can be shown that

\[
J(r,p) = p(p-1)\lambda_{r,n} = 1 - 2 \sum_{s=0}^{r-1} \left( \frac{p}{s} \right)^{2-p}, \ 1 \leq r \leq p.
\]

(IV) **Logistic distribution:** \( F(x) = (1+e^{-x})^{-1}, \ -\infty < x < \infty \). Here, we have

\( f(x) = F(x)[1-F(x)] \), and hence, it can be shown that

\[
\lambda_{r,n} - 2(r - \frac{p+1}{2})/p(p-1) \Rightarrow J(r,p) = r \text{ for } 1 \leq r \leq p.
\]

Thus the optimum \( S_n \) reduces to Friedman's \( \chi^2_r \).

(V) **Normal distribution:** \( f(x) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}x^2\right\}, \ -\infty < x < \infty \). We denote by \( X_{s,p-1} \) the \( s \)th smallest observation in a sample of size \( p-1 \) from the standard normal distribution, \( s=1,\ldots,p-1 \). Then from (3.7), we have

\[
(p-1)\beta_{s-1,p-2} = E[f(X_{s,p-1})], \ s=1,\ldots,p-1.
\]

It may be difficult to evaluate (4.10) for various values of \( p \) and \( s \). Let \( \mu_{s,p-1} \) and \( \sigma^2_{s,p-1} \) be respectively the mean and variance of \( X_{s,p-1} \). Then, for \( p \geq 4 \), (4.10)
may be approximated (crudely) as

\[(4.11) \quad (p-1)\varepsilon_{s-1,p-2} = f(\mu_{s,p-1})\{1+\frac{1}{2}(\mu_{s,p-1}^2-1)\sigma_{s,p-1}^2\}, \ s=1,\ldots,p-1.\]

As tables for the values of \(\mu_{s,p-1}\) and \(\sigma_{s,p-1}^2\) are available (cf [6]), (4.11) can be computed for various values of \(s\) and \(p\).

It may be noted that for logistic cdf, the rank-sum test is asymptotically optimum (in one way layouts) and the same property holds for the method of \(n\) rankings. However, for double exponential cdf, in one way layout, the median test (cf. [4]) is asymptotically optimum, but the statistic (1.2) (with \(a = [p/2]\) or \([p/2+1]\)) is not so for the method of \(n\) rankings. There is also difference in the scores for the normal distribution for the two cases.

We have so far considered the case of complete layouts where each observer ranks all the \(p\) objects. Certain balanced incomplete layouts are considered by Durbin (1951) and Bhapkar (1961); the test statistics are straightforward generalizations of (1.1) and (1.2). If \(F_1 = \ldots = F_n = F\) then all what has been discussed earlier this section also holds for such (balanced) incomplete layouts.

For brevity, the details are omitted. If, however, \(F_1, \ldots, F_n\) are not all identical, it is quite difficult to simplify the expression for the power-efficiency of such a scheme and to consider results analogous to (4.1) to (4.11).

REFERENCES


