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ABSTRACT

The theory of linear representation of projective planes was developed by Bruck and one of the authors (Bose) in two earlier papers [J. Algebra, 1 (1964), 85-102 and 4 (1966), 117-172]. Bose and Barlotti obtained some new linear representations by generalizing the concept of incidence in the representation [Ann. Mat. Pura. Appli., (197)]. In this paper, we show that the Δ-planes of Bose and Barlotti are semi-translation planes, and obtain the ternary rings of these planes, where the ternary function of Δ is expressed explicitly in terms of the addition and multiplication in the Veblen-Wedderburn system, coordinatizing the translation plane T, from which Δ can be obtained by dualization and derivation.

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§ 1.

1. A (right) Veblen-Wedderburn system (or quasifield) is a set of elements $R$, with two binary operations $+$ and $\times$ satisfying the following axioms:

   (i) $(R,+)$ is an Abelian group with zero element $0$.

   (ii) $x \times 0 = 0 \times x = 0$ for any $x$ in $R$.

   (iii) If $R_1$ is the set of non-zero elements of $R$ then $(R_1, \times)$ is a loop with identity element $1$.

   (iv) $(x+y)z = xz + yz$ for all $x, y, z$ in $R$.

   (v) If $a, b, c$ are elements of $R$, with $a \neq b$, there exists one and only one element $x$ of $R$ such that $x \times a = x \times b + c$.

   We shall denote the Veblen-Wedderburn system satisfying (1)-(v) by $(R,+,\times)$, or in short $R$.

   Let $K$ be the subset of all elements $f$ of $R$ for which

   \[
   f(x+y) = f(x) + f(y), \quad (f \times x) \times y = f \times (x \times y)
   \]

   for all $x, y$ in $R$. Then we know that [see for example (5)], that $K$ is a skew field, which is defined to be the left operator skew field (or kernel) of $(R,+,\times)$.

2. It is well known [8] that we can coordinatize any affine translation plane $\alpha T$ by using a suitable Veblen-Wedderburn system $(R,+,\times)$. The ternary ring of $T$ is

   \[
   (1.2) \quad F(x,m,b) = x \times m + b.
   \]

   The points of $T$ are ordered pairs $(x,y)$ where $x$ and $y$ are in $R$. The lines of $T$ are of two types. Lines of the first type are specified by ordered pairs $[m,b]$, where $m$ and $b$ are in $R$. The lines of the second type are specified by a single element $[c]$ of $R$. $m$ is called the slope and $b$ the
y-intercept of the line \([m, b]\) of the first type. \(c\) is called the x-intercept of the line \([c]\) of the second type. The point \((x, y)\) is incident with the line \([m, b]\) of the first type if and only if

\[(1.3) \quad y = F(x, m, b) = x \times m + b.\]

The point \((x, y)\) is incident with the line \([c]\) of the second type if and only if \(x = c\). The affine plane \(\mathcal{A}\) can be extended to a projective plane \(\mathcal{P}\) in the usual manner.

We shall show that if \((R, +, \times)\) is two dimensional over some skew field contained in \(K\) and satisfies an additional condition, then we can use \(R\) to coordinatize another affine plane \(\mathcal{A}\) which when extended to a projective plane \(\Delta\), is a semi-translation plane in the sense of Ostrom [9].

3. If \((R, +, \times)\) is two dimensional over \(F\), we can without loss of generality choose the basis of \(R\) over \(F\) to be 1, \(i\) where 1 is the unit element of \(F\) and \(R\), and \(i\) is a suitably chosen element of \(R\), not in \(F\). Then any element \(x\) in \(R\) can be expressed as \(x = x_1 + x_2 i\) where \(x_1, x_2\) are in \(F\), and \(x_2 i\) is \(x_2 \times i\).

To each element \(x\) of \(R\), there corresponds the binary vector \((x_1, x_2)\) with coordinates from \(F\). This is a \((1,1)\) correspondence. If \(x + y = z\), and the vectors corresponding to \(x, y\) and \(z\) are \((x_1, x_2), (y_1, y_2)\) \((z_1, z_2)\) then

\[(1.4) \quad (z_1, z_2) = (x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2).\]

The vectors corresponding to the elements 0, 1 and \(i\) of \(R\) are \((0,0)\), \((1,0)\) and \((0,1)\). Let \(m = m_1 + m_2 i\) and \(im = m_1^* + m_2^* i\), where \(m_1, m_2, m_1^*, m_2^*\) are in \(F\). We shall say that the 2×2 matrix

\[A_m = \begin{bmatrix} m_1 & m_2 \\ m_1^* & m_2^* \end{bmatrix},\]
corresponds to the element \( m \) of \( R \). The first row of \( A_m \) is the vector corresponding to \( m \), and the second row of \( A_m \) is the vector corresponding to \( im \).

Let \( C \) be the collection of all \( 2 \times 2 \) matrices over \( F \), which correspond in this way to the elements of \( R \). Then \( C \) is a subset of the set of \( 2 \times 2 \) matrices over \( F \). Now if \( y = x \times m \),

\[
(1.5) \quad y_1 + y_2 i = (x_1 + x_2 i) \times (m_1 + m_2 i) \\
= x_1 (m_1 + m_2 i) + x_2 i (m_1 + m_2 i) \\
= x_1 (m_1 + m_2 i) + x_2 (m^*_1 + m^*_2 i) \\
= x_1 m_1 + x_2 m^*_1 + (x_1 m_2 + x_2 m^*_2) i.
\]

Hence the vector corresponding to \( x \times m \) is \( (x_1 m_1 + x_2 m^*_1, x_1 m_2 + x_2 m^*_2) \). We can therefore alternatively represent \( (R, +, \times) \) by binary vectors over \( F \) where the addition is the ordinary vector addition given by (1.4) and multiplication is defined by

\[
(1.6) \quad (x_1, x_2) \times (m_1, m_2) = (x_1, x_2) \begin{bmatrix}
  m_1 & m_2 \\
  m^*_1 & m^*_2
\end{bmatrix}.
\]

In particular

\[
(0,1) \times (m_1, m_2) = (m^*_1, m^*_2).
\]

The vector corresponding to the product \( x \times m \) is the ordinary matrix product of the vector corresponding to \( x \) and the matrix corresponding to \( m \). We shall write the elements of \( R \) as \( x_1 + x_2 i \) or \( (x_1, x_2) \) according to convenience.

4. A \( 2 \times 2 \) matrix

\[
(1.7) \quad U = \begin{bmatrix}
  a_1 & a_2 \\
  b_1 & b_2
\end{bmatrix},
\]

with elements from the skew field \( F \) is defined to be singular, if its row vectors are independent from the left, i.e. there exist elements \( r_1, r_2 \) of \( F \),
not both zero, such that

\[ r_1a_1 + r_2b_1 = 0, \quad r_1a_2 + r_2b_2 = 0. \]

\( U \) is defined to be non-singular if \( U \) is not singular. We state here, for subsequent use, the following two obvious Lemmas.

**Lemma 1.1.** The column vectors of the matrix \( U \), given by (1.7) are dependent from the right, i.e., there exist elements \( s_1 \) and \( s_2 \) of \( F \), not both zero, such that

\[ a_1s_1 + a_2s_2 = 0, \quad b_1s_1 + b_2s_2 = 0 \]

if and only if \( U \) is singular.

In other words, the dependence (independence) of the row vectors of \( U \) from the left is equivalent to the dependence (independence) of the column vectors of \( U \) from the right.

**Corollary.** If \( U \) is non-singular, it cannot have a null row or a null column.

**Lemma 1.2.** If the matrix \( U \) given by (1.7) is non-singular, then the equations

\[ x_1a_1 + x_2b_1 = c_1, \quad x_1a_2 + x_2b_2 = c_2 \]

have a unique solution \( x_1, x_2 \) and the equations

\[ a_1y_1 + a_2y_2 = d_1, \quad b_1y_1 + b_2y_2 = d_2 \]

have a unique solution \( y_1, y_2 \).

5. The collection \( C \) of 2×2 matrices corresponding to the elements of \( R \) has the properties given by the following Lemma:
Lemma (1.3). (a) The null matrix \( 0 \) and the unit matrix \( I \) belong to \( C \).

(b) Each binary vector of \( R \), appears as the first (second) row of a unique matrix of \( C \).

(c) If \( A_m \) and \( A_m' \) are two distinct matrices of \( C \) then their difference \( A_m - A_m' \) is non-singular. In particular, if \( A_m \neq 0 \), then \( A_m \) is non-singular.

(d) If \( (x_1, x_2) \neq (0,0) \) and \( (y_1, y_2) \) are binary vectors of \( R \), there is a unique matrix \( A_m \) of \( C \) such that

\[
(1.8) \quad (y_1, y_2) = (x_1, x_2)A_m = (x_1, x_2) \begin{bmatrix} m_1 & m_2 \\ m_1^* & m_2^* \end{bmatrix}.
\]

(e) If \( (y_1, y_2) \) and \( (m_1, m_2) \neq (0,0) \) are given binary vectors of \( R \) and \( A_m \) is the unique matrix of \( C \) whose first row is \( (m_1, m_2) \), then there is a unique vector \( (x_1, x_2) \) of \( R \) for which (1.8) holds.

The property (a) is obvious since the null matrix

\[
0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\]

corresponds to the zero element of \( R \), and the unit matrix

\[
I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

corresponds to the unity of \( R \). The first part of (b) follows from definition. The second part follows by noting that if the vector \( (m_1^*, m_2^*) \) with elements from \( F \) is given, and \( m^* = m_1^* + m_2^* i \), then the relation \( im = m^* \) uniquely determines the element \( m = m_1 + m_2 i \) of \( R \). Since (1.8) can be written as

\[
y_1 + y_2 i = (x_1 + x_2 i) \times (m_1 + m_2 i),
\]

(d) and (e) follow from axiom (iii), which states that the non-zero elements of \( R \) form a loop with respect to multiplication. We thus only need to prove the property (c). Let
be two distinct matrices belonging to \( C \). From (a), \((m_1,m_2) \neq (m'_1,m'_2)\). If \( A_m - A_{m'} \) is singular, then by definition there exists a vector \((r_1,r_2) \neq (0,0)\) with coordinates from \( F \), such that

\[(1.9) \quad (r_1,r_2)(A_m - A_{m'}) = 0,\]

which is equivalent to

\[(1.10) \quad (r_1,r_2) \times (m_1,m_2) = (r_1,r_2) \times (m'_1,m'_2).\]

Since \((r_1,r_2) \neq (0,0)\), it follows from (iii), that \((m_1,m_2) = (m'_1,m'_2)\) which is a contradiction. This proves (c) and completes the proof of the Lemma.

6. We shall now assume that the Veblen-Wedderburn system \( R \) which is two dimensional over the skew field \( F \) contained in the left operator skew field (kernel) \( K \) of \( R \), satisfies the following additional axiom:

(vi) For any given elements \( x = x_1 + x_2i \neq 0 \) and \( y = y_1 + y_2i \) of \( R \), there exists a unique element \( m = m_1 + m_2i \) of \( R \), such that

\[(1.12) \quad m_1x_1 + m_2x_2 + (m_1x_1 + m_2x_2)i = y_1 + y_2i,\]

where \( i = m_1 + m_2i \), and \( x_1, x_2, y_1, y_2, m_1, m_2, m'_1, m'_2 \) are in \( F \).

Axiom (vi) may alternatively be written as:

(vi,a) given the binary vectors \((x_1,x_2) \neq (0,0)\) and \((y_1,y_2)\) with coordinates from \( F \), there exists a unique matrix \( A_m \) of \( C \) such that

\[(1.13) \quad A_m \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} m_1 & m_2 \\ m'_1 & m'_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.\]
**Lemma (1.4).** Given any binary vector with coordinates from $F$, it appears as the first (second) column of a unique matrix $A_m$ of $C$, if axiom (vi) is satisfied.

Let the binary vector be $(y_1, y_2)$. In (1.13), choose $x_1 = 1$, $x_2 = 0$. Then $m_1 = y_1$, $m_1^* = y_2$. Hence there is one matrix of $C$ with the first column $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. There cannot be two such matrices, since their difference would be singular from the corollary to Lemma (1.1), contradicting Lemma (1.3), part (c). Similarly by choosing $x_1 = 0$, $x_2 = 1$, it follows that there is a unique matrix of $C$ whose second column is $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$.

7. In view of Lemma (1.3) part b, we can define the functions $F_1$, $F_2$ of elements $x_1, x_2$ of $F$ by setting

\[ F_1(x_1, x_2) = x_1^*, \quad F_2(x_1, x_2) = x_2^* \tag{1.14} \]

where

\[
A_x = \begin{bmatrix} x_1 & x_2 \\ x_1^* & x_2^* \end{bmatrix}
\]

is the uniquely defined matrix of $C$, whose first row is $(x_1, x_2)$.

Similarly, when axiom (vi) is satisfied, we can define functions $G_1$, $G_2$ of elements $x_1, x_2$ of $F$ by setting

\[ G_1(x_1, x_1^*) = x_2, \quad G_2(x_1, x_1^*) = x_2^* \tag{1.15} \]

where $A_x$ is the uniquely defined matrix of $C$ whose first column is $\begin{bmatrix} x_1 \\ x_1^* \end{bmatrix}$.

**Lemma (1.5).** If axiom (vi) is satisfied and $m_1, m_2 \neq 0, b_1, b_2$ belonging to $F$ are given, and

\[ (y_1, y_2) = (m_1, m_2) \times (x_1, x_2) + (b_1, b_2), \tag{1.16} \]

then any one of the pairs $(x_1, y_1)$, $(x_2, y_2)$ uniquely determines the other.
The relation (1.16) is equivalent to

\[(1.17) \quad (y_1, y_2) = (m_1, m_2) \begin{bmatrix} x_1 & x_2 \\ x_1^* & x_2^* \end{bmatrix} + (b_1, b_2), \]

where \( x_1^* + x_2^* i = i (x_1 + x_2 i) \), i.e. the matrix on the right in (1.17) belongs to \( C \). Hence

\[(1.18) \quad y_1 = m_1 x_1 + m_2 x_1^* + b_1, \quad y_2 = m_1 x_2 + m_2 x_2^* + b_2. \]

Hence \( x_1^* = m_2^{-1} (y_1 - m_1 x_1 - b_1) \), and therefore from Lemma (1.4) \( x_2 \) and \( x_2^* \) are uniquely determined if \( x_1, y_1 \) are given. Then \( y_2 \) is determined by the second equation in (1.18). In terms of the functions \( F_1, F_2, G_1, G_2 \) given by (1.14) and (1.15) we can write

\[(1.19) \quad x_2 = G_1(x_1, m_2^{-1}(y_1 - m_1 x_1 - b_1)), \]

\[(1.20) \quad y_2 = m_1 G_1(x_1, m_2^{-1}(y_1 - m_1 x_1 - b_1)) + m_2 G_2(x_1, m_2^{-1}(y_1 - m_1 x_1 - b_1)) + b_2. \]

In the same way, we can show that if \( x_2, y_2 \) are given, \( x_1, y_1 \) are uniquely determined.

8. We note that if \( F \) is finite, i.e., \( F \) is a Galois field \( GF(q) \), then axiom (vi) is automatically satisfied. This follows by observing that the right hand side of (1.13) can assume exactly \( q^2 \) values and there are exactly \( q^2 \) matrices in \( C \). If for some \( (y_1, y_2) \) there is no corresponding \( A_m \), then there must exist distinct matrices \( A_m \) and \( A_m' \) belonging to \( C \) such that

\[ A_m \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A_m' \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \]

Hence

\[ (A_m' - A_m) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0. \]
Since \((x_1, x_2) \neq (0, 0)\), it follows from Lemma (1.1), that \(A_m', A_m''\) is singular, which contradicts Lemma (1.3), part (c). Hence when \(F\) is finite, and \((x_1, x_2) \neq (0, 0)\) and \((y_1, y_2)\) are given, there is a unique \(A_m\) belonging to \(C\) for which (1.13) is satisfied.

In §3, we shall give a geometrical interpretation of axiom (vi).

§ 2.

1. We shall now use the (right) Veblen-Wedderburn system \((R, +, \times)\) considered in §1, to obtain an affine plane \(a\Delta\) different from the affine plane \(aT\) with the ternary ring (1.2). It will be shown in §5 that \(a\Delta\) is a semi-translation plane in the sense of Ostrom [9]. As before, we take \(R\) to be two dimensional over the skew field \(F\) contained in the kernel \(K\) of \(R\), and having the basis elements \(1\) and \(i\), where \(1\) is the unit element of \(F\) and \(R\), and \(i\) is not in \(F\). We shall assume that \(R\) satisfies the standard axioms (i)-(v) and the additional axiom (vi).

2. Let us consider an incidence structure \([P, L, I]\) where \(P\) is a set of points, \(L\) is a set of lines and \(I\) is an incidence relation between points and lines, defined as follows:

Each point of \(P\) is coordinatized by an ordered pair of elements of \(R\). The coordinates of a point can be taken as \((x_1^1y_1^1, x_2^1+y_2^1)\), where \(x_1, y_1, x_2, y_2\) are elements of \(F\). When \(F\) is the Galois field \(GF(q)\), there are \(q^4\) points.

Lines of \(L\) are of several types. Each line of type 1 is coordinatized by an ordered pair of elements of \(R\), the first of which does not belong to \(F\). The coordinates of a line of type 1 can be taken as \([m_1^1+m_2^1i, b_1^1+b_2^1i]\), where \(m_2 \neq 0\). Lines of type 2A are coordinatized by an ordered pair of elements of \(R\),
the first of which belongs to $F$. The coordinates of a line of type 2a can be taken as $[m_1, b_1+b_2i]$. Each line of type 2b is coordinatized by a single element of $R$. Thus a line of type 2b has a coordinate $[c_1+c_2i]$. When $F$ is the Galois field $GF(q)$, there are $q^4-q^3$ lines of type 1, $q^3$ lines of type 2a and $q^2$ lines of type 2b.

The line $[m_1+m_2i, b_1+b_2i]$ of type 1, $m_2 \neq 0$, is defined to be incident with the point $(x_1+y_1i, x_2+y_2i)$ if and only if

\[(2.1) \quad y_1 + y_2i = (m_1+m_2i) \times (x_1+x_2i) + (b_1+b_2i).\]

The line $[m_1, b_1+b_2i]$ of type 2a is defined to be incident with the point $(x_1+y_1i, x_2+y_2i)$ if and only if

\[(2.2) \quad x_2 + y_2i = (x_1m_1+b_1) + (y_1m_1+b_2)i.\]

The line $[c_1+c_2i]$ of type 2b is said to be incident with the point $(x_1+y_1i, x_2+y_2i)$ if and only if

\[(2.3) \quad x_1 + y_1i = c_1 + c_2i.\]

We shall show that the incidence structure $(P, L, I)$ so defined is an affine plane.

We have already noted that the elements of $R$ can alternatively be represented by binary vectors over $F$ through the mapping $(x_1+x_2i) \rightarrow (x_1, x_2)$. The point $\{(x_1+y_1i, (x_2+y_2i))$ then becomes $\{(x_1, y_1), (x_2, y_2)\}$. The line $[m_1+m_2i, b_1+b_2i]$ becomes $\{[m_1, m_2], (b_1, b_2)\}$, and the line $[c_1+c_2i]$ becomes the line $[c_1, c_2]$.

The condition for the line $\{[m_1, m_2], (b_1, b_2)\}$, $m_2 \neq 0$ of type 1 to be incident with point $\{(x_1, y_1), (x_2, y_2)\}$ then becomes

\[(2.4) \quad (y_1, y_2) = (m_1, m_2) \begin{bmatrix} x_1 & x_2 \\ x_1^* & x_2^* \end{bmatrix} + (b_1, b_2),\]
where \((0,1)x(x_1,x_2) = (x_1^*,x_2^*), \text{ i.e.,}\)

\[
A_x = \begin{bmatrix}
x_1 & x_2 \\
x_1^* & x_2^*
\end{bmatrix}
\]

belongs to \(C\).

The condition for the line \([(m_1,0),(b_1,b_2)]\) of type 2a to be incident with the point \([(x_1,y_1),(x_2,y_2)]\) becomes

\[
x_2 = x_1m_1 + b_1, \quad y_2 = y_1m_1 + b_2.
\]

The condition for the line \([c_1,c_2]\) of type 2b to be incident with the point \([(x_1,y_1),(x_2,y_2)]\) becomes

\[
x_1 = c_1, \quad y_1 = c_2.
\]

3. Two lines \([(m_1,m_2),(b_1,b_2)]\) and \([(m'_1,m'_2),(b'_1,b'_2)]\) both of type 1 or both of type 2a will be defined to belong to the same parallel class if and only if \((m_1,m_2) = (m'_1,m'_2)\). We may, therefore, speak of parallel classes \((m_1,m_2)\). The parallel class \((m_1,m_2)\) is said to be type 1 if \(m_2 \neq 0\), and type 2a if \(m_2 = 0\). All lines of type 2b are defined to belong to the same parallel class which may be denoted by \((\infty)\). Two lines of different types do not belong to the same parallel class. The two succeeding Lemmas provide the justification for this nomenclature.

**Lemma (2.1)**. Two distinct lines of the same parallel class are not both incident with the same point.

Case I. Let \([(m_1,m_2),(b_1,b_2)]\) and \([(m_1,m_2),(b'_1,b'_2)\), \(m_2 \neq 0\) be two distinct lines of type 1 belonging to the parallel class \((m_1,m_2)\) of type 1. If they are incident with the same point \([(x_1,y_1),(x_2,y_2)]\), then from (2.4), \((b_1,b_2) = (b'_1,b'_2)\). This is a contradiction since the two lines are distinct.
Case II. Let \( [(m_1,0),(b_1,b_2)] \) and \( [(m_1',0),(b_1',b_2')] \) be two distinct lines of type 2a belonging to the parallel class \((m_1,0)\). If they are incident with the same point \( \{(x_1,y_1),(x_2,y_2)\} \), then from (2.6), \( b_1 = b_1' \), \( b_2 = b_2' \), which is a contradiction.

Case III. From (2.7), two distinct lines \( [c_1,c_2], [c_1',c_2'] \) of type 2b cannot be incident with the same point \( \{(x_1,y_1),(x_2,y_2)\} \).

**Lemma (2.2).** Two lines belonging to different parallel class are both incident with a single point.

Case I. It follows from (2.6) and (2.7) that the line \( [c_1,c_2] \) of type 2b, and the line \( [(m_1,0),(b_1,b_2)] \) of type 2a are both incident with the unique point \( \{(c_1,c_2),(c_1m_1+b_1, c_2m_1+b_2)\} \).

Case II. If \( [c_1,c_2] \) is a line of type 2b, and \( [(m_1,m_2),(b_1,b_2)] \), \( m_2 \neq 0 \) is a line of type 1, then if \( \{(x_1,y_1),(x_2,y_2)\} \) is a point common to the two lines, \( x_1 = c_1, y_1 = c_2 \) and (2.4) is satisfied. It follows from Lemma (1.5), that there is a single point with which both lines are incident.

Case III. Let \( [(m_1,m_2),(b_1,b_2)], [(m_1',m_2'),(b_1',b_2')] \), \( m_2 \neq 0, m_2' \neq 0 \), be two given lines of type 1, not belonging to the same parallel class. Then \( (m_1,m_2) \neq (m_1',m_2') \). If \( \{(x_1,y_1),(x_2,y_2)\} \) is a point incident with both lines, it follows from (2.4) that

\[
(2.8) \quad (0,0) = (m_1-m_1', m_2-m_2')[\begin{array}{cc} x_1 & x_2 \\ x_1' & x_2' \end{array}] + (b_1-b_1', b_2-b_2').
\]

If \( (b_1,b_2) = (b_1',b_2') \) then from (2.8) and Lemma (1.3) part (d), \( x_1 = 0, x_2 = 0 \) is the only solution of (2.8). Hence from (2.4), \( \{(0,0),(b_1,b_2)\} \) is the unique point incident with both lines.

If \( (b_1,b_2) \neq (b_1',b_2') \), then from (2.8), \( (x_1,x_2) \neq (0,0) \). Hence from Lemma (1.3) part (c), the matrix (2.5) is non-singular. Hence from Lemma (1.3) part (d), \( (x_1,x_2) \) is uniquely determined. Then \( (y_1,y_2) \) is determined by (2.4).
Thus there is a unique point \((x_1, y_1, (x_2, y_2))\) incident with both lines.

Case IV. Let \([(m_1, 0), (b_1, b'_2)], [(m'_1, 0), (b'_1, b'_2)]]\) be two lines of type 2a, not belonging to the same parallel class. Then \(m_1 \neq m'_1\). If \((x_1, y_1, (x_2, y_2))\) is a point incident with both lines, it follows from (2.5), that

\[
\begin{align*}
(2.9) \quad x_2 &= x_1 m_1 + b_1 = x_1 m'_1 + b'_1, \\
y_2 &= y_1 m_1 + b_2 = y_1 m'_1 + b'_2.
\end{align*}
\]

Hence \(x_1 = -(b_1 - b'_1)(m_1 - m'_1)^{-1}, y_1 = -(b_2 - b'_2)(m_2 - m'_2)^{-1}\). Then \(x_2\) and \(y_2\) are uniquely determined by (2.9).

Case V. Let \([(m_1, m_2), (b_1, b'_2)], m_2 \neq 0\) be a line of type 1, and \([(m'_1, 0), (b'_1, b'_2)]]\) be a line of type 2a. If \((x_1, y_1, (x_2, y_2))\) is a point incident with both lines, it follows from (2.4) and (2.6)

\[
\begin{align*}
(2.10) \quad y_1 &= m_1 x_1 + m_2 x^* + b_1, \\
(2.11) \quad y_2 &= m_1 x_2 + m_2 x^* + b_2, \\
(2.12) \quad x_2 &= x_1 m'_1 + b'_1, \\
(2.13) \quad y_2 &= y_1 m'_1 + b'_2.
\end{align*}
\]

Let \(\lambda = m_2^{-1}(b_1 m'_1 - m_1 b'_2 - b_2 + b'_2)\). From axiom (vi, a) formula (1.13), we can find unique matrix \(A_x\) belonging to \(C\), such that

\[
(2.14) \quad A_x \begin{bmatrix} -m'_1 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \\ x^*_1 & x^*_2 \end{bmatrix} \begin{bmatrix} -m'_1 \\ 1 \end{bmatrix} = \begin{bmatrix} b'_1 \\ \lambda \end{bmatrix}
\]

Then (2.12) is satisfied, and \(y_1, y_2\) can be uniquely determined from (2.10) and (2.11). We shall show that (2.13) is automatically satisfied. Multiplying (2.10) by \(-m'_1\) from the right and adding (2.11) we have

\[
-y_1 m'_1 + y_2 = m_1 (-x_1 m'_1 + x_2) + m_2 (-x^*_1 m_1 + x^*_2) - b_1 m'_1 + b_2.
\]
Using (2.14) and substituting for \( \lambda \), this reduces to (2.13). Thus there is exactly one point incident with both the given lines.

4. **Lemma (2.3).** Given any point \( \{(x_1, y_1), (x_2, y_2)\} \), there is exactly one line in any given parallel class incident with the point.

   **Case I.** In the parallel class \( (\omega) \) consisting of all lines of type 2b, there is only one line, viz \([x_1, y_1]\) which is incident with the given point.

   **Case II.** Let \([ (m_1, 0), (b_1, b_2) ] \) be a line of type 2a belonging to the parallel class \( (m_1, 0) \). If it is incident with the point \( \{(x_1, y_1), (x_2, y_2)\} \), then \((b_1, b_2)\) is uniquely determined by (2.6).

   **Case III.** Let \([ (m_1, m_2), (b_1, b_2) ] \), \( m_2 \neq 0 \) be a line of type 1 incident with the point \( \{(x_1, y_1), (x_2, y_2)\} \). Then \((b_1, b_2)\) is uniquely determined by (2.4).

   **Lemma (2.4).** For given \( (x_1, y_1), (x_2, y_2), (x'_1, y'_1), (x'_2, y'_2) \), where \((x_1, y_1) \neq (x'_1, y'_1)\), the equations

\[
(2.15) \quad (y_1, y_2) = (m_1, m_2) \begin{bmatrix} x_1 & x_2 \\ x'_1 & x'_2 \end{bmatrix} + (b_1, b_2)
\]

\[
(2.16) \quad (y'_1, y'_2) = (m_1, m_2) \begin{bmatrix} x'_1 & x'_2 \\ x'^*_1 & x'^*_2 \end{bmatrix} + (b_1, b_2)
\]

in which the matrices appearing on the right belong to \( C \), uniquely determine \((m_1, m_2), (b_1, b_2)\). Further \( m_2 = 0 \) if and only if the matrix

\[
(2.17) \begin{bmatrix} x_1 - x'_1 & x_2 - x'_2 \\ y_1 - y'_1 & y_2 - y'_2 \end{bmatrix}
\]

is singular.

From (2.15) and (2.16) we have

\[
(2.18) \quad (y_1 - y'_1, y_2 - y'_2) = (m_1, m_2) \begin{bmatrix} x_1 - x'_1 & x_2 - x'_2 \\ x^*_1 - x'^*_1 & x^*_2 - x'^*_2 \end{bmatrix}.
\]

Now \((x_1, x_2) \neq (x_1', x_2')\), otherwise \((x_1, y_1) = (x_1', y_1')\) contradicting the hypothesis. From Lemma (1.3), part (c), the matrix appearing on the right in (2.18) is non-singular. From Lemma (1.2), \(m_1, m_2\) are uniquely determined. Then \((b_1, b_2)\) is determined from either (2.15) or (2.16). If \(m_2\) so determined is zero, then

\[
(2.19) \quad m_1(x_1 - x_1') - (y_1 - y_1') = 0, \quad m_1(x_2 - x_2') - (y_2 - y_2') = 0,
\]

Then by definition, the matrix (2.17) is singular.

Conversely, if (2.17) is singular, we can find \((r_1, r_2) \neq (0,0)\) such that

\[
(2.20) \quad r_1(x_1 - x_1') + r_2(x_2 - x_2') = 0, \quad r_1(y_1 - y_1') + r_2(y_2 - y_2') = 0.
\]

Since \((x_1, y_1) \neq (x_1', y_1')\), \(r_2 \neq 0\). If we set \(m_1 = -r_2^{-1}r_1\), (2.19) is satisfied. Hence (2.18) is satisfied with \(m_2 = 0\). Since \((m_1, m_2)\) is uniquely determined by (2.18), \(m_2\) must be zero. Thus in this case (2.15), (2.16) uniquely determine \((m_1, m_2), (b_1, b_2)\) with \(m_2 = 0\).

5. Lemma (2.5). Any two distinct points \(\{(x_1, y_1), (x_2, y_2)\}\) and \(\{(x_1', y_1'), (x_2', y_2')\}\) are incident with exactly one line.

Case I. Let \((x_1, y_1) = (x_1', y_1')\). Then the line \([x_1, y_1]\) of type 2b is clearly incident with both points. If the line \([m_1, m_2], (b_1, b_2)\], \(m_2 \neq 0\) of type 1 is incident with both points, then from (2.4) and Lemma (1.5), the two points would coincide. Similarly, if the line \([m_1, 0], (b_1, b_2)\] of type 2a is incident with both points, then from (2.6) the two points with coincide. Hence the line \([x_1, y_1]\) of type 2b is the only line incident with both points.

Case II. Suppose \((x_1, y_1) \neq (x_1', y_1')\), and the matrix (2.17) is singular. From Lemma (1.1) we can find \((s_1, s_2) \neq (0,0)\) such that

\[
(2.20) \quad (x_1 - x_1')s_1 + (x_2 - x_2')s_2 = 0, \quad (y_1 - y_1)s_1 + (y_2 - y_2')s_2 = 0.
\]
Now $s_2$ is non-zero, otherwise $(x_1',y_1') = (x_1',y_1')$ contradicting the hypothesis. Let $m_1 = -s_2s_1^{-1}$, $b_1 = x_2 - x_1m_1 = x_2' - x_1'm_1$, $b_2 = y_2 - y_1m_1 = y_2' - y_1'm_1$. Then from (2.6), $[(m_1,0),(b_1,b_2)]$ is the only line of type 2a incident with both points. From (2.7) there is no line of type 2b incident with both points. If the line $[(m_1,m_2),(b_1,b_2)]$, $m_2 \neq 0$ of type 1 is incident with both points, then from (2.4), equations (2.15) and (2.16) are satisfied. Hence from Lemma (2.4), $(m_1,m_2)$, $(b_1,b_2)$ are uniquely determined with $m_2 = 0$ which is a contradiction. Thus there is exactly one line which is incident with both the points. It is of type 2a.

Case III. Suppose $(x_1,y_1) \neq (x_1',y_1')$ and the matrix (2.17) is non-singular. From Lemma (2.4), we can uniquely determine $(m_1,m_2)$, $(b_1,b_2)$, $m_2 \neq 0$ satisfying (2.15) and (2.16). Hence from (2.4) there is a unique line $[(m_1,m_2),(b_1,b_2)]$ of type 1 incident with both points. From (2.7) there is no line of type 2b incident with both points. Again, if there is a line of type 2a incident with both points then from (2.6), the equation (2.20) is satisfied with $s_1 = m_1$, $s_2 = -1$. Hence from Lemma (1.1), the matrix (2.17) is singular, which is a contradiction. Thus there is exactly one line incident with both points. It is of type 1.

7. **Theorem (2.1).** The incidence structure $[P, L, I]$ defined in §2, paragraph 2, is an affine plane.

In view of the Lemmas (2.1), (2.2), (2.3), (2.5) it is sufficient to verify the existence of three non-collinear points. The points $\{(0,0),(0,0)\}$, $\{(0,0),(1,0)\}$ are from (2.6), incident with the line $[(1,0),(0,0)]$ of type 2a. The point $\{(1,0),(0,0)\}$ is not incident with this line.

The affine plane of Theorem (2.1) will be denoted by $\alpha \Delta$. Its points will be called $\alpha \Delta$-points, and its lines will be called $\alpha \Delta$-lines. It can be completed to a projective plane $\Delta$ in the usual manner. Corresponding to each parallel
class \((m_1, m_2)\) of type 1 or 2a, we take a point at infinity denoted by \(\{m_1, m_2\}\). Similarly corresponding to the parallel class \(\{\infty\}\) which consists of all \(a \Delta\)-lines of type 2b, we take a point at infinity denoted by \(\{\infty\}\). Finally, we take a line at infinity denoted by \([\infty]\). Each point at infinity is defined to be incident with the line at infinity, and all lines of the parallel class corresponding to it (and with no other line). Thus \(\Delta\)-points consist of \(a \Delta\)-points and the points at infinity, and \(\Delta\)-lines consist of \(a \Delta\)-lines and the line at infinity.

§ 3,

1. Let \(F\) be a skewfield. Let \(S_4\) be a four dimensional projective space based on \(F\), regarded as a vector space \(V\), with \(F\) as a ring of left operators. We can then assign coordinates to the points of \(S_4\), and write down the equations of linear subspaces of \(S_4\) in the usual manner, by choosing a specified basis \(e_1, e_2, e'_1, e'_2, e_0\) of \(V\). Points of \(S_4\) correspond to one dimensional vector subspaces of \(V\). Thus the coordinates of a point \(P\) which corresponds to the vector subspace \(V(P)\) with basis vector \(x_1 e_1 + x_2 e_2 + y_1 e'_1 + y_2 e'_2 + ze_0\) are \((x_1, x_2, y_1, y_2, z)\) where \(x_1, x_2, y_1, y_2, z\) are elements of \(F\) not all zero. The coordinates of a point are arbitrary up to a non-zero multiple of an element of \(F\) from the left. Thus if \(r \neq 0\), then the point with coordinates \((rx_1, rx_2, ry_1, ry_2, rz)\) is identical with \(P\). A linear equation \(\xi_1 a_1 + \xi_2 a_2 + \eta_1 b_1 + \eta_2 b_2 + \xi c = 0\) (where \(a_1, a_2, b_1, b_2, c\) are elements of \(F\) not all zero) represents a 3-space \(S_3\) which contains the point \((x_1, x_2, y_1, y_2, z)\) if and only if \(x_1 a_1 + x_2 a_2 + y_1 b_1 + y_2 b_2 + zc = 0\). The equation of a 3-space is arbitrary up to a non-zero multiple of an element of \(F\) from the right. Thus if \(k \neq 0\), then the equation
\[ \xi_1 a_1 k + \xi_2 a_2 k + \eta_1 b_1 k + \eta_2 b_2 k + \zeta k = 0 \]

also represents \( S_3 \). Dependence and independence of vectors or equations can be defined in the usual manner with the convention that multiples of vectors are to be taken from the left and multiples of equations from the right. Dependence or independence of points is defined by the dependence or independence of the corresponding coordinate vectors. A \( t \) dimensional subspace \( S_t \) of \( S_4 \) \((0 \leq t \leq 4)\) corresponds to a \( t+1 \) dimensional vector subspace \( V(S_t) \) of \( V \), and may therefore be defined as the set of points dependent on a given set of \( t+1 \) independent points corresponding to a basis of \( V(S_t) \). Alternatively \( S_t \) can be represented by a set of \( 4-t \) independent equations. A point belongs to \( S_t \) if and only if its coordinates satisfy the equations of \( S_t \), i.e., the point is contained in \( S_t \).

2. Let \( E \) be a fixed 3-space of \( S_4 \). A spread \( S \) is defined as a set of lines contained in \( E \), such that every point of \( E \) is contained in exactly one line of \( S \). A plane in \( E \) cannot contain more than one line of \( S \) (otherwise they would intersect in a point). The spread \( S \) is called a dual spread if each plane of \( E \) contains at least (and therefore exactly) one line of \( S \). If \( S_4 \) is a finite space, \( F \) can be taken as the Galois field \( GF(q) \). In this case, the spread \( S \) must necessarily be a dual spread. In the infinite case, \( S \) may or may not be a dual spread \([7]\). If a plane contains the line \( s_\infty \) belonging to \( S \), then every other line \( s \) of \( S \) meets the plane in a unique point \( S \), not in \( s_\infty \); and conversely through every point \( S \) of the plane, not in \( s_\infty \), there passes a unique line of \( S \). In the finite case, \( S \) contains exactly \( q^2+1 \) lines. For further properties of spreads, see references \([4]\), \([5]\), \([6]\) and \([12]\).

3. Let us identify \( F \), with skew field \( F \) of \( \mathbb{S} \), which is contained in the left operator skew field (kernel) \( K \) of the Veblen-Wedderburn system \((R, \times, +)\) which is two dimensional over \( F \), and which satisfies the standard axioms.
(i)-(v). We shall continue to use the notation of §1, i.e., if \( m = m_1 + m_2 i \), then \( \text{im} = m_1^* + m_2^* i \), where \( m_1, m_2, m_1^*, m_2^* \) are in \( F \). We shall also continue to identify the element \( x_1 + x_2 i \) of \( F \), where \( x_1 \) and \( x_2 \) are in \( F \), with the binary vector \( (x_1, x_2) \).

Let \( E_1, E_2, E_1', E_2' \) be four independent points of \( \Sigma \), and let \( E_0 \) and \( U \) be distinct points of \( S_4 \), not contained in \( \Sigma \) where the line \( E_0 U \) meets \( \Sigma \) in a point \( U_0 \) which is not contained in any of the planes \( E_1 E_2 E_1', E_1 E_2 E_2' \), \( E_1 E_1' E_2', E_2 E_1' E_2' \). We can now choose a basis of \( V \) so that the coordinates of \( E_1, E_2, E_1', E_2', E_0 \) and \( U \) are \( (1,0,0,0,0) \), \( (0,1,0,0,0) \), \( (0,0,1,0,0) \), \( (0,0,0,1,0) \), \( (0,0,0,0,1) \) and \( (1,1,1,1,1) \) respectively. Then the equation of \( \Sigma \) is \( \zeta = 0 \). Let the planes \( E_1 E_1' E_2' \) and \( E_2 E_1' E_2' \), contained in \( \Sigma \) be denoted by \( \Pi \) and \( \Pi^* \) respectively. \( \Pi \) is represented by the equations \( \xi_2 = 0 \), \( \zeta = 0 \) and \( \Pi^* \) by the equations \( \xi_1 = 0 \), \( \zeta = 0 \). \( \Pi \) and \( \Pi^* \) contain the common line \( s_\infty = E_1 E_2' \), represented by the equations \( \xi_1 = 0 \), \( \xi_2 = 0 \), \( \zeta = 0 \). Let \( \Pi_0 \) be the set of points in \( \Pi \) which are not in \( s_\infty \). Then the \( \xi_1 \)-coordinate of any point \( Y \) in \( \Pi_0 \) is non-zero. Hence the coordinates of \( Y \) can be written in the canonical form \( (1,0,y_1,y_2,0) \). Similarly, if \( \Pi_0^* \) is the set of points in \( \Pi^* \) but not in \( s_\infty \), then the coordinates of any point \( Y^* \) in \( \Pi_0^* \) can be written in the canonical form \( (0,1,y_1^*,y_2^*,0) \). In particular, let \( U_1 \) be the point of \( \Pi_0 \) with coordinates \( (1,0,1,0,0) \) and \( U_1^* \) the point of \( \Pi_0^* \) with coordinates \( (0,1,0,1,0) \).

Let \( m = m_1 + m_2 i = (m_1, m_2) \) be any element of \( R \), and let

\[
(3.1) \quad A_m = \begin{bmatrix} m_1 & m_2 \\ m_1^* & m_2^* \end{bmatrix},
\]

be the unique matrix of \( C \), corresponding to \( m \). Then \( \text{im} = m_1^* + m_2^* i \). To each element, \( m \) of \( R \) then corresponds a unique point \( M \) of \( \Pi_0 \) with coordinates \( (1,0,m_1,m_2,0) \) and a unique point \( M^* \) of \( \Pi_0^* \) with coordinates \( (0,1,m_1^*,m_2^*,0) \).
To any element \( m \) of \( R \), therefore corresponds a unique line \( M^m \) in \( C \), which we shall denote by \( s_m \). In particular if we choose \( m \) to be the zero element of \( R \), then \( M \) and \( M^* \) become \( E_1 \) and \( E_2 \) respectively. The line corresponding to the zero element of \( R \) is thus \( E_1 E_2 \) which can be denoted by \( s_0 \). Again if we choose \( m \) to be the unit element of \( R \), then \( M \) and \( M^* \) become \( U_1 \) and \( U_1^* \) respectively. The line corresponding to the unit element of \( R \) is thus \( U_1 U_1^* \) which may be denoted by \( s_1 \).

Let \( S' \) be the set of all lines corresponding to different elements of \( R \), and let \( S \) be obtained from \( S' \) by adjoining the additional line \( s_\infty \). We shall show that \( S \) is a spread.

Clearly \( s_m \) cannot intersect \( s_\infty \), and is represented by the equations

\[
\eta_1 = \xi_1 m_1 + \xi_2 m^*_1, \quad \eta_2 = \xi_1 m_2 + \xi_2 m^*_2, \quad \zeta = 0,
\]

which may be written as

\[
(\eta_1, \eta_2) = (\xi_1, \xi_2) \begin{bmatrix} m_1 & m_2 \\ m^*_1 & m^*_2 \end{bmatrix} = (\xi_1, \xi_2)A_m, \quad \zeta = 0.
\]

If \( s_m' \) is another line of \( S \), corresponding to the element \( m' \neq m \) of \( R \), then the equations of the corresponding line \( s_m' \) of \( S \) are

\[
(\eta_1, \eta_2) = (\xi_1, \xi_2) \begin{bmatrix} m'_1 & m'_2 \\ m^*_1 & m^*_2 \end{bmatrix} = (\xi_1, \xi_2)A_{m'}, \quad \zeta = 0,
\]

where \( A_{m'} \) is the matrix of \( C \) corresponding to \( m' \). The lines \( s_m \) and \( s_m' \) cannot have any point in common. If there is a common point \( P \), then its coordinates \( (a_1, a_2, b_1, b_2, c) \) must satisfy both (3.3) and (3.4). Hence \( c = 0 \), and

\[
(b_1, b_2) = (a_1, a_2)A_m = (a_1, a_2)A_{m'}.
\]
Hence

\[(3.6) \quad (a_1, a_2)[A_m - A_m'] = (0, 0).\]

Since \( P \) is not in \( s_{\infty} \), \((a_1, a_2) \neq (0, 0)\). Hence the matrix \( A_m - A_m' \) is singular, which contradicts Lemma (1.3) part (c). We have thus shown that any two lines of \( S \) are non-intersecting.

Again, let \( Q \) be any point of \( E \) with coordinates \((x_1, x_2, y_1, y_2, 0)\). If \((x_1, x_2) = (0, 0)\), then \((y_1, y_2) \neq (0, 0)\) and \( Q \) is contained in the line \( s_{\infty} \) of \( S \). It cannot be contained in any other line \( s_m \) of \( S \), otherwise its coordinates would satisfy (3.3), making \((y_1, y_2) = (0, 0)\). If \((x_1, x_2) \neq (0, 0)\), then from Lemma (1.3) part (d), there is a unique matrix \( A_m \) of \( C \), such that the coordinates of \( Q \) satisfy (3.3). Hence there is a unique line \( s_m \) belonging to \( S \), which contains \( Q \). This completes the proof that \( S \) is a spread.

We have now shown that starting from \((R, +, \times)\) we can construct a corresponding spread \( S \) in \( E \). This is a special case of a result proved in [5].

Note that the necessary and sufficient condition for the point \((x_1, x_2, y_1, y_2, 0)\) to be contained in the line \( s_m \) of \( S \) is

\[(3.8) \quad (y_1, y_2) = (x_1, x_2)A_m = (x_1, x_2) \begin{bmatrix} m_1 & m_2 \\ m_1^* & m_2^* \end{bmatrix} = (x_1, x_2) \times (m_1, m_2)\]

where \( A_m \) is the matrix of \( C \) corresponding to \( m \).

4. We shall now show that \( R \) will satisfy axiom (vi) of §1, precisely when \( S \) is a dual spread.

Let \( S \) be a dual spread and let \( \Pi_m \) be the plane represented by the equations

\[(3.9) \quad \xi_1 y_1 + \xi_2 y_2 = \eta_1 x_1 + \eta_2 x_2, \quad \xi = 0.\]
If \((x_1, x_2) \neq (0, 0)\), then \(\Pi_m\) does not contain \(s_\infty\). Hence it must contain a unique line \(s_m\) of \(S\), corresponding to the element \(m\) of \(R\). Therefore \(\Pi_m\) contains the points \(M\) and \(M^*\) with coordinates \((1,0,m_1,m_2,0), (0,1,m^*_1,m^*_2,0)\), where \(A_m\) given by (3.1) is the matrix of \(C\) corresponding to \(m\). Hence

\[
y_1 = \bar{m}_1 x_1 + \bar{m}_2 x_2, \quad y_2 = \bar{m}^*_1 x_1 + \bar{m}^*_2 x_2.
\]

Hence given the binary vectors \((x_1, x_2) \neq (0, 0)\) and \((y_1, y_2)\), with coordinates from \(F\), there exists a unique matrix \(A_m\) of \(C\), such that

\[
A_m \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} m_1 & m_2 \\ m^*_1 & m^*_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.
\]

We have thus proved the alternative form of axiom (vi). Conversely if axiom (vi) or alternatively (vi,a) holds, and \(\Pi_m\) is an arbitrary plane of \(\Sigma\), represented by the equations (3.9), then if \((x_1, x_2) = (0, 0), \ \Pi_m\) contains the line \(s_\infty\) of \(S\). Otherwise (3.11) determines a unique matrix \(A_m\) of \(C\), and \(\Pi_m\) contains the corresponding line \(s_m\) of \(S\).

Note that the necessary and sufficient condition that the plane

\[
\xi_1 y_1 + \xi_2 y_2 = \eta_1 x_1 + \eta_2 x_2, \quad \zeta = 0
\]

contains the line \(s_m\) of the spread \(S\) is

\[
\begin{pmatrix} m_1 & m_2 \\ m^*_1 & m^*_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.
\]
§ 4.

1. We have shown in §3 how to construct a spread $S$ corresponding to a Veblen-Wedderburn system $(R, +, \times)$ which is two dimensional over a skew field $F$ contained in its kernel. This process is reversible.

Let $S_4$ be a four dimensional projective space based on the skew field $F$, $\Sigma$ a particular 3-space contained in $S_4$, and $S$ a spread in $\Sigma$. Let $s_\infty$ be a particular line of $S$, and let $S_0$ be the set of lines of $S$ other than $s_\infty$. Let $\Pi$ be a plane contained in $\Sigma$, and passing through $s_\infty$, and $\Omega$ a 3-space in $S_4$ meeting $\Sigma$ in the plane $\Pi$. Let $E'_1, E'_2$ be any two distinct points on $s_\infty$. Let $E_1$ be any point of $\Pi$ not in $s_\infty$, and $U_1$ any point of the line $E_1'E'_1$ distinct from $E_1$ and $E'_1$. Let $s_0$ and $s_1$ be the lines of $S$ passing through the points $E_1$ and $U_1$ respectively. There is a unique line $\ell$ passing through $E'_2$ which intersects $s_0$ and $s_1$. Let $\Pi^*$ be the plane containing $\ell$ and $s_\infty$, and let $E_1$ and $U^*_1$ be the points of intersection of $\ell$ with $s_0$ and $s_1$ respectively. Let $E_0$ by any point of $\Omega$ not in $\Sigma$, and $U_0$ any point of $s_1$ distinct from $U_1$ and $U^*_1$. Finally, let $U$ be a point of $S_4$ on the line $E_0U_0$ distinct from $E_0$ and $U_0$. We can then choose a basis of $V$, such that the points $E_1, E_2, E'_1, E'_2, E_0$ and $U$ have the coordinates $(1,0,0,0,0), (0,1,0,0,0), (0,0,1,0,0), (0,0,0,1,0), (0,0,0,0,1)$ and $(1,1,1,1,1)$ respectively. Then the equation of $\Omega$ is $\xi_2 = 0$, and the equations of $\Sigma$, $\Pi$, $\Pi^*$, $s_\infty$ and the coordinates of $U_1$ and $U^*_1$ are as in paragraph 3 of §3. If $s_m$ is any line of $S_0$, and if the coordinates of the points $M$ and $M^*$ in which $s_m$ intersects the planes $\Pi$ and $\Pi^*$ are $(1,0,m_1,m_2,0)$ and $(0,1,m^*_1,m^*_2,0)$ respectively, then we can associate with the line $s_m$ the matrix

$$(4.1) \quad A_m = \begin{bmatrix} m_1 & m_2 \\ m^*_1 & m^*_2 \end{bmatrix}.$$
Let $C$ be the set of all possible matrices $A_m$ corresponding in this way to all possible lines of $S_0$. Then the matrices of $C$ satisfy the properties (a)-(e) of Lemma (1.3).

The property (a) is obvious since the null matrix $0$ corresponds to the line $s_0$ and the unit matrix $I$ corresponds to the line $s_1$. For (b), we observe that for any binary vector $(m_1, m_2)$ there is a unique point $M$ of $\Pi$ with coordinates $(1, m_1, m_2, 0)$. Let $s_m$ be the line of $S_0$ through $M$, then the matrix $A_m$ corresponding to $s_m$ is the unique matrix of $C$, with $(m_1, m_2)$ as its first row. Similarly, any binary vector $(m_1^*, m_2^*)$ occurs as the second row of a unique matrix of $C$.

If $A_m$ and $A_m'$ are two distinct matrices of $C$, then equations of the corresponding lines $s_m$ and $s_{m'}$ are given by (3.3) and (3.4) respectively. If $A_m - A_m'$ is singular, we can find $(a_1, a_2) \neq (0, 0)$ such that (3.6) is satisfied. Let $(b_1, b_2)$ be determined by (3.5). Then the point $(a_1, a_2, b_1, b_2, 0)$ satisfies the equations of both $s_m$ and $s_{m'}$. This is a contradiction since any two lines of $S$ are non-intersecting. This establishes (c). In particular, if $A_m = 0$, then $A_m$ is non-singular.

The property (d) follows because given $(x_1, x_2) \neq (0, 0)$ and $(y_1, y_2)$ there is a unique line $s_m$ of $S_0$ which passes through the point $(x_1, x_2, y_1, y_2, 0)$.

The property (e) is equivalent to the assertion that the equations

$$x_1 m_1 + x_2 m_2^* = y_1, \quad x_1 m_2 + x_2 m_2^* = y_2$$

have a unique solution $(x_1, x_2)$, given $(m_1, m_2) \neq 0$ and $(y_1, y_2)$. This follows from Lemma (1.2) and the non-singularity of $A_m$.

We can now take $R$ to be the set of binary vectors $(m_1, m_2)$ over $F$. Consider the system $(R, +, \times)$ with addition and multiplication defined by

$$(4.2) \quad (m_1, m_2) + (m'_1, m'_2) = (m_1 + m'_1, m_2 + m'_2)$$
\[(4.3) \quad (x_1, x_2) \times (m'_1, m'_2) = (x_1, x_2) A_m = (x_1 m'_1 + x_2 m'_2, x_1 m'_2 + x_2 m'_2)\]

where \(A_m\) is the unique matrix of \(C\) whose first row is \((m'_1, m'_2)\). Then 
\((R, +, \times)\) is a right Veblen-Wedderburn system with zero element \((0,0)\) and unity \((1,0)\).

The axioms (i) and (ii) follow directly from definitions. Axiom (iii) follows from properties (d) and (e) of Lemma (1.3). Axiom (iv) follows directly from definition. Axiom (v) is a consequence of the property (c) of Lemma (1.3). It is also readily seen that the subset of elements \((m'_1, 0)\) form a skew field contained in the kernel of \(R\), and isomorphic with \(F\). The elements \((1,0)\) and \((0,1)\) form a basis of \(R\) over \(F\) since

\[(4.4) \quad (m'_1, m'_2) = (m'_1, 0) \times (1,0) + (m'_2, 0) \times (0,1).\]

We can alternatively write \((m'_1, m'_2) = m'_1 + m'_2 i\). Then

\[(4.5) \quad (1,0) = 1, \quad (0,1) = i, \quad \text{im} = (0,1) \times (m'_1, m'_2) = (m^*_1, m^*_2)\]

\[= m^*_1 + m^*_2 i.\]

Finally, as we have shown in §3, axiom (vi) or (vi,a) is satisfied by 
\((R, +, \times)\) if \(S\) is a dual spread. In what follows, we shall always take \(S\) to be a dual spread.

2. We shall now obtain a linear representation of the affine plane \(\alpha A\) obtained in §2, in the projective space \(S_4\), where the spread \(S\) and the corresponding Veblen-Wedderburn system \((R, +, \times)\) are as in paragraph 1.

Let \(X\) and \(X^*\) be points of \(S_4\) with coordinates \((1,0, x_1, x_2, 0)\) and 
\((0,1, x^*_1, x^*_2, 0)\). Then \(X\) is a point of \(\Pi\), and \(X^*\) is point of \(\Pi^*\). The line 
\(s_x = XX^*\) belongs to \(S_0\), and from (3.2) its equation is

\[(4.6) \quad \eta_1 = \xi_1 x_1 + \xi_2 x^*_1, \quad \eta_2 = \xi_1 x_2 + \xi_2 x^*_2, \quad \zeta = 0,\]
which may be written as

\[(4.7) \quad (\eta_1, \eta_2) = (\xi_1, \xi_2) \begin{bmatrix} x_1 & x_2 \\ x_1^* & x_2^* \end{bmatrix} = (\xi_1, \xi_2) A_x, \quad \zeta = 0 \]

where \( A_x \) is the matrix belonging to \( C \), corresponding to the element \( x = x_1 + x_2 \cdot \mathbf{i} \) of \( R \). Denote by \((s_x, Y)\) the plane of \( S_4 \) passing through \( s_x \) and meeting the plane \( E_1' E_2' E_0' \) with equations \( \eta_1 = 0, \eta_2 = 0 \), in the point \( Y \) with coordinates \((0,0,-y_1,-y_2,1)\). Let the \( \alpha \Delta \)-point \((x_1, y_1), (x_2, y_2)\) be represented by the plane \((s_x, Y)\). This sets up a \((1,1)\) correspondence between \( \alpha \Delta \)-points, and planes of \( S_4 \) passing through the lines of \( S_0 \), and not contained in \( S_4' \). The equations of the plane \((s_x, Y)\) are

\[(4.8) \quad (\eta_1, \eta_2) = (\xi_1, \xi_2) \begin{bmatrix} x_1 & x_2 \\ x_1^* & x_2^* \end{bmatrix} - \zeta (y_1, y_2). \]

Let the \( \alpha \Delta \)-line \([(m_1, m_2), (b_1, b_2)]\) of type 1, be represented by the point \((m_1, m_2-b_1, -b_2, 1), \quad m_2 \neq 0, \quad \text{of} \quad S_4 \). This sets up a \((1,1)\) correspondence between \( \alpha \Delta \)-lines of type 1, and points of \( S_4 \) not contained in \( \Sigma \) or \( \Omega \).

Again let the \( \alpha \Delta \)-line \([(m_1, 0), (b_1, b_2)]\) of type 2a be represented by the plane

\[(4.9) \quad \eta_1 m_1 - \eta_2 = -\xi_1 b_1 + \zeta b_2, \quad \xi_2 = 0 \]

of \( S_4 \). This sets up a \((1,1)\) correspondence between \( \alpha \Delta \)-lines of type 2a, and planes of \( S_4 \) which are contained in \( \Omega \), and which do not pass through the point \( E_2' \).

Finally, let the \( \alpha \Delta \)-line \([c_1, c_2]\) of type 2b be represented by the plane

\[(4.10) \quad \eta_1 = \xi_1 c_1 - \zeta c_2, \quad \xi_2 = 0 \]

of \( S_4 \). This sets up a \((1,1)\) correspondence between \( \alpha \Delta \)-lines of type 2b, and
planes of $S_4$, which are contained in $\Omega$, and pass through the point $E'_2$ but do not contain the line $s_\infty$.

Thus the a$\Delta$-lines of type 2a and 2b taken together are represented by planes of $S_4$ contained in $\Omega$ but not containing the line $s_\infty$.

**Lemma (4.1).** An a$\Delta$-point $\{(x_1,y_1),(x_2,y_2)\}$ is incident with the a$\Delta$-line $\{(m_1,m_2),(b_1,b_2)\}$, $m_2 \neq 0$ of type 1, if and only if the plane of $S_4$, representing the a$\Delta$-point, passes through the point of $S_4$, representing the a$\Delta$-line.

The condition for the plane $(s_x,Y)$ of $S_4$ with equations (4.8), representing the a$\Delta$-point $\{(x_1,y_1),(x_2,y_2)\}$ to pass through the point $(m_1,m_2,-b_1,-b_2,1)$ of $S_4$ representing the a$\Delta$-line $\{(m_1,m_2),(b_1,b_2)\}$ is

$$-(b_1,b_2) = (m_1,m_2) \begin{bmatrix} x_1 & x_2 \\ x^*_1 & x^*_2 \end{bmatrix} - (y_1,y_2)$$

where $(x^*_1,x^*_2)$ is the uniquely determined matrix of $C$, with the first row $(x_1,x_2)$. This is precisely the same as the condition (2.4) for the a$\Delta$-point $\{(x_1,y_1),(x_2,y_2)\}$ to be incident with the a$\Delta$-line $\{(m_1,m_2),(b_1,b_2)\}$, $m_2 \neq 0$ of type 1.

**Lemma (4.2).** An a$\Delta$-point $\{(x_1,y_1),(x_2,y_2)\}$ is incident with the a$\Delta$-line $\{(m_1,0),(b_1,b_2)\}$ of type 2a if and only if the plane of $S_4$ representing the a$\Delta$-point intersects the plane of $S_4$ representing the a$\Delta$-line.

The a$\Delta$-point $\{(x_1,y_1),(x_2,y_2)\}$ is represented by the plane $(s_x,Y)$ with equations (4.8), and the a$\Delta$-line $\{(m_1,0),(b_1,b_2)\}$ is represented by the plane with equations (4.9). These planes intersect in a line if and only if there exist elements $r_1, r_2, s_1, s_2$ of $F$, not all zero, such that the result of adding the four equations after multiplying them successively from the right by $r_1, r_2, s_1, s_2$ is an identity. This gives

$$r_1 + m_1 s_1 = 0, \quad r_2 - s_1 = 0$$

(4.11)
\[(4.12)\quad x_1 r_1 + x_2 r_2 - b_1 s_1 = 0, \quad -y_1 r_1 - y_2 r_2 + b_2 s_1 = 0\]

\[(4.13)\quad x^* r_1 + x^* r_2 + s_2 = 0.\]

If the above conditions are satisfied, \(s_1 \neq 0\), otherwise \(r_1, r_2, s_1, s_2\) are all zero. Hence without loss of generality, we can take \(s_1 = 1\). This gives \(r_1 = -m_1, r_2 = 1, s_1 = 1\). Then the equations (4.12) reduce to (2.6). Conversely, if the conditions (2.6) are satisfied, we can choose \(r_1 = -m_1, r_2 = 1, s_1 = 1, s_2 = x^* m_1 - x^*\). Then (4.11), (4.12), (4.13) are satisfied. This proves the Lemma.

**Lemma (4.3).** An \(\alpha\Delta\)-point \(\{(x_1, y_1), (x_2, y_2)\}\) is incident with the \(\alpha\Delta\)-line \([c_1, c_2]\) of type 2b, if and only if the plane of \(S_4\) representing the \(\alpha\Delta\)-point, intersects the plane representing the \(\alpha\Delta\)-line in a line.

Proof is similar to Lemma (4.2).

The affine plane \(\alpha\Delta\) can be completed to the projective plane \(\Delta\). We shall now obtain linear representations for the elements at infinity in \(\Delta\), denoting them as in §2, paragraph 7.

Let the point at infinity \(\{m_1, m_2\}, m_2 \neq 0\), corresponding to the parallel class \((m_1, m_2)\) of type 1, be represented by the plane \(\xi_1 = \zeta m_1, \xi_2 = \zeta m_2\) of \(S_4\), where \(m_2 \neq 0\). This sets up a \((1,1)\) correspondence between the points at infinity in \(\Delta\) which correspond to parallel classes of type 1, and planes of \(S_4\), which pass through the line \(s_\infty\) and are not contained in \(E\) or \(\Omega\).

The point at infinity \(\{m_1, 0\}\) which corresponds to the parallel class \((m_1, 0)\) of type 2a, will be represented by the point \((0, 0, 1, m_1, 0)\) of \(S_4\). Again the point at infinity \(\{\infty\}\), which corresponds to the parallel class \(\{\infty\}\) consisting of all lines of type 2b, will be represented by the point \((0, 0, 0, 1, 0)\) of \(S_4\), viz. the point \(E'_2\). We thus have a \((1,1)\) correspondence between the points...
at infinity in \( \Delta \), which correspond to parallel classes of type 2a or 2b, and the points of the line \( s_\infty \) of \( S_4 \). Finally \(*\) the line at infinity in \( \Delta \) will be represented by the line \( s_\infty \) of \( S_4 \). A point at infinity in \( \Delta \), may also be called a \( \Delta \)-point at infinity, and the line at infinity in \( \Delta \), may be called the \( \Delta \)-line at infinity.

The following Lemmas are now easy to verify.

**Lemma (4,4).** A \( \Delta \)-point at infinity is incident with an \( \alpha \Delta \)-line if and only if the linear space of \( S_4 \) representing the \( \Delta \)-point at infinity is contained in or contains the linear space of \( S_4 \) representing the \( \alpha \Delta \) line, except in the case when both the \( \Delta \)-point at infinity and the \( \alpha \Delta \)-line are represented by planes in which case they are incident if and only if the planes representing them intersect in a line.

**Lemma (4,5).** The line \( s_\infty \) representing the \( \Delta \)-line at infinity either contains or is contained in the linear space of \( S_4 \) representing any \( \Delta \)-point at infinity.

Thus we have arrived at the same linear representation of a projective plane \( \Delta \), which was obtained by Bose and Bartolli in [3]. We thus have an alternative proof of the following theorem due to them.

**Theorem (4,1).** Let \( S_4 \) be a projective space based on a skew field \( F \). Let \( \Sigma \) and \( \Omega \) be distinct 3-spaces of \( S_4 \), intersecting in the plane \( \Pi \). Let \( S \) be a spread of lines in \( \Sigma \), which is also a dual spread. Let \( s_\infty \) be the line of \( S \) contained in \( \Pi \), and let \( S_0 \) be the set of lines of \( S \) not contained in \( \Pi \). Let us consider an incidence structure \( \Delta \) whose points and lines are represented by linear spaces of \( S_4 \) as follows:

\( \Delta \)-points are of the following types: (i) Type \( \emptyset \), represented by planes of \( S_4 \) passing through lines of \( S_0 \), and not contained in \( \Sigma \); (ii) Type \( I_1 \),
represented by planes of $S_4$, passing through $s_\infty$ and not contained in \(\Sigma\) or \(\Omega\); (iii) Type $I_2$, represented by points of $s_\infty$.

\(\Delta\)-lines are of the following types: (i) Type 1, represented by points of $S_4$, not contained in \(\Sigma\) or \(\Omega\); (ii) Type 2, represented by planes of $S_4$ contained in \(\Omega\), but not containing the line $s_\infty$; (iii) A single \(\Delta\)-line represented by the line $s_\infty$.

A \(\Delta\)-point and \(\Delta\)-line both of which are not represented by planes are incident if and only if the linear space of $S_4$ representing the \(\Delta\)-point either contains or is contained in the linear space representing the \(\Delta\)-line. A \(\Delta\)-point and a \(\Delta\)-line both of which are represented by planes are incident, if the planes representing them intersect in a line.

Note that the \(\Delta\)-points of type 0 are the \(\alpha\Delta\)-points. \(\Delta\)-points of type $I_1$ are the \(\Delta\)-points at infinity corresponding to parallel classes of type 1, and \(\Delta\)-points of type $I_2$ are \(\Delta\)-points at infinity corresponding to parallel classes of type 2a or 2b.

Bose and Barlotti [3] showed that the plane $\Delta$ is the derived plane in the sense of Ostrom [10,11], and Albert [1] of the dual translation plane $\delta T$ which can be linearly represented as follows: Points of $\delta T$ are of two types: (i) $\delta T$-points of type 1 are represented by planes of $S_4$ passing through the lines of $S$, and not contained in $\Sigma$; (ii) There is only a single $\delta T$ point of type 2 represented by $\Sigma$.

Lines of $\delta T$ are of two types: (i) $\delta T$-lines of type 1 are represented by points of $S_4$ not belonging to $\Sigma$; (ii) $\delta T$-lines of type 2 are represented by the lines of $S$. Incidence in $\delta T$ is given by the containing contained relation. The derivation set consists of all $\delta T$ points represented by planes passing through $s_\infty$ and contained in $\Omega$ but not in $\Sigma$, and the $\delta T$-point represented by $\Sigma$.  

§ 5.

1. In §2, we have obtained a projective plane $\Delta$, based on a Veblen-Wedderburn system $(R, +, \times)$ and in §4, we have obtained a linear representation of $\Delta$, in a projective space $S_4$. Any linear transformation $\sigma$ of $S_4$ which fixes $\Sigma, \Omega, s_\infty$ and $S_0$ ($\sigma$ may interchange lines of $S_0$) will give a collineation of $\Delta$, in terms of the representation in $S_4$.

We want to consider in particular a linear transformation of $S_4$ which gives a group of elations of $\Delta$, with axis represented by $s_\infty$ and centres represented by the points on $s_\infty$. Consider the group $G$ of linear transformations $\sigma(k_1, k_2)$ given by

\[(5.1) \quad \xi'_1 = \xi_1, \quad \xi'_2 = \xi_2, \quad \eta'_1 = \eta_1 - \zeta k_1, \quad \eta'_2 = \eta_2 - \zeta k_2, \quad \zeta' = \zeta.\]

The point $(u_1, u_2, v_1, v_2, w) \rightarrow (u_1, u_2, v_1 - \omega k_1, v_2 - \omega k_2, w)$ and the 3-space

\[\xi_1 a_1 + \xi_2 a_2 + \eta_1 b_1 + \eta_2 b_2 + \zeta c = 0\]

to

\[\xi_1 a_1 + \xi_2 a_2 + \eta_1 b_1 + \eta_2 b_2 + \zeta(k_1 b_1 + k_2 b_2 + c) = 0.\]

Thus $\sigma(k_1, k_2)$ fixes $\Sigma, \Omega, s_\infty$ and $S_0$ and transforms a linear space of $S_4$ representing an element of $\Delta$ into a linear space representing some element of $\Delta$.

The equations of $s_\infty$ are $\xi_1 = 0, \xi_2 = 0, \zeta = 0$. Hence $[\sigma]$, the line at infinity in $\Delta$ remains fixed. $\Delta$-points at infinity of type $I_1$ are represented by planes $\xi_1 = \zeta m_1, \xi_2 = \zeta m_2$, and $\Delta$ points at infinity of type $I_2$ are represented by points on $s_\infty$. They remain fixed, i.e., the $\Delta$-points $\{\omega_1, \omega_2\}$ and $\{\omega\}$ remain fixed. A $\Delta$-point of type $0$, i.e., an $a\Delta$-point represented by the plane,

\[(5.2) \quad (\eta_1, \eta_2) = (\xi_1, \xi_2) \begin{bmatrix} x_1 & x_2 \\ x_1 & x_2 \end{bmatrix} - \zeta(y_1, y_2)\]

is transformed by $\sigma(k_1, k_2)$ to the $\Delta$-point represented by the plane.
\( (n_1, n_2) = (\xi_1, \xi_2) \begin{bmatrix} x_1 & x_2 \\ x_1^* & x_2^* \end{bmatrix} - \zeta(y_1^{+k_1}, y_2^{+k_2}) \).

Hence the \( \Delta \)-point \( ((x_1, y_1), (x_2, y_2)) \) is transformed to \( ((x_1, y_1^{+k_1}), (x_2, y_2^{+k_2})) \). It is fixed only by the identity of \( G \).

The \( \Delta \)-line \( \{(m_1, m_2), (b_1, b_2)\}, m_2 \neq 0 \) of type 1, represented by the point \( (m_1, m_2, -b_1, -b_2, 1) \) of \( S_4 \) is transformed to the \( \Delta \)-line \( \{(m_1, m_2), (b_1^{+k_1}, b_2^{+k_2})\} \) represented by the point \( (m_1, m_2, -b_1^{+k_1}, -b_2^{+k_2}, 1) \). It is fixed only by the identity of \( G \).

The \( \Delta \)-line \( \{(m_1, 0), (b_1, b_2)\} \) of type 2a represented by the plane

\[
\eta_1 m_1 - \eta_2 = -\xi_1 b_1 + \xi b_2, \quad \xi_2 = 0
\]

is transformed by \( \sigma(k_1, k_2) \) to the \( \Delta \)-line \( \{(m_1, 0), (b_1, k_2^{+k_1} m_1 + b_2)\} \) represented by the plane \( \eta_1 m_1 - \eta_2 = -\xi_1 b_1 + \xi (k_2^{+k_1} m_1 + b_2), \xi_2 = 0 \). It is fixed by the subgroup \( G_{m_1} \) of \( G \) for which \( k_2 = k_1 m_1 \), where \( m_1 \) is fixed, but \( k_1 \) is arbitrary.

Now the \( \Delta \)-line \( \{(m_1, 0), (b_1, b_2)\} \) is incident with the \( \Delta \)-point at infinity \( \{m_1, 0\} \) which is represented by the point \( (0, 0, 1, m_1, 0) \) of \( S_\infty \), and conversely every \( \Delta \)-line incident with this \( \Delta \)-point at infinity is of the form \( \{(m_1, 0), (b_1, b_2)\} \) where \( m_1 \) is a fixed and \( b_1, b_2 \) are arbitrary elements of \( F \). Thus \( G_{m_1} \) is a subgroup of elations for which the line at infinity in \( \Delta \) is the axis and the \( \Delta \)-point at infinity \( \{m_1, 0\} \) is the center. Corresponding to each \( m_1 \) belonging to \( F \), we get a subgroup \( G_{m_1} \).

The \( \Delta \)-line \( \{c_1, c_2\} \) of type 2b represented by the plane

\[
\eta_1 = \xi_1 c_1 - \xi c_2, \quad \xi_2 = 0
\]

is transformed by \( \sigma(k_1, k_2) \) to the line \( \{c_1, c_2^{+k_1}\} \), represented by the plane \( \eta_1 = \xi_1 c_1 - \xi (k_1^{+k_2} c_2), \xi_2 = 0 \). It is fixed by the subgroup \( G_\infty \) of \( G \) for which \( k_1 = 0 \), but \( k_2 \) is arbitrary. Now the \( \Delta \)-line \( \{c_1, c_2\} \) is incident with the
\( \Delta \)-point \( \{ \infty \} \), represented by the point \((0,0,0,1,0)\) of \( s_\infty \), and conversely every \( \Delta \)-line incident with \( \{ \infty \} \) is a line \([c_1, c_2]\) of type \(2b\) for some \(c_1, c_2\) belonging to \( F\). Hence the subgroup under consideration is a subgroup of elations for which the line at infinity in \( \Delta \) is the axis and the \( \Delta \)-point \( \{ \infty \} \) is the center.

Any two subgroups have only the identity in common. Each point of \( S_4 \) on the line \( s_\infty \) represents a \( \Delta \)-point at infinity, which is the center for a subgroup of elations, for which the axis is the line at infinity in \( \Delta \), represented by \( s_\infty \). When \( F \) is the field \( \text{GF}(q) \), \( \Delta \) is a finite plane. The group \( G \) is of order \( q^2 \), and there are \( q+1 \) subgroups each of order \( q \). Hence \( \Delta \) is a semitranslation plane with respect to the line at infinity in \( \Delta \) in the sense of Ostrom [9]. Of course, the situation is completely analogous in the infinite case.

2. We shall now obtain the ternary ring of the projective plane \( \Delta \). All points and lines considered in this paragraph belong to \( \Delta \). Hence we shall speak only of points and lines instead of \( \Delta \)-points and \( \Delta \)-lines. Let us choose the line \([0,0,0,0]\) of type \(2a\) to be the \( x \)-axis and the line \([0,0]\) of type \(2b\) as the \( y \)-axis. From (2.6) and (2.7), the point \((0,0,0,0)\) is incident both with the \( x \)-axis and \( y \)-axis. This will be chosen to be the origin. The \( x \)-axis belongs to the parallel class \((0,0)\) of type \(2a\). Lines parallel to the \( x \)-axis have coordinates \([0,0),(b_1,b_2)\]). The \( y \)-axis belongs to the parallel class \( \{ \infty \} \) consisting of all lines of type \(2b\). Hence lines parallel to the \( y \)-axis have coordinates \([c_1,c_2]\). Let us choose the line \([1,0,0,0]\) of type \(2a\) to be the unit line. From (2.6) it is incident with the origin. If the point \([(x_1,y_1), (x_2,y_2)]\) is an incident with the unit line, then from (2.6), \( x_1 = x_2, y_1 = y_2 \). Hence a general point incident with the unit line has coordinates \([(x_1,y_1), (x_1,y_1)]\). We may associate to this point the element \((x_1,y_1)\) of \( R \). Thus there is a \((1,1)\) correspondence between the elements of \( R \) and the points on the unit
line, the element \((x, y)\) of \(R\) corresponding to the point with coordinates 
\[
(x, y), (x, y)
\]

3. **Lemma (5.1).** Given two points \(((x_1, y_1), (x_1, y_1))\) and \(((x_2, y_2), (x_2, y_2))\) incident with the unit line corresponding to the elements \((x_1, y_1), (x_2, y_2)\) of \(R\) the line through the first point parallel to the y-axis, meets the line through the second point parallel to the x-axis in the point whose coordinates are \(((x_1, y_1), (x_2, y_2))\)

The line parallel to the y-axis and incident with \(((x_1, y_1), (x_2, y_2))\) is from (2.7), \([x_1, y_1]\). Again the line parallel to the x-axis and incident with \(((x_2, y_2), (x_2, y_2))\) is from (2.6), \([(0, 0), (x_2, y_2)]\). Then from (2.6) and (2.7), we see that the point incident with both \([x_1, y_1]\) and \([(0, 0), (x_2, y_2)]\) is \(((x_1, y_1), (x_2, y_2))\). This proves the Lemma.

Thus we may appropriately call \((x_1, y_1)\) the x-coordinate and \((x_2, y_2)\) the y-coordinate of the point \(((x_1, y_1), (x_2, y_2))\). (See Albert [1].)

4. However, \((m_1, m_2)\) and \((b_1, b_2)\) are not the slope and the y-intercept of the line \([m_1, m_2], (b_1, b_2)]\) in the standard sense. We will call \((m_1, m_2)\) the m-coordinate, and \((b_1, b_2)\) the b-coordinate of the line \([m_1, m_2], (b_1, b_2)]\).

Since \((1, 0)\) is the unity of \(R\), we may take the point \(((1, 0), (1, 0))\) on the unit line, to be the unit point. The line \([1, 0]\) of type 2b, parallel to the y-axis through the unit point will be called the slope line. Let \([m_1, m_2], (b_1, b_2)]\) be any line of type 1 or 2a. Then the line parallel to this and incident with the origin is \([m_1, m_2], (0, 0)]\). If it meets the slope line in the point \((1, 0), (u_2, v_2)\), then \((u_2, v_2)\) is defined to be the slope of the line \([m_1, m_2], (b_1, b_2)]\). We have to consider two separate cases.

Case I. Let the line \([m_1, m_2], (b_1, b_2)]\) be of type 1, i.e., \(m_2 \neq 0\). If its slope is \((u_2, v_2)\), then the point \(((1, 0), (u_2, v_2))\) of the slope line is incident with the line \([m_1, m_2], (0, 0)]\). From (2.4) the condition for this is
\begin{equation}
(0, \nu_2) = (m_1, m_2) \begin{bmatrix}
1 \\
u_1^* \\
\nu_2^*
\end{bmatrix},
\end{equation}

where the matrix appearing on the right hand side of (5.6) belongs to \(C\), i.e.,
\((\nu_1^*, \nu_2^*) = (0, 1) \times (1, \nu_2)\) or in terms of the functions \(F_1, F_2, G_1, G_2\) defined by (1.14) and (1.15)

\begin{equation}
\nu_1^* = F_1(1, \nu_2), \quad \nu_2^* = F_2(1, \nu_2).
\end{equation}

From Lemma (1.5), \(\nu_1, \nu_2\) are uniquely determined. From (1.19) and (1.20),

\begin{equation}
\nu_2 = G_1(1, -m_2^{-1} m_1), \quad \nu_2 = m_1 G_2(1, -m_2^{-1} m_1) + m_2 G_2(1, -m_2^{-1} m_1).
\end{equation}

Also \(\nu_2 \neq 0\). If \(\nu_2 = 0\), then from Lemma (1.3), part (d), the matrix on the right in (5.6) would be the null matrix which is a contradiction.

Conversely, given \((\nu_2, \nu_2)\), from Lemma (1.3) part (e), \((m_1, m_2)\) is uniquely determined, and since \(m_2 \neq 0\), it follows that \(\nu_2 \neq 0\). We can explicitly express \(m_1, m_2\) as

\begin{equation}
(m_1, m_2) = \nu_2 d^{-1}(\nu_1^*, -1)
\end{equation}

where \(\nu_1^*, \nu_2^*\) are given by (5.7) and \(d = \nu_1^* \nu_2^* - \nu_2^*\).

Note that \(d \neq 0\), otherwise the column vectors of the matrix on the right in (5.6), would be dependent from the right. From Lemma (1.1), the matrix would then be singular. This would contradict Lemma (1.3) part (c), since the matrix is non-null. Thus there is a \((1, 1)\) correspondence between \(m\)-coordinates \((m_1, m_2)\), \(m_2 \neq 0\) characterizing parallel classes of type 1, and the corresponding slopes \((\nu_2, \nu_2), \nu_2 \neq 0\).

Case II. Consider the line \([(m_1, 0), (b_1, b_2)]\) of type 2a. If its slope is \((\nu_2, \nu_2)\), then the point \((1, 0), (\nu_2, \nu_2)\) of the slope line is incident with the line \([(m_1, 0), (0, 0)]\). Hence from (2.6), \(\nu_2 = m_1, \nu_2 = 0\). Hence the slope of the line \([(m_1, 0), (b_1, b_2)]\) of type 2a is \((m_1, 0)\).
The slope of any line \([c_1,c_2]\) of type 2b may be defined to be \(\infty\).

5. Given any two elements \((x_1,y_1), (\nu_2,\nu_2)\) of \(R\) we can define a new multiplication \(\odot\) by setting

\[
(5.10) \quad (x_2, y_2) = (x_1, y_1) \odot (\nu_2, \nu_2),
\]

if \((x_2, y_2)\) is the \(y\)-coordinate of the uniquely determined point with \(x\)-coordinate \((x_1, y_1)\) and incident with the line through the origin whose slope is \((\nu_2, \nu_2)\). Then this multiplication is a loop with unity \((1,0)\).

We can express \((x_1, y_1) \odot (\nu_2, \nu_2)\) in terms of the operations \(\times\) and \(+\) in \((R, \times, +)\) as follows: Let \([[(m_1, m_2), (0,0)]\) be the line through the origin with slope \((\nu_2, \nu_2)\). Then \(\nu_2 = 0, (m_1, m_2) = (\nu_2, 0)\), from the Case II of the previous paragraph. If \(\nu_2 \neq 0, (m_1, m_2)\) is given by \((5.9)\). Now the point \(((x_1, y_1), (x_2, y_2))\) is incident with \([[(m_1, m_2), (0,0)]]\). Hence from \((2.4)\), and formulae \((1.19)\) \((1.120)\) of Lemma \((1.5)\), if \(\nu_2 \neq 0,

\[
(5.11) \quad (x_1, y_1) \odot (\nu_2, \nu_2) = (x_2, y_2) = [G_1(x_1, k), m_1G_1(x_1, k) + m_2G_2(x_2, k)],
\]

where

\[
(5.12) \quad k = m_2^{-1}(y_1 - m_1x_1),
\]

and the functions \(G_1, G_2\) are defined by \((1.15)\). Substituting for \(m_1, m_2\) in terms of \(\nu_2, \nu_2\) from \((5.9)\), we have an explicit expression for \((x_1, y_1) \odot (\nu_2, \nu_2)\) when \(\nu_2 \neq 0\).

If \(\nu_2 = 0\), then \((m_1, m_2) = (\nu_2, 0)\). Then from \((2.6)\),

\[
(5.13) \quad (x_1, y_1) \odot (\nu_2, \nu_2) = (x_1, y_1) \nu_2.
\]

Note that from Lemma \((2.4)\) and \((2.5)\) the line through the origin with slope \((\nu_2, \nu_2)\) is of type 1 or 2a according as \(\nu_2 \neq 0\) or \(\nu_2 = 0\). Hence if we wish to
determine \((u_2, v_2)\) given \((x_1, y_1), (x_2, y_2)\), the formula (5.11) or (5.13) is valid according as the matrix
\[
\begin{bmatrix}
x_1 & x_2 \\
y_1 & y_2
\end{bmatrix}
\]
is non-singular or singular.

6. The equation of any line of \(\Delta\), passing through the origin and having the slope \((u_2, v_2)\) is given by (5.8).

Let \([m_1, m_2], (b_1, b_2)\) be any line of type 1. From (2.4), the necessary and sufficient condition for the point \(\{(x_1, y_1), (x_2, y_2)\}\) to be incident with it is

\[(5.14) \quad (y_1 - b_1, y_2 - b_2) = (m_1, m_2) \begin{bmatrix} x_1 & x_2 \\ x_1' & x_2' \end{bmatrix},\]

where \((x_1', x_2') = (0, 1)(x_1, x_2)\). It follows that the point \(\{(x_1, y_1 - b_1), (x_2, y_2 - b_2)\}\) is incident with the line \([m_1, m_2], (0, 0)\] through the origin. If \((u_2, v_2)\) is the slope of the line, we have

\[(5.15) \quad (x_2, y_2 - b_2) = (x_1, y_1 - b_1) \oplus (u_2, v_2).\]

This is the equation of a line of type 1, with slope \((u_2, v_2)\) and \(b\)-coordinate \((b_1, b_2)\).

Similarly, the equation of a line of type 2a with slope \((u_2, 0)\) and \(b\)-coordinate \((b_1, b_2)\) is

\[(5.16) \quad (x_2, y_2) = (x_1, y_1)u_1 + (b_1, b_2).\]

7. Finally we note that the \(b\)-coordinate of a line is not the \(y\)-intercept of a line in the standard sense. We may define the \(y\)-intercept of a line of type 1 or type 2a, to be the \(y\)-coordinate of the point in which the line intersects the \(y\)-axis.
Let \((\beta_2, \gamma_2)\) be the \(y\)-intercept of the line \([((m_1, m_2), (b_1, b_2))].\) We have to consider two cases.

Case I. The line \([((m_1, m_2), (b_1, b_2))\] is of type 1. Hence \(m_2 \neq 0.\) The point \(((0, 0), (\beta_2, \gamma_2))\) is incident with it. Hence from (2.4),

\[
(5.17) \quad (-b_1, \gamma_2 - b_2) = (m_1, m_2) \begin{bmatrix} 0 & \beta_2 \\ \beta_1^* & \beta_2^* \end{bmatrix},
\]

where \((\beta_1^*, \beta_2^*) = (0, 1) \times (0, \beta_2)\) or alternatively

\[
\beta_1^* = F_1(0, \beta_2), \quad \beta_2^* = F_2(0, \beta_2),
\]

where the functions \(F_1\) and \(F_2\) are given by (1.14).

From Lemma (1.5), \((\beta_2, \gamma_2)\) is uniquely determined by \((b_1, b_2)\) since \((m_1, m_2)\) is given. Conversely given \((\beta_2, \gamma_2)\), (5.17) determines \((b_1, b_2)\).

Case II. The line \([((m_1, 0), (b_1, b_2))\] is of type 2a. Since \(((0, 0), (\beta_2, \gamma_2))\) is incident with it, we have from (2.6), \((\beta_2, \gamma_2) = (b_1, b_2)\). Thus in this case, the \(y\)-intercept is \((b_1, b_2)\).

We are now in a position to write down the ternary ring of \(\Delta.\) Let

\[
(5.18) \quad \phi[(x_1, y_1), (\mu_2, \nu_2), (\beta_2, \gamma_2)] = (x_1, y_1 - b_1) \otimes (\mu_2, \nu_2) + b_2 \quad \text{if} \quad \nu_2 \neq 0
\]

\[
= (x_1, y_1)u_1 + (\beta_2, \gamma_2) \quad \text{if} \quad \nu_2 = 0,
\]

where in the case \(\nu_2 \neq 0,\) we determine \((b_1, b_2)\) from (5.15) and (5.7).

Then (5.18) is the ternary ring of \(\Delta.\) The equations of lines of type 1 or type 2a are given by

\[
(5.19) \quad (x_2, y_2) = \phi[(x_1, y_1), (\mu_2, \nu_2), (\beta_2, \gamma_2)],
\]

where \((x_1, y_1), (x_2, y_2)\) are the \(x\) and \(y\) coordinates of a point incident with a line which has slope \((\mu_2, \nu_2)\) and \(y\)-intercept \((\beta_2, \gamma_2)\).
The lines of type 2b, parallel to the y-axis have equations

\[(5.20) \quad (x_1, y_1) = (c_1, c_2).\]

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