COMBINATORIAL MATHEMATICS YEAR

February 1969 - June 1970

A GEOMETRIC CHARACTERIZATION OF THE LINE GRAPH OF A SYMMETRIC BALANCED INCOMPLETE BLOCK DESIGN

by

Martin Aigner　　　　Thomas A. Dowling

Department of Statistics
University of North Carolina
Chapel Hill, N.C.

Institute of Statistics Mimeo Series No. 600.5

FEBRUARY 1969

Research partially sponsored by the United States Air Force under AFOSR Grant No. 68-1406, under AFOSR Grant No. 68-1415; and the National Science Foundation, under Grant No. 68-8624.

A GEOMETRIC CHARACTERIZATION OF THE LINE GRAPH OF

A SYMMETRIC BALANCED INCOMPLETE BLOCK DESIGN

by

Martin Aigner Thomas A. Dowling

I. INTRODUCTION

A symmetric balanced incomplete block (SBIB) design $\pi(v,k,\lambda)$ is an arrangement of v objects, called treatments, into v sets, called blocks, such that each block consists of k distinct treatments, each treatment appears in k blocks, and each pair of distinct treatments appears in λ blocks. The parameters satisfy $\lambda < k$ and $\lambda(v-1) = k(k-1)$. The graph of $\pi(v,k,\lambda)$ is defined as the bipartite graph $H(\pi)$ whose vertices are the 2v treatments and blocks of π, with two vertices adjacent if and only if one is a block and the other is a treatment contained in the block. The line graph $G(\pi)$ of $\pi(v,k,\lambda)$ is the line graph of $H(\pi)$, i.e. the graph whose vertices are the edges of $H(\pi)$, with two vertices in $G(\pi)$ adjacent if and only if the corresponding edges in $H(\pi)$ have a common end point.

Let $\{G(\pi)\}$ denote the set consisting of the line graphs of all SBIB designs with parameters v,k,λ. In a recent paper, Hoffman and Ray-Chaudhuri [4] proved that $G \in \{G(\pi)\}$ if and only if G is a regular connected graph on vk vertices and the distinct

Research partially sponsored by the United States Air Force under AFOSR Grant No. 68-1406, under AFOSR Grant No. 68-1415; and the National Science Foundation, under Grant No. 68-8624.

eigenvalues of the adjacency matrix of G are -2, $2k-2$, and $k-2\pm(k-\lambda)^{1/2}$, unless $v = 4$, $k = 3$, $\lambda = 2$, when the sufficiency of these conditions fails due to the existence of a single exceptional graph. We give here a characterization of $\{G(\pi)\}$ in terms of several geometric properties of these graphs, with again one exceptional case (Figure 1) for $v = 7$, $k = 4$, $\lambda = 2$. This result generalizes an earlier characterization [2] of the line graph of a finite projective plane $(\lambda=1)$, but the conditions for $\lambda=1$ given here are slighly different from those of [2]. (See Remarks below.)

II. DEFINITIONS

By a <u>graph</u> G we mean a finite undirected graph without loops or multiple edges. $V(G)$, $E(G)$ denote, respectively, the <u>vertex set</u> and the <u>edge set</u> of G. We denote by $d(x,y)$ the <u>distance</u> between vertices x and y and define further

$$D_i(x) = \{y \in V(G) : d(x,y) = i\}, \quad i = 0,1,\ldots .$$

The <u>degree</u> of $x \in V(G)$ is written <u>deg</u> x $(= |D_1(x)|)$. $\Delta(x,y) = |D_1(x) \cap D_1(y)|$ is the number of vertices adjacent to both x and y. G is <u>regular</u> if <u>deg</u> x is constant for all $x \in V(G)$, and <u>edge-regular</u> if, further, $\Delta(x,y)$ is constant for all $(x,y) \in E(G)$. A <u>clique</u> K is a set of vertices, any two of which are adjacent.

III. THE THEOREM

THEOREM. Let v,k,λ be positive integers with $\lambda < k$ and $\lambda(v-1) = k(k-1)$, and let $\{G(\pi)\}$ be the set of line graphs of all symmetric balanced incomplete block designs with parameters v,k,λ. If $G \in \{G(\pi)\}$, then G is connected and has the following properties:

(P1) $|V(G)| = vk$,

(P2) $\underline{\deg}\ x = 2k - 2$ for all $x \in V(G)$,

(P3) $\Delta(x,y) = k - 2$ if $(x,y) \in E(G)$,

(P4) $\Delta(x,y) \leqq 2$ if $(x,y) \notin E(G)$,

(P5) $|D_3(x) \cap D_1(y)| = k - \lambda$ if $d(x,y) = 2$, $\Delta(x,y) = 1$,

(P6) $d(x,y) \leqq 3$ for all $x,y \in V(G)$.

Conversely, if G is a connected graph satisfying (P1) - (P6), then $G \in \{G(\pi)\}$ or else $v = 7$, $k = 4$, $\lambda = 2$, and G is the graph shown in Figure 1.

Remarks. Properties (P1) - (P5) are the counterparts of the characterizing properties of the line graph of a finite projective plane $(\lambda = 1)$ given in [2], except that the analogue of (P4) in that paper is $\Delta(x,y) \leqq 1$ if $(x,y) \notin E(G)$. The increased upper bound for $\Delta(x,y)$ required when $\lambda > 1$ necessitates the addition of (P6) here. That neither (P5) nor (P6) is redundant in all cases is demonstrated by the example given in [2] and the graph of Figure 2.

On comparing our characterization with that of Hoffman and Ray-Chaudhuri [4], we meet with the intriguing problem of the relationship of the eigenvalues of the graph to its geometric properties. Apart from the fact that in the case of regular graphs the dominant eigenvalue is the degree of regularity, very little is known. An important tool is the polynomial of a graph [3], which for regular graphs gives an upper bound to the diameter. For example, the eigenvalues of $G(\pi)$ other than the degree are -2, $k-2\pm(k-\lambda)^{1/2}$, and the fact that there are just three immediately yields property (P6) of the theorem. Using certain "impossible subgraphs" of [4], the fact that -2 is the minimal eigenvalue is easily shown to imply (P4) except for one possible configuration. Hoffman and Ray-Chaudhuri directly prove edge-regularity (P3) for $k>4$. The exact nature of the relationship between eigenvalues $k-2\pm(k-\lambda)^{1/2}$ and condition (P5) is unknown.

IV. PROOF OF THE THEOREM

The necessity of (P1) - (P6) when $G \in \{G(\lambda)\}$ is easily verified and the proof is omitted here. We therefore assume that G is a connected graph and satisfies (P1) - (P6).

By (P3) it is evident that $|K| \leqq k$ for any clique K in G. Since we shall only consider cliques K in G such that $|K| = k$, let us agree to use the term "clique" in this restricted sense. We then have

<u>LEMMA 1.</u> Each vertex of G is contained in exactly two cliques, and each edge of G is contained in exactly one clique.

Proof. If $k > 4$, the lemma follows from (P2) - (P4) by a theorem of Bose and Laskar [1] on edge-regular graphs. The cases $k = 2, 3, 4$ must be examined separately, and it is easily verified that (P2) - (P4) imply the lemma except for $k = 4$, when these three conditions alone do not rule out the possibility that a vertex $x \in V(G)$ exists such that the subgraph generated by $D_1(x)$ is a 6-cycle. If this is the case, connectivity implies that the subgraph generated by $D_1(y)$ is a 6-cycle for all $y \in V(G)$. We can then show with little difficulty that no such graph satisfies all the conditions (P1) - (P6). (We do encounter one graph that satisfies (P1) - (P6) if $\lambda = 4$ ($= k$). This graph is the exceptional case in Shrikhande's characterization [5] of the L_2 association scheme (square lattice graph). To avoid some special arguments in the proof of the theorem, we have assumed $\lambda < k$. However, the theorem remains valid when $\lambda = k = v$ provided we include this additional exception.) As the proof involves a case by case investigation and bears little connection to the main argument, it has been omitted.

COROLLARY 1.1. The number of cliques in G is $2v$.

Let $C(G)$ denote the set of cliques in G, and $I(G)$ the set of unordered pairs (K,L) of distinct cliques such that $K \cap L \neq \emptyset$. Then by Lemma 1, $(K,L) \in I(G)$ if and only if $|K \cap L| = 1$. We denote the single vertex $x \in K \cap L$ by $x = \langle K,L \rangle$.

A <u>clique chain</u> $\mathcal{C} = (K_0, K_1, \ldots, K_m)$ is a sequence of distinct cliques such that $(K_i, K_{i+1}) \in I(G)$ for $i=0,1,\ldots m-1$. The <u>length</u> of $\mathcal{C}$ is the number m of vertices $x_i = \langle K_i, K_{i+1} \rangle$. If $K_0 = K_m$ and no other clique appears twice, $\mathcal{C}$ is a <u>clique cycle</u>.

<u>LEMMA 2.</u> G contains no clique cycle of length three.

<u>Proof.</u> Suppose $\mathcal{C} = (K_0, K_1, K_2, K_0)$ is a clique cycle in G. Let $x_i = \langle K_i, K_{i+1} \rangle$, (subscripts modulo 3). Then $x_0 \in D_1(x_1) \cap D_1(x_2)$ and $x_0 \notin K_2$, which implies $\Delta(x_1, x_2) \geqq k-1$, contradicting (P3).

For x V(G), we define

$$D_2^{(j)}(x) = \{y : d(x,y) = 2, \ \Delta(x,y) = j\}$$

for $j = 1,2$. Then by (P4), the sets $D_2^{(1)}(x)$, $D_2^{(2)}(x)$ partition $D_2(x)$.

<u>LEMMA 3.</u> Let $x = \langle K_0, K_1 \rangle$, $y = \langle L_0, L_1 \rangle$ be two vertices of G. Then by properly labeling the four cliques K_i, L_j, $i,j = 0,1$, we have

(i) $y \in D_0(x)$ iff $K_0 = L_0$, $K_1 = L_1$.

(ii) $y \in D_1(x)$ iff $K_0 = L_0$, $K_1 \neq L_1$.

(iii) $y \in D_2^{(2)}(x)$ iff $(K_0, L_0) \in I(G)$, $(K_1, L_1) \in I(G)$.

(iv) $y \in D_2^{(1)}(x)$ iff $(K_0, L_0) \in I(G)$, $(K_1, L_1) \notin I(G)$.

<u>Proof.</u> Follows immediately from Lemmas 1 and 2.

<u>LEMMA 4.</u> If $x = \langle K_0, K_1 \rangle$, $y = \langle L_0, L_1 \rangle$, where $(K_0, L_0) \in I(G)$, $(K_1, L_1) \notin I(G)$, then $y \in D_2^{(1)}(x)$ and there are

exactly λ cliques, including L_0, which intersect L_1 and one of K_0, K_1.

Proof. Clearly $y \in D_2^{(1)}(x)$ by Lemma 3 - (iv). By (P5) there are $k-2$ vertices in $D_3(x) \cap D_1(y)$, and these must be in L_1. Hence the remaining λ vertices of L_1 are in $D_2^{(1)}(x)$. If $z = \langle L_1, K \rangle$ is one of these, then either $(K, K_0) \in I(G)$ or $(K, K_1) \in I(G)$.

LEMMA 5. G contains no clique cycle of length five.

Proof. Suppose $\mathcal{C} = (K_0, K_1, K_2, K_3, K_4, K_0)$ is a clique cycle in G . Let $x_i = \langle K_i, K_{i+1} \rangle$ (subscripts modulo 5) and define α_i as the number of cliques intersecting both K_{i-1} and K_{i+1}, including K_i. Then $x_i \in D_2^{(1)}(x_{i+2})$, so by the preceeding lemma, $\alpha_{i-1} + \alpha_{i+1} = \lambda$. Since $(i+1) - (i-1) = 2$ is prime to 5 , we infer that all $\alpha_i = \alpha$, say, and $\lambda = 2\alpha$.

If $y \in D_2^{(1)}(x_0)$, $y = \langle K, L \rangle$, where K intersects neither K_0 nor K_1 , and L intersects exactly one of K_0, K_1. Then $|K \cap D_2^{(1)}(x_0)| = \lambda = 2\alpha$. Since $2k$ cliques intersect K_0 or K_1 (including K_0, K_1) and G contains $2v$ cliques in all,

$$2k + (2\alpha)^{-1} |D_2^{(1)}(x_0)| \leqq 2v. \tag{1}$$

To show (1) is impossible, let L_0 be one of the $k-\alpha$ cliques intersecting K_0, but not K_2. Then if $y_0 = \langle K_0, L_0 \rangle$, we find $x_1 \in D_2^{(1)}(y_0)$, and so by Lemma 4, there are 2α cliques meeting K_2 which meet K_0 or L_0. But there are exactly $\alpha_1 = \alpha$ such cliques meeting K_2 and K_0, so also there are α cliques meeting K_2 and L_0. Then by the same argument, there must be α cliques meeting L_0 and K_1. It follows that L_0 contains exactly $k-\alpha$ vertices of $D_2^{(1)}(x_0)$. We can repeat the argument for any of the $k-\alpha$ cliques L_1 meeting K_1, but not K_4, obtaining finally

$$|D_2^{(1)}(x_0)| \geqq 2(k-\alpha)^2,$$

which is easily seen to contradict (1).

COROLLARY 5.1. If K_0, K_1 are two disjoint cliques in G, and L is a clique intersecting both K_0 and K_1, then the number of cliques intersecting both K_0 and K_1, including L, is either λ or k.

Proof. Let $x_i = \langle K_i, L \rangle$, $i=0,1$. If there are fewer than k cliques meeting K_0 and K_i, then there exists $y_1 \in K_1$ such that $x_0 \in D_2^{(1)}(y_1)$. Now if $y_2 = \langle K_1, L_1 \rangle$, there are λ cliques meeting K_0 which meet one of K_1, L_1. But if such a clique were to meet K_0 and L_1, G would contain a clique cycle of length five.

Consider now a graph H, the clique graph of G, defined by $V(H) = C(G)$, $E(H) = I(G)$. The mapping $\phi : (K,L) \to \langle K,L \rangle$ is a one-to-one function of $E(H)$ onto $V(G)$ such that ϕ maps adjacent edges of H into adjacent vertices of G and ϕ^{-1} maps adjacent vertices of G into adjacent edges of H. It follows that G is the line graph of H. Thus $G \in \{G(\pi)\}$ if and only if $H \in \{H(\pi)\}$. We shall restrict our attention henceforth to H . We first summarize some properties of H in

<u>LEMMA 6.</u> H is connected and has the following properties:

(Q1) $|V(H)| = 2v$,

(Q2) $\deg K = k$ for all $K \in V(H)$,

(Q3) $\Delta(K,L) = \lambda$ or k if $d(K,L) = 2$,

(Q4) H contains no cycles of length three or five.

<u>Proof.</u> (Q1) is Corollary 1.1, (Q2) follows from Lemma 1, (Q3) is Corollary 5.1, and (Q4) follows from Lemmas 2 and 5, since a clique cycle in G corresponds to a cycle in H .

<u>LEMMA 7.</u> If $\Delta(K,L) < k$ for all $K,L \in V(H)$ such that $d(K,L) = 2$, then $H \in \{H(\pi)\}$.

<u>Proof.</u> In view of (Q3) we have $d(K,L) = \lambda$ for all K,L such that $d(K,L) = 2$. Let $K \in V(H)$ and define $V_1 = \{K\} \cup D_2(K)$. Then (Q2) - (Q4) imply

$$|V_1| = 1 + k(k-1)/\lambda = v.$$

If two vertices of V_1 are adjacent, then H contains a 5-cycle. Thus V_1 is an independent set, and so the vk edges incident with vertices of V_2 are all distinct. Since these are all the edges of H by (Q1) and (Q2) , the complementary set $V_2 = V(H) - V_1$ is also independent. Thus V is bipartite with vertex sets V_1, V_2.

Finally let n be the number of unordered pairs K_0, K_1 in V_1 such that $\Delta(K_0,K_1) = \lambda$. Then $n\lambda = vk(k-1)/2$, since each of the v vertices in V_2 is in $D_1(K_0) \cap D_1(K_1)$ for exactly $k(k-1)/2$ such pairs K_0,K_1. Thus $n = v(v-1)/2$, i.e. $\Delta(K_0,K_1) = \lambda$ for all $K_0,K_1 \in V_1, K_0 \neq K_1$. If we identify the vertices of V_1 with treatments and those of V_2 with blocks, and define the treatment to be contained in a block if and only if the corresponding vertices are adjacent in H, then it is clear that H is the bipartite graph of an SBIB design $\pi(v,k,\lambda)$, i.e. $H \in \{H(\pi)\}$. [1)]

We consider now the case where there exists two vertices, K,L in H such that $d(K,L) = 2$, $\Delta(K,L) = k$. Let us define two vertices K_0,K_1 to be <u>equivalent</u> and write $K_0 \equiv K_1$, if $K_0=K_1$ or $d(K_0,K_1) = 2$, $\Delta(K_0,K_1) = k$. By (Q2) we then have $K_0 \equiv K_1$ if and only if $D_1(K_0) = D_1(K_1)$. Let $\overline{K}$ denote the equivalence class containing $K \in V(H)$ ($\equiv$ is readily seen to be, in fact, an equivalence relation.)

<u>LEMMA 8.</u> Any two equivalence classes in $V(H)$ contain the same number $t \geqq 2$ of vertices.

<u>Proof.</u> Let $K_0,K_1 \in V(H)$, $t_i = |\overline{K}_i|$, i=0,1. Suppose first that $(K_0,K_1) \in E(H)$. If we can show that in this case $t_0 = t_1 = t$, then the lemma will follow by the connectedness of H. Consider then the number n of edges $(L_0,L_1) \in E(H)$ such that $(K_0,L_0) \in E(H)$, $(K_1,L_1) \in E(H)$, $L_0 \neq K_1$, $L_1 \neq K_0$. If $L_1 \in \overline{K}_0 - \{K_0\}$, L_0 can be chosen in $k-1$ ways, while if $L_1 \notin \overline{K}_0 - \{K_0\}$, L_0 can be chosen in $\lambda-1$ ways. Hence

1) It is interesting to note that so far we have not made use of (P6).

$n = (t_0-1)(k-1) + (k-t_0)(\lambda-1)$. Similarly if we first fix L_0 and choose L_1, we obtain $n = (t_1-1)(k-1) + (k-t_1)(\lambda-1)$. Since $k-2 > 0$, this implies $t_0 = t_1$, and the proof is complete. (Note $t \geqq 2$ by hypothesis.)

Suppose $(K,L) \in E(H)$. Then $\overline{K} \neq \overline{L}$, and if $K_1 \in \overline{K}$, $L_1 \in \overline{L}$, then $(K,L_1) \in E(H)$ and therefore $(K_1,L_1) \in E(H)$. Hence the subgraph of H on the vertices of $\overline{K} \cup \overline{L}$ is the complete bipartite graph on $t + t$ vertices. The equivalence relation $\equiv$ defined on $V(H)$ induces a homomorphism $H \to \overline{H}$, where $\overline{H}$ is the graph defined by

$$V(\overline{H}) = \{\overline{K} : K \in V(H)\},$$
$$E(\overline{H}) = \{(\overline{K},\overline{L}) : (K,L) \in E(H)\}.$$

<u>LEMMA 9.</u> $\overline{H}$ is connected and has the following properties, where $\overline{v} = v/t$, $\overline{k} = k/t$, $\overline{\lambda} = \lambda/t$:

(Q1) $|V(\overline{H})| = 2\overline{v}$,

(Q2) $\deg \overline{K} = \overline{k}$ for all $\overline{K} \in V(H)$,

(Q3) $\Delta(\overline{K},\overline{L}) = \overline{\lambda}$ if $d(\overline{K},\overline{L}) = 2$,

(Q4) $\overline{H}$ contains no cycles of length three or five.

<u>Proof.</u> $(\overline{Q1})$ and $(\overline{Q2})$ are obvious. $(\overline{Q3})$ follows from (Q3) of Lemma 6, since now $\Delta(\overline{K},\overline{L}) = \overline{k}$ would imply $K \equiv L$, and hence $\overline{K} = \overline{L}$. We finally observe that if $\overline{H}$ were to contain a cycle of length three or five, then so would H, contradicting (Q4) of Lemma 6.

The equation relating $\overline{v}$, $\overline{k}$, $\overline{\lambda}$ reads

$$\overline{\lambda}(\overline{vt}-1) = \overline{k}(\overline{kt}-1). \tag{2}$$

(Note that $\overline{v}$ need not be an integer.)

Consider now a fixed edge $(\overline{K}_0,\overline{L}_0) \in E(\overline{H})$ and define

$$\begin{aligned} A_0 &= \{\overline{K}_0\} = D_0(\overline{K}_0), & B_o &= \{\overline{L}_0\} = D_0(\overline{L}_0), \\ B_1 &= D_1(\overline{K}_0) - D_0(\overline{L}_0), & A_1 &= D_1(\overline{L}_0) - D_0(\overline{K}_0), \\ A_2 &= D_2(\overline{K}_0) - D_1(\overline{L}_0), & B_2 &= D_2(\overline{L}_0) - D_1(\overline{K}_0). \end{aligned}$$

Then

$$\begin{aligned} D_0(\overline{K}_0) &= A_0, & D_0(\overline{L}_0) &= B_0, \\ D_1(\overline{K}_0) &= B_0 \cup B_1, & D_1(\overline{L}_0) &= A_0 \cup A_1, \\ D_2(\overline{K}_0) &= A_1 \cup A_2. & D_2(\overline{L}_0) &= B_1 \cup B_2. \end{aligned}$$

It follows from $(\overline{Q4})$ that the six sets A_i,B_i $(i=0,1,2)$ are pairwise disjoint, and thus the two sets $D_i(\overline{K}_0)$, $D_i(\overline{L}_0)$ are disjoint for each $i=0,1,2$. Also by $(\overline{Q4})$ we see that no two vertices of $D_i(\overline{K}_0)$ or of $D_i(\overline{L}_0)$ can be adjacent for $i=0,1,2$. Using $(\overline{Q2})$ and $(\overline{Q3})$ we can then easily determine the number of vertices in each set, obtaining

$$\begin{aligned} |A_0| &= |B_0| = 1, \\ |B_1| &= |A_1| = \overline{k}-1, \\ |A_2| &= |B_2| = (\overline{k}-1)(\overline{k}-\overline{\lambda}) \;/\; \overline{\lambda}. \end{aligned} \tag{3}$$

Let C denote the set of vertices not in any of these sets. Then using $(\overline{Q1})$, (2), and (3) we have

$$|C| = 2(\overline{k}-\overline{\lambda})(t-1) \;/\; \overline{\lambda}t. \tag{4}$$

Thus t divides $2(\overline{k}-\overline{\lambda})$, and so

$$2 \leqq t \leqq 2(\overline{k}-\overline{\lambda}). \tag{5}$$

<u>LEMMA 10.</u> No two vertices of C are adjacent.

<u>Proof.</u> Clearly we have $d(\overline{K}_0,\overline{K}) = 3$, $d(\overline{L}_0,\overline{K}) = 3$ for all $\overline{K} \in C$. If $\overline{K},\overline{L} \in C$ and $(\overline{K},\overline{L}) \in E(\overline{H})$, then $(K,L) \in E(H)$. Let x be the vertex of G defined by $x = \langle K,L \rangle$, and let $x_0 = \langle K_0,L_0 \rangle$. Then by (P6), $d(x_0,x) \leqq 3$ in G. If $d(x_0,x) = 3$, there exists a clique L_1 meeting one of K_0,L_0 and one of K,L, say K_0 and K. Then $d(K_0,K) = 2$ in H, which implies either $\overline{K}_0 = \overline{K}$ or $d(\overline{K}_0,\overline{K}) = 2$ in $\overline{H}$, a contradiction. If $d(x_0,x) \leqq 2$, the result is even simpler.

It follows from Lemma 10 that $D_1(\overline{K}) \subseteqq A_2 \cup B_2$ for all $\overline{K} \in C$. Hence $C \subseteqq D_3(\overline{K}_0) \cup D_3(\overline{L}_0)$.

<u>LEMMA 11.</u> If $d(\overline{K},\overline{L}) = 3$, then

$$|D_2(\overline{K}) \cap D_1(\overline{L})| = |D_1(\overline{K}) \cap D_2(\overline{L})| .$$

<u>Proof.</u> Let $r = |D_2(\overline{K}) \cap D_1(\overline{L})|$, $s = |D_1(\overline{K}) \cap D_2(\overline{L})|$. Consider the number n of edges joining $D_2(\overline{K}) \cap D_1(\overline{L})$ and $D_1(\overline{K}) \cap D_2(\overline{L})$. Using (Q3), we have $n = r\overline{\lambda} = s\overline{\lambda}$, i.e. $r = s$.

Let $\overline{K}$ be a fixed vertex of C and define

$$a = |D_1(\overline{K}) \cap A_2| ,$$
$$b = |D_1(\overline{K}) \cap B_2| .$$

Then $a + b = \overline{k}$, since $A_2 \cap B_2 = \emptyset$. If we define

$$a_1 = |D_2(\overline{K}) \cap B_1| \ ,$$
$$b_1 = |D_2(\overline{K}) \cap A_2| \ ,$$

then it follows from Lemma 11 that $a_1 = a$, since clearly $D_1(\overline{K}) \cap D_2(\overline{K}_0) = D_1(\overline{K}) \cap A_2$ and $D_2(\overline{K}) \cap D_1(\overline{K}_0) = D_2(\overline{K}) \cap B_1$. Similarly $b = b_1$. Since $a = |D_2(\overline{K}) \cap B_1| \leqq |B_1| = \overline{k}-1$, we have $b \geqq 1$. Similarly $a \geqq 1$. Hence $C = D_3(\overline{K}_0) \cap D_3(\overline{L}_0)$. Again, $a \geqq 1$ implies $D_2(\overline{K}) \cap B_1 \neq \phi$, and hence there exists $\overline{L}_1 \in D_2(\overline{K}) \cap B_1$ with $\Delta(\overline{K},\overline{L}_1) = \overline{\lambda}$. Each of these vertices must be in $D_1(\overline{K}) \cap A_2$, and therefore $a \geqq \overline{\lambda}$. Similarly $b \geqq \overline{\lambda}$ and we have

$$\overline{\lambda} \leqq a,b \leqq \overline{k}-\overline{\lambda} \ , \qquad (6)$$

where $a + b = \overline{k}$.

Consider $D_2(\overline{K})$. We wish to establish an upper bound on the number of vertices $\overline{L} \in D_2(\overline{K})$ which are not in C . Since $D_1(\overline{K}) = (D_1(\overline{K}) \cap A_2) \cup (D_1(\overline{K}) \cap B_2)$, such an $\overline{L}$ is adjacent to some vertex of $D_1(\overline{K}) \cap A_2$ or to some vertex of $D_1(\overline{K}) \cap B_2$. Assume without loss of generality the former and call the vertex $\overline{K}_2$. It then follows that $\overline{L} \in D_2(\overline{L}_0)$, and hence $\overline{L} \in D_1(\overline{K}_1)$ for some vertex $\overline{K}_1 \in D_1(\overline{L}_0)$. If $\overline{K}_1 \in D_2(\overline{K}) \cap A_1$, then $\overline{H}$ contains a cycle of length five. Hence $\overline{K}_1 \in A_0 \cup (A_1-D_2(\overline{K}))$, a set of a vertices. Thus we have, since $\overline{K}_1 \in D_2(\overline{K}_2)$, $|D_2(\overline{K}_2) \cap D_1(\overline{L}_0)| \leqq a$. By Lemma 11, $|D_1(\overline{K}_2) \cap D_2(\overline{L}_0)| \leqq a$. Thus at most a of the $\overline{k}$ vertices adjacent to $\overline{K}_2 \in D_1(\overline{K}) \cap A_2$

are in $D_2(\overline{K}) \cap D_2(\overline{L}_0)$. Since there are a vertices in $D_1(\overline{K}) \cap A_2$, and since any vertex $\overline{L}_a \in D_2(\overline{K}) \cap D_2(\overline{L}_0)$ adjacent to one vertex of $D_1(\overline{K}) \cap A_2$ is adjacent to exactly $\overline{\lambda}$ of them (by (Q4), $\overline{L}_a \in D_2(\overline{K}) \cap D_2(\overline{L}_0)$ cannot be adjacent to any vertex of $D_1(\overline{K}) \cap B_2)$, we have at most $a^2/\overline{\lambda}$ vertices $\overline{L}_a \in D_2(\overline{K})$ which are also in $D_1(\overline{K}_2)$ for some $\overline{K}_2 \in D_1(\overline{K}) \cap A_2$. A similar argument shows that there are at least $b^2/\overline{\lambda}$ vertices $\overline{K}_b \in D_2(\overline{K})$ which are also in $D_1(\overline{L}_2)$ for some $\overline{L}_2 \in D_1(\overline{K}) \cap B_2$. This now gives us

$$|D_2(\overline{K}) - C| \leqq (a^2 + b^2)/\overline{\lambda} ,$$

and thus together with (3)

$$|C| \geqq 1 + \overline{\lambda}^{-1} \cdot [\overline{k}(\overline{k}-1) - (a^2 + b^2)] . \tag{7}$$

Then by (4)

$$2(\overline{k}-\overline{\lambda})(1-1/t) \geqq \overline{\lambda} + \overline{k}(\overline{k}-1) - (a^2 + b^2) .$$

The inequality remains valid on replacing t by the upper bound $2(\overline{k}-\overline{\lambda})$ of (5) , and we then have

$$a^2 + b^2 = \overline{k}^2 - 3\overline{k} + 3\overline{\lambda} + 1 .$$

Substituting $\overline{k}-a$ for b and simplifying,

$$a^2 - \overline{k}a + 1/2[3(\overline{k}-\overline{\lambda}) - 1] \geqq 0 . \tag{8}$$

Assuming with no loss of generality that $a \leq b$, it is easily verified that the only values of $a, b, \bar{k}, \bar{\lambda}$ satisfying both (6) and (8) are

(i) $\bar{k} \geq 2$, $\bar{\lambda} = 1$, $a = 1$, $b = \bar{k}-1$,

(ii) $\bar{k} = 4$, $\bar{\lambda} = 1$, $a = b = 2$.

If $\bar{k} = 2$, $\bar{\lambda} = 1$, $a = b = 1$, then $t = 2$ by (5), and from (3) and (4) , we have exactly one vertex in each of the sets A_i, B_i $(i = 0,1,2)$, and C. In this case $\bar{H}$ is a cycle of length seven, $v = 7$, $k = 4$, $\lambda = 2$, and G is the graph of Figure 1 below. The lines in the figure represent cliques, two vertices being adjacent in G if and only if they are collinear.

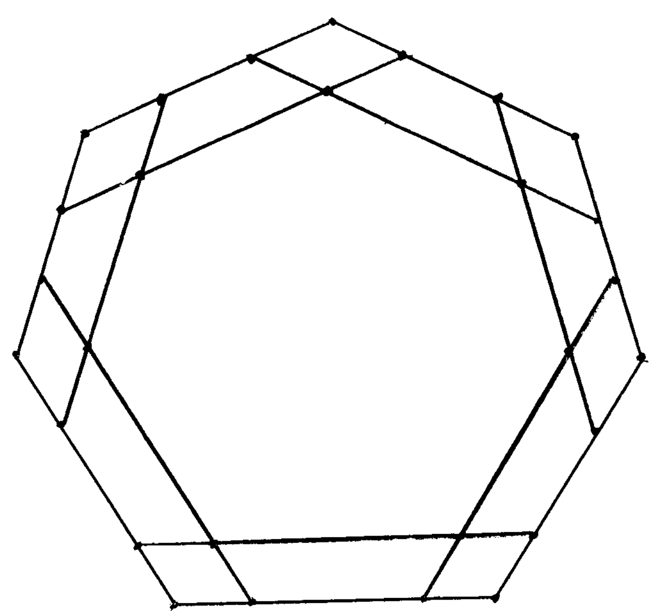

Figure 1. A graph $G \notin \{G(\pi)\}$ satisfying the properties (P1) - (P6) with $v = 7$, $k = 4$, $\lambda = 2$.

Consider next the case $\overline{k} \geqq 3$, $\overline{\lambda} = 1$, $b = \overline{k}-1$. Then if $\overline{K}_2$ is the vertex of A_2 adjacent to $\overline{K} \in C$, there are exactly $\overline{k}-1$ vertices in $D_1(\overline{K}_2) \cap C$, including $\overline{K}$, since $\overline{K}_2$ is adjacent to exactly one vertex of $D_2(\overline{L}_0)$ by our earlier argument. Since $a = 1$, $b = \overline{k}-1$ or $a = \overline{k}-1$, $b = 1$ for every choice of $\overline{K} \in C$, it follows that each vertex $\overline{K} \in C$ is contained in at least one set C_0 of $\overline{k}-1$ vertices in C such that either $C_0 \subseteqq D_1(\overline{K}_2)$ or $C_0 \subseteqq D_1(\overline{L}_2)$ for some $\overline{K}_2 \in A_2$, $\overline{L}_2 \in B_2$. By (4) and (5) we have

$$\overline{k}-1 \leqq |C| \leqq 2\overline{k}-3 , \tag{9}$$

so that if C_0, C_1 are two such sets, they must have a non-empty intersection. If $|C_0 \cap C_1| \geqq 2$, and $\overline{K}, \overline{L} \in C_0 \cap C_1$, then $\Delta(\overline{K}, \overline{L}) \geqq 2$, contradicting $(\overline{Q3})$, since $\overline{\lambda} = 1$. Hence it follows from (9) that either there is only one such set $C_0 \subseteqq C$, or else there are two sets $C_0, C_1 \subseteqq C$ with $|C_0 \cap C_1| = 1$. In the former case, $|C| = \overline{k}-1$, while in the latter $|C| = 2\overline{k}-3$.

We can easily verify, using $(\overline{Q3})$, that the number of edges joining A_2 and C is equal to the number of edges joining B_2 and C . For $\overline{K}_i \in C$, set $a_i = |D_1(\overline{K}_i) \cap A_2|$ and $b_i = |D_2(\overline{K}_i) \cap B_2|$, and let $\overline{a}$ be the number of $\overline{K}_i$'s for which $a_i = 1$, $b = \overline{k}-1$. Then since $\sum a_i = \sum b_i$, we obtain

$$\overline{a} + (|C|-\overline{a})(\overline{k}-1) = \overline{a}(\overline{k}-1) + (|C|-\overline{a}) ,$$

and since $\overline{k} \geqq 3$,

$$|C| = 2\overline{a}. \tag{10}$$

This rules out the possibility $|C| = 2\bar{k}-3$. As to the case $|C| = \bar{k}-1$, it follows from (10) that we must have two vertices $\bar{K}_1, \bar{K}_2 \in C$ with, say $a_1 = 1$ and $b_2 = 1$. Let $\bar{L}_2 \in A_2$ and $\bar{L}_1 \in B_2$ be the vertices adjacent to $\bar{K}_1, \bar{K}_2$, respectively, then by our earlier argument $\bar{L}_1, \bar{L}_2$ are both adjacent to all of C , which implies $\Delta(\bar{K}_1, \bar{K}_2) \geqq 2$, a contradiction.

The remaining case $\bar{k} - 4$, $\bar{\lambda} = 2$, $a = b = 2$ can be shown to be impossible by a rather involved argument which demonstrates that $\bar{H}$ must contain a cycle of length five. We have omitted the proof here.

Concluding Remarks. The question whether the conditions (P1) - (P6) are redundant is difficult to answer in general. The example given in [2] with $v = 7$, $k = 3$, $\lambda = 1$ shows that (P5) cannot be dropped without admitting additional exceptions, while the graph in Figure 2 below with $v = 16$, $k = 6$, $\lambda = 2$ demonstrates that the same holds true for (P6). For ease of exposition we have drawn the graph $\bar{H}$ in Figure 2 with $t = 2$, and thus $\bar{v} = 8$, $\bar{k} = 3$, $\bar{\lambda} = 1$. The two vertices of G corresponding to the edges e and f in $\bar{H}$ are readily seen to be at distance 4 from each other.

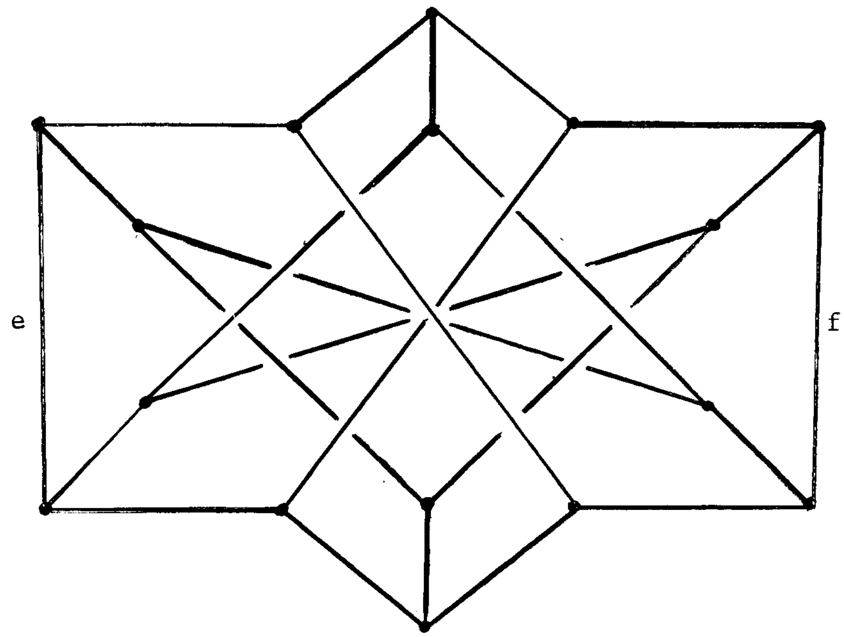

Figure 2. A graph $\overline{H}$ whose corresponding graph $G \notin \{G(\pi)\}$ and satisfies the properties (P1) - (P5) with $v = 16$, $k = 6$, $\lambda = 2$.

REFERENCES

[1] Bose, R. C. and Laskar, R.: A Characterization of Tetrahedral Graphs, J. Combinatorial Theory 3 (1967), 366-385

[2] Dowling, T. A. and Laskar, R.: A Geometric Characterization of the Line Graph of a Projective Plane, J. Combinatorial Theory 3 (1967), 402-410

[3] Hoffman, A. J.: On the Polynomial of a Graph, Amer. Math. Monthly 70 (1963), 30-36

[4] Hoffman, A. J. and Ray-Chaudhuri, D. K.: On the Line Graph of a Symmetric Balanced Incomplete Block Design, Trans. Amer. Math. Soc. 11 (1965), 238-252.

[5] Shrikhande, S. S.: The Uniqueness of the L_2 Association Scheme, Ann. Math. Stat. 30 (1959), 39-47.