A RENEWAL THEOREM FOR SUMS OF I.I.D. VECTORS

by

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Abstract

The theorem we prove gives an approximation for the expected number of visits a drifting (n+1)-dimensional random walk makes to a given set.

Running Head: Renewal Theory

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Although a great deal of research has been devoted to renewal limit theorems in one dimension, the literature concerning renewal theory in several dimensions is rather sparse. Doney [4] and Stam [11] have studied the asymptotic renewal measure of a translated set and Mode [8] has derived a related renewal density theorem.

In our theorem we obtain a finite asymptotic limit for the renewal measure of a set as it is translated in the direction of the mean and scaled in the other directions. The theorems of Doney, Stam and Mode are related to our result much as the local limit theorem is related to the central limit theorem. Other limit theorems in the same general area have been given by Smith [10], Bickel and Yahav [1], Farrell [5,6], and Ney and Spitzer [9].

Let X be a random variable and Y an n-dimensional random vector, with joint law P. Define the renewal measure

$$U = \sum_{i=0}^{\infty} P^{(i)}$$

where $P^{(i)}$ is the i-fold convolution of P with itself. For a set $S \in \mathbb{R}^{n+1}$ let

$$S_\tau = \{(x,y):\ (x,y/\tau) \in S\} .$$

The following result is a multi-dimensional generalization of Blackwell's [2,3] renewal theorem.

Theorem. *If* X *is non-lattice, has a finite second moment and a finite positive mean* μ*, and if* Y *has zero mean and finite covariance* $\Sigma = EYY'$*, then for any Borel set* S *whose projection onto the* x*-axis is bounded we have*

$$\lim_{\tau\to\infty} U((\tau^2,0)+S_\tau) = \int_S \frac{dx}{\mu} N(0\,,\Sigma/\mu)(dy)$$

where $N(\cdot,\cdot)$ *is the multivariate normal measure.*

Proof. For constants $b = 0,1$ and $r < 1$, define the measures V_τ by

$$V_\tau(S) = \sum_{i=0}^{\infty} r^i [P^{(i)}((b\tau^2,0)+S_\tau) + P^{(i)}((-b\tau^2,0)-S_\tau)] \ .$$

If ϕ is the characteristic function

$$\phi(p,q) = E\exp(ipX + iq'Y) \ ,$$

then the Fourier transform of V_τ is

$$\int e^{ipx+iq'y}\, V_\tau(dx,dy) = 2e^{-ipb\tau^2}\, \mathrm{Re}\left[\frac{1}{1-r\phi(p,q/\tau)}\right] \ .$$

If g is an L^1 function such that

$$\hat{g}(z) = \int g(x)e^{-izx}\, dx$$

is L^1, then Parseval's relation gives

$$\int g(x)e^{iq'y}\, V_\tau(dx,dy) = \frac{1}{\pi} \int \hat{g}(z)e^{-ib\tau^2 z}\, \mathrm{Re}\left[\frac{1}{1-r\phi(z,q/\tau)}\right]dz \ . \tag{1}$$

If we choose

$$g(x) = \frac{3}{2h}\left(1-\frac{|x|}{h}\right)^2 \quad \text{for} \quad |x| < h$$

$$= 0 \qquad\qquad \text{otherwise}$$

and $b = 0$, we have

$$\frac{3}{2h}\int_{-h}^{h}\left(1-\frac{|x|}{h}\right)^2 e^{iq'y}\, V_\tau(dx,dy) = \frac{6}{\pi h^2}\int\left(1-\frac{\sin(zh)}{zh}\right)\mathrm{Re}\left[\frac{1}{1-r\phi(z,q/\tau)}\right]\frac{dz}{z^2} \quad ,$$

which implies

$$\frac{2}{h}\int_{|z|>\frac{2}{h}} \mathrm{Re}\left[\frac{1}{1-r\phi(z,q/\tau)}\right]\frac{dz}{1+z^2} < \pi \lim_{r\to 1} V_\tau((-h,h)\times\mathbb{R}^n) \ .$$

Since X has a positive mean, this last expression is uniformly bounded for positive τ; so as $h \downarrow 0$,

$$\text{(2)} \qquad \int_{|z|>\frac{2}{h}} \mathrm{Re}\left[\frac{1}{1-r\phi(z,q/\tau)}\right]\frac{dz}{1+z^2} = O(h)$$

uniformly in τ, q and r. For the rest of our argument we will take $b = 1$,

$$g(x) = e^{ipx}/(\pi(1+x^2)) \ ,$$

$$\hat{g}(z) = e^{-|z-p|} \ ,$$

and let $r < 1$ be a function of τ, whose difference from 1 is $o(1/\tau^2)$ as $\tau \to \infty$. From Taylor's Theorem, if $z \to 0$ and $\tau \to \infty$ in a manner such that $z = O(1/\tau^2)$,

(3) $$\mathrm{Re}\left[\frac{1}{1-r\phi(z,q/\tau)}\right] = \frac{2\tau^2 q'\Sigma q}{4\mu^2 z^2\tau^4 + (q'\Sigma q)^2} + o(\tau^2) ,$$

and if $z \to 0$ and $\tau \to \infty$ in a manner such that $1/\tau^2 = o(z)$,

(4) $$\mathrm{Re}\left[\frac{1}{1-r\phi(z,q/\tau)}\right] = 0\left(1+\frac{1}{\tau^2 z^2}\right) .$$

We now fix p, q and $\varepsilon > 0$. Feller and Orey [7] have shown that

$$\frac{1}{z^2+1}\ \mathrm{Re}\left[\frac{1}{1-\phi(z,0)}\right]$$

is L^1. Using this fact and equations (2) and (4) it is possible to choose $\delta = 1/M$ so that for $\tau > M$,

$$\left[\int_{|z|>M} + \int_{|z|<\delta}\right] \frac{\hat{g}(z)}{\pi}\ \mathrm{Re}\left[\frac{1}{1-\phi(z,0)}\right] dz \le \frac{\varepsilon}{6} ,$$

$$\left[\int_{|z|>M} + \int_{\frac{M}{\tau^2}}^{\delta} + \int_{-\delta}^{-\frac{M}{\tau^2}}\right] \frac{\hat{g}(z)}{\pi}\ \mathrm{Re}\left[\frac{1}{1-r\phi(z,q/\tau)}\right] dz \le \frac{\varepsilon}{6} ,$$

and

$$\int_{|z|>M} \frac{\hat{g}(0)}{\pi}\ \frac{2q'\Sigma q}{4\mu^2 t^2 + (q'\Sigma q)^2}\ dt \le \frac{\varepsilon}{6} .$$

Using the Riemann-Lesbegue Lemma, equation (3) and the fact that X is non-lattice, we see that for τ sufficiently large we will have

$$\left|\int \frac{\hat{g}(z)}{\pi}\, e^{-iz\tau^2}\, \mathrm{Re}\left[\frac{1}{1-\phi(z,0)}\right] dz\right| \le \frac{\varepsilon}{6} \;,$$

$$\left|\frac{1}{\pi}\int_{-\frac{M}{\tau^2}}^{\frac{M}{\tau^2}} \left\{\hat{g}(z)e^{-iz\tau^2}\, \mathrm{Re}\left[\frac{1}{1-r\phi(z,q/\tau)}\right] - \hat{g}(0)e^{-iz\tau^2}\, \frac{2\tau^2 q'\Sigma q}{4\mu^2 z^2\tau^4 + (q'\Sigma q)^2}\right\} dz\right| \le \frac{\varepsilon}{6} \;,$$

and

$$\left|\left\{\int_{\delta}^{M} + \int_{-M}^{-\delta}\right\} \frac{\hat{g}(z)}{\pi}\, e^{-iz\tau^2}\, \mathrm{Re}\left[\frac{1}{1-r\phi(z,q/\tau)} - \frac{1}{1-\phi(z,0)}\right] dz\right| \le \frac{\varepsilon}{6} \;.$$

Combining the last 6 inequalities we see that

$$\left|\int \frac{\hat{g}(z)}{\pi}\, e^{-iz\tau^2}\, \mathrm{Re}\left[\frac{1}{1-r\phi(z,q/\tau)}\right] dz - \frac{\hat{g}(0)}{\pi}\int \frac{2q'\Sigma q}{4\mu^2 t^2 + (q'\Sigma q)^2}\, e^{-it}\, dt\right| < \varepsilon \;.$$

As ε was arbitrary, this proves that

$$\lim_{\tau\to\infty} \int \frac{e^{ipx+iq'y}}{\pi(1+x^2)}\, V_\tau(dx,dy) = \frac{1}{\mu}\exp\left(-\frac{q'\Sigma q}{2\mu} - |p|\right)$$

$$= \int \frac{e^{ipx+iq'y}}{\pi(1+x^2)}\, \frac{dx}{\mu}\, N(0\,,\Sigma/\mu)(dy) \;.$$

As the rate at which $r(\tau) \to 1$ was arbitrary, this equation must also hold with $r = 1$, and by the Lévy Continuity Theorem, the laws with density $1/(\pi(1+x^2))$ with respect to V_τ converge weakly. Hence, if S is a Borel set whose projection onto the x-axis is bounded we have

$$\lim_{\tau\to\infty} \{U((\tau^2,0)+S_\tau) + U((-\tau^2,0)-S_\tau)\} = \int_S \frac{dx}{\mu}\, N(0\,,\Sigma/\mu)(dy) \;.$$

This completes the proof as

$$\lim_{\tau\to\infty} U((-\tau^2,0)-S_\tau) = 0 \quad .$$

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