OPERATOR-VALUED MEASURES AND PERTURBATIONS OF SEMI-GROUPS

Igor Kluvánek

Institute of Statistics Mimeo Series #1205
OPERATOR-VALUED MEASURES AND PERTURBATIONS OF SEMI-GROUPS

by Igor Kluvánek

Introduction

Let \( l \) be a natural number. The \( \sigma \)-algebra of Borel sets in \( \mathbb{R}^l \) is denoted by \( \mathcal{B}(\mathbb{R}^l) \) or, shortly, by \( \mathcal{B} \). Let \( X \) be the Banach space of all finite (real- or complex-valued) \( \sigma \)-additive measures on \( \mathcal{B} \). The space \( L^1 = L^1(\mathbb{R}^l) \) will be considered a subspace of \( X \) consisting of all measures in \( X \) which are absolutely continuous with respect to the Lebesgue measure.

Let \( D \) be a strictly positive number. Denote

\[
(0.1) \quad p(x,t) = \frac{1}{(4\pi Dt)^{\frac{l}{2}}} \exp\left(-\frac{|x|^2}{4Dt}\right),
\]

for every \( x \in \mathbb{R}^l \) and \( t > 0 \).

For \( t > 0 \) and \( \mu \in X \), let

\[
(0.2) \quad (S(t)\mu)(B) = \int_B dy \int_{\mathbb{R}^l} p(y-x,t) d\mu(x), \quad B \in \mathcal{B}.
\]

In other words, the measure \( S(t)\mu \) is the convolution of \( \mu \) with the measure whose density is \( x \mapsto p(x,t), \ x \in \mathbb{R}^l \). By the customary abuse of notation, we extend this definition also to \( t = 0 \), interpreting \( x \mapsto p(x,0), \ x \in \mathbb{R}^l \), as the 'density' of the unit mass at 0. Of course, \( S(0)\mu = \mu \), for every \( \mu \in X \).

It is well-known that, for every \( t \geq 0 \), the map \( S(t): X \to X \) is linear, of norm \( \leq 1 \), and positive in the sense that \( S(t)\mu \) is a real non-negative measure for every real non-negative \( \mu \in X \). Furthermore, \( S(t)X \subset L^1 \), for every \( t > 0 \).

It is also known that \( t \mapsto S(t), \ t \in [0,\infty) \), is a continuous semigroup of operators.

Let \( t \geq 0 \). Denote by \( \Gamma_t \) the set \( C([0,t],\mathbb{R}^l) \) of all continuous maps (paths) \( \gamma: [0,t] \to \mathbb{R}^l \). Let \( \mathcal{S}_t \) be the \( \sigma \)-algebra of sets in \( \Gamma_t \) generated by all sets of the form \( \{ \gamma \in \Gamma_t: \gamma(s) \in B \}, \ for \ s \in [0,t] \) and \( B \in \mathcal{B} \).

Given \( \mu \in X \), let \( \omega^\mu_t \) be the \( (\sigma \)-additive, finite, real or complex) measure on \( \mathcal{S}_t \) such that
\[ \omega_\mu(B) = \int_\mathbb{R} \cdots \int_\mathbb{R} p(y-x_1, t-t_1) \cdots p(x_n-x_{n-1}, t_{n-1} - t_{n-1}) \cdots \]
\[ \cdots p(x_2-x_1, t_2-t_1) p(x_1-x_1, t_1) d\mu(x) dx_1 dx_2 \cdots dx_n dy, \]
whenever

\[ (0.3) \quad E = \{ \gamma \in \Gamma : \gamma(t_j) \in B_j, j = 1, 2, \ldots, n \}, \]

for some natural number \( n \), sets \( B_j \in B \), and numbers \( t_j \in [0,t] \), \( j = 1, 2, \ldots, n \), such that \( t_{j-1} < t_j \), for \( j = 2, 3, \ldots, n \).

If \( \mu \) is a non-negative measure of the total mass one, then \( \omega_\mu \) is the Wiener measure corresponding to the Brownian motion with diffusion coefficient \( D \) (variance \( 2D \) per unit time) of a particle with the initial distribution \( \mu \).

If \( \mu \) is the unit mass measure concentrated at the origin and \( D = \frac{1}{2} \), then \( \omega_\mu \) is the standard Wiener measures.

If \( \nu \) is a Borel function on \( \mathbb{R}^l \), by the same letter \( \nu \) we denote the operator which associates with a measure \( \mu \in X \) the indefinite integral of \( \nu \) with respect to \( \mu \). If the function \( \nu \) is unbounded, then also the corresponding operator is unbounded. To be sure, a measure \( \mu \in X \) belongs to the domain of the operator \( \nu \) only if \( \nu \) is \( \mu \)-integrable.

By \( \Delta \) is denoted the Laplacian on \( \mathbb{R}^l \). It also will be interpreted as an operator in (a subset of) \( X \). A measure \( \mu \in X \) will belong to the domain of \( \Delta \) if its density \( \phi \) is everywhere twice differentiable and the function \( \psi = \Delta \phi \) is in \( L^1 \). Then \( \Delta \mu \) will be the measure with the density \( \psi \). The domain of \( \Delta \) is not norm dense in \( X \). It is norm dense in \( L^1 \). In a suitably weakened topology, such as the vague topology, the domain of \( \Delta \) will become dense also in \( X \).

If \( \nu \) is a Borel function on \( \mathbb{R}^l \), and the indicated integral exists, the formula

\[ f(\gamma) = \exp( \int_0^t \nu(\gamma(s)) ds ), \quad \gamma \in \Gamma, \]
defines a $S_t$-measurable function $f$ on $\Gamma_t$. If $V$ happens to be bounded above, then $f$ is bounded and, hence, integrable with respect to $\omega_\mu$, for any $\mu \in \mathcal{X}$. The integral is given by

$$\int_{\Gamma_t} f(\gamma) \, d\omega_\mu(\gamma) = (\exp(t(D\Delta + V))\mu)(\mathbb{R}^d) \, ,$$

where $\exp(t(D\Delta + V))\mu$ is the measure whose density $x \mapsto u(x,t)$, $x \in \mathbb{R}^d$, is obtained as the solution of the problem

$$\frac{\partial u}{\partial t} = D\Delta u + Vu, \quad t > 0, \quad x \in \mathbb{R}^d ; \quad \lim_{t \to 0^+} \int_B u(x,t) \, dx = \mu(B), \quad B \in \mathcal{B} \, .$$

More generally, for any $B \in \mathcal{B}$,

$$\int_B f(\gamma) \, d\omega_\mu(\gamma) = (\exp(t(D\Delta + V))\mu)(B) = \int_B u(x,t) \, dx \, ,$$

where $E(t;B) = \{\gamma \in \Gamma_t : \gamma(t) \in B\}$.

The values $u(x,t)$ can be recovered using formula (0.6). Namely,

$$\mu(x,t) = \lim_{r \to 0^+} \frac{\Gamma(\frac{d+1}{2})}{2\pi^{\frac{d}{2}} r^d} \int_{E(t;B(x,r))} f(\gamma) \, d\omega_\mu(\gamma) \, ,$$

where $B(x,r)$ is the ball of radius $r$ centered at $x$, and $2\pi^{\frac{d}{2}} r^d / \Gamma(\frac{d}{2})$ is its $d$-dimensional volume.

A nice reference for these results, due to M. KAC, is the Chapter IV of [1]. The calculations in [1] are carried out for the case when $\mu$ is the unit mass at the origin. The general case is a simple consequence of this special one.

These results can be rephrased so that their operator-theoretic content becomes more salient. Let

$$M_t(E) = S(t-t_n)\chi_{B_n} S(t_{n-1}-t_n)\ldots\chi_{B_2} S(t_2-t_1)\chi_{B_1} S(t_1) \, ,$$

for every set $E$ given in the form (0.3), assuming that $0 < t_1 < t_2 < \ldots < t_n < t$ and the sets $B_1, B_2, \ldots, B_n$ are in $\mathcal{B}$. For every such set $E$, $M_t(E)$ is a continuous linear operator on $X$. Moreover, there exists a unique measure $M_t$ on the
\( \sigma \)-algebra \( S_t \), whose values are continuous operators on \( X \), which is \( \sigma \)-additive in the strong operator topology such that the formula (0.8) holds whenever the set \( E \) is given by (0.3). If we restrict the values of this measure to \( L^1 \), we obtain a measure \( \sigma \)-additive in the strong operator topology whose values are operators on \( L^1 \).

Given \( \mu \in X \), let \( (M_t \mu)(E) = m_t(E) \mu \), for every \( E \in S_t \). It follows that, for every \( \mu \in X \), the application \( M_t : S_t \to X \) is an \( X \)-valued \( \sigma \)-additive measure on \( S_t \). If \( \mu \in L^1 \), the vector measure \( M_t \mu \) is \( L^1 \)-valued. Actually, if \( t > 0 \), then \( M_t \mu \) is \( L^1 \)-valued for every \( \mu \in X \).

We may note that \( \nu_{\mu}(E) = (M_t(E)\mu)(\mathbb{R}^d) = (M_t(E))((\mathbb{R}^d)^1) \), for every \( E \in S_t \).

Our aim is to show that, using a single exchange of two integral signs (Fubini theorem), we obtain the formula

\[
(0.9) \quad \int_{\Gamma} f(\gamma) dM_t(\gamma) \mu = \exp(t(D\Delta + \mathcal{V}))\mu.
\]

For \( t > 0 \), this integral is an element of \( L^1 \). If we represent, as customary, the elements of \( L^1 \) by their densities, the formula (0.9) gives the solution of the problem (0.5) directly without going through the limiting procedure (0.7).

The Kac formula (0.4) is obtained from (0.9) by substituting \( \mathbb{R}^d \) to both sides of (0.9).

So, the operator or vector approach provides for a new proof of the Kac formula. Such a proof gives, perhaps, a better insight into the mechanism of the formula and a possibility to examine more precisely conditions for its validity. Indeed, cases of the function \( \mathcal{V} \) may occur, when the integral on the left-hand side of (0.9) exists in a good sense but it is not possible, or easy, using only scalar integration, to show its connection with the problem (0.5). What is more, solutions of (0.5) may not exist at all. In such cases, the measure \( \exp(t(D\Delta + \mathcal{V}))\mu \) can be defined by (0.9) and it can be shown that such a definition is a natural extension of the definition by (0.5) and that it has a good physical or probabilistic meaning.
It seems that the operator approach may prove successful in the analysis of perturbations of semigroups of operators other than the diffusion semigroup. The approach may, at least, indicate the limitations of some methods used for solving problems related to such perturbations and, hopefully, suggest new ones.

It should be noted that not every perturbation can be treated using this approach. On the other hand, an example will be given in the last Section indicating that the perturbations which can be so treated are abundant.

1. **Operator-valued measures**

Let $X$ be a locally convex topological vector space. The dual space (all continuous linear functionals) is denoted by $X'$ and the algebraic dual space (all linear functionals) by $X^*$. The pairing of elements $\phi \in X$ and $\phi' \in X'$ (or $\phi' \in X^*$) is denoted interchangeably by $\langle \phi, \phi' \rangle$ or $\langle \phi', \phi \rangle$. No special symbol is used for the natural injection of $X$ into $X''$ and into $X''^*$. We write simply $X \subset X''$ and $X \subset X''^*$.

The space $X$ will always be assumed quasi-complete (bounded closed sets are complete). This assumption eliminates some difficulties (otherwise trivial from our point of view) with the existence of various objects. The identity operator on $X$ is denoted by $I$.

The space of all continuous linear maps from $X$ to $X$ is denoted by $L(X)$. The elements of $L(X)$ will be called operators, although also some maps defined only on linear subspaces of $X$ will be called so; this will be clear from the context.

In this Section, we introduce conventions and definitions, concerning integration of vector-valued functions with respect to scalar-valued measures and scalar-valued functions with respect to vector-valued measures, to be used in the sequel.

Following the custom, in formulas involving integration with respect to the Lebesgue measure (on the space $\mathbb{R}^J$ or a measurable part thereof), the symbol for
the measure will be omitted. However, if a need for a specific symbol for the Lebesgue measure arises, this measure will be denoted by "leb".

For integration of vector-valued functions with respect to scalar-valued measures, the Pettis integral will be used. Let us recall the definition.

Let $T$ be a space, let $T$ be a $\sigma$-algebra of subsets of the space $T$ and let $\mu: T \rightarrow \mathbb{R}$ be a finite real-valued ($\sigma$-additive) measure. Let $f: T \rightarrow X$ be a vector-valued function. We say that the function $f$ is integrable with respect to the measure $\mu$ if, for every $E \in T$, there exists a vector $\mu_{E}(f)$ in the space $X$ such that

\[
(1.1) \quad \langle \mu_{E}(f), \phi' \rangle = \int_{E} \langle f(u), \phi' \rangle d\mu(u),
\]

for every $\phi' \in X'$. We use the standard notations

\[
\int_{E} f d\mu = \int_{E} f(u) d\mu(u) = \mu_{E}(f); \quad \mu(f) = \int f d\mu = \int f(u) d\mu(u) = \mu_{T}(f).
\]

The integrability of a function $f: T \rightarrow X'^{*}$, with values in the algebraic dual of the space $X'$, will be understood with respect to the topology $\sigma(X'^{*}, X')$ of the space $X'^{*}$. Actually, for settling questions of integrability, the topology itself is not needed; the knowledge of the dual space suffices. The dual space of $X'^{*}$ with respect to the topology $\sigma(X'^{*}, X')$ is $X'$. Hence, a function $f: T \rightarrow X'^{*}$ is $\mu$-integrable if, for every $E \in T$, there is a vector $\mu_{E}(f)$ in $X'^{*}$ such that (1.1) holds for every $\phi' \in X'$. It is immediate that a function $f: T \rightarrow X'^{*}$ is $\mu$-integrable if and only if, for every $\phi' \in X'$, the function $u \rightarrow \langle f(u), \phi' \rangle$, $u \in T$, is $\mu$-integrable.

It is interesting and important to note that the indefinite integral of an integrable function $f: T \rightarrow X'^{*}$ may very well have its values in the space $X$ even if $f(u) \notin X$, for every $u \in T$. This is to say, the vectors satisfying (1.1) may belong to $X$, for every $E \in T$. The Orlicz-Pettis lemma then implies that the indefinite integral, that is, the assignment $E \mapsto \mu_{E}(f)$, $E \in T$, is an $X$-valued vector measure.
E. Thomas ([3]) demonstrated, on many interesting examples, that this phenomenon is far from a mere curiosity and that it is of considerable importance (at least potentially) in the context of classical problems of mathematical physics.

Let $\Omega$ be an abstract set and let $\mathcal{S}$ be a $\sigma$-algebra of subsets of $\Omega$. A map $m: \mathcal{S} \rightarrow X$ is called a vector measure or, more specifically, an $X$-valued measure, if it is $\sigma$-additive with respect to the topology of the space $X$. By an operator-valued measure is meant an $L(X)$-valued measure interpreting $L(X)$ as a locally convex space with respect to the strong operator topology. Explicitly, a map $M: \mathcal{S} \rightarrow L(X)$ is an operator-valued measure if, for every $\phi \in X$ and disjoint sets $E_k$, $k = 1, 2, \ldots$, from $\mathcal{S}$ whose union is $E$, the equality

$$M(E)\phi = \sum_{k=1}^{\infty} M(E_k)\phi$$

holds in the sense of convergence in the space $X$.

An operator-valued measure $P: \mathcal{S} \rightarrow L(X)$ such that $P(E \cap F) = P(E)P(F)$, for every $E \in \mathcal{S}$ and $F \in \mathcal{S}$, is called a projection-valued measure. A spectral measure is a projection-valued measure $P: \mathcal{S} \rightarrow L(X)$ such that $P(\Omega) = I$. Spectral measures are, perhaps, the most prominent examples of an operator-valued measure.

Given a vector measure $m: \mathcal{S} \rightarrow X$ and a continuous linear functional $\phi' \in X'$, by $\langle m, \phi' \rangle$ is denoted the scalar-valued measure $E \mapsto \langle m(E), \phi' \rangle$, $E \in \mathcal{S}$. Given an operator-valued measure $M: \mathcal{S} \rightarrow L(X)$ and an element $\phi \in X$, the symbol $M\phi$ represents the vector measure $E \mapsto M(E)\phi$, $E \in \mathcal{S}$.

A scalar-valued function $f$ on $\Omega$ is said to be integrable with respect to the vector measure $m: \mathcal{S} \rightarrow X$, shortly, $m$-integrable, if it is $\langle m, \phi' \rangle$-integrable, for every $\phi' \in X'$, and if, for every set $E \in \mathcal{S}$, there exists a vector $m_E(f)$ in the space $X$ such that

$$\langle m_E(f), \phi' \rangle = \int_E f(\omega) d\langle m(\omega), \phi' \rangle,$$  \hspace{1cm} (1.2)
for every \( \phi' \in X' \). The vector \( m_E(f) \), determined uniquely by the condition (1.2), is called the integral of the function \( f \) on the set \( E \) with respect to the measure \( m \). We use the standard notations

\[
\int_E f \, dm = \int_E f(\omega) \, dm(\omega) = m_E(f),
\]

and

\[
m(f) = \int f \, dm = \int f(\omega) \, dm(\omega) = m_\Omega(f).
\]

If the function \( f \) is \( m \)-integrable, the application \( E \mapsto m_E(f) \), \( E \in S \), from \( S \) to \( X \), is a vector measure. This follows directly from the Orlicz-Pettis lemma. The vector measure \( E \mapsto m_E(f) \), \( E \in S \), is called the indefinite integral of the function \( f \) with respect to the vector measure \( m \).

Instead of saying that the function \( f \) is \( m \)-integrable we may say, for stylistical purposes, that the integral \( m(f) \) exists.

A function \( f \) is said to be \( m \)-integrable on a set \( E \in S \), if the function \( f|_E \) is \( m \)-integrable. Clearly, if a function is integrable on a set \( E \in S \) and \( F \in S \), \( F \subseteq E \), then the function is integrable also on the set \( F \).

A set \( E \in S \) is said to be \( m \)-null or \( m \)-negligible, if \( m(F) = 0 \), for every \( F \in S \), \( F \subseteq E \). This comes to the same as saying that \( m_E(f) = 0 \), for every \( m \)-integrable function \( f \). In connection with \( m \)-null sets we shall use the standard \( m \)-almost-everywhere-type jargon.

A treatment of integration with respect to vector measures can be found, for example, in [2], Chapter II. Let us mention here that every bounded \( S \)-measurable function is \( m \)-integrable. This is a consequence of quasi-completeness of the space \( X \).

For a large class of spaces, which included all weakly sequentially complete ones, the definition of integrability can be simplified: A scalar function \( f \) is integrable with respect to a measure \( m \) taking values in such a space \( X \) if
and only if it is \((m, \phi')\)-integrable, for every \(\phi' \in X'\). In general, however, if \(f\) is \((m, \phi')\)-integrable, for every \(\phi' \in X'\), it may not be possible to find vectors \(m_E(f)\) in \(X\) such that (1.2) holds for every \(\phi' \in X'\). Such vectors could be found only in \(X''\).

In some contexts, there is a need for a definition of integral relaxed even more than suggested by the last remark. We may have to allow the integral even to be outside of \(X''\) and to belong to \(X''\), the algebraic dual of the space \(X'\). Then \(f\) must be permitted to be non-integrable with respect to \((m, \phi')\), for some \(\phi' \in X'\). To obtain a useful concept, we should require \((m, \phi')\)-integrability for a total set of functionals \(\phi' \in X'\). So, we adopt the following definition.

A scalar-valued function \(f\) on \(\Omega\) is said to be \(m\)-integrable in a relaxed sense if the set of functionals \(\phi' \in X'\), such that the function \(f\) is \((m, \phi')\)-integrable, is total on \(X\).

If the function \(f\) is \(m\)-integrable in the relaxed sense, an element \(m_E(f)\) of \(X''\) can be chosen for each set \(E\) in \(\mathcal{S}\) such that, for every \(\phi' \in X'\), the assignment \(E \mapsto \langle m(E), \phi' \rangle\), \(E \in \mathcal{S}\), is a \(\sigma\)-additive scalar measure, and the equality (1.2) holds for every \(E \in \mathcal{S}\), for every \(\phi' \in X'\) such that the function \(f\) is \((m, \phi')\)-integrable.

Although the choice of \(m_E(f)\) is not unique, we shall still call it the integral in the relaxed sense of the function \(f\) on the set \(E\) with respect to the measure \(m\). Also, we shall maintain the notations (1.3) and (1.4) for the integral in the relaxed sense.

If there is a danger of confusion, an \(m\)-integrable function \(f\) shall be described as \(m\)-integrable in the strict sense, or we shall say that the integral \(m(f)\) exists in the strict sense.

Integration of scalar-valued functions with respect to operator-valued measures is a special case of integration with respect to vector-valued measures; the space \(L(X)\) of operators is a locally convex space under its strong operator
topology. However, integration with respect to operator-valued measures has some specific features worth mentioning and requires the introduction of further conventions.

Explicitly stated, a scalar-valued function \( f \) on \( \Omega \) is integrable (in the strict sense) with respect to an operator-valued measure \( M: S \rightarrow L(X) \) if, for every \( E \in S \), there is an operator \( M_E(f) \) in \( L(X) \) such that

\[
\langle M_E(f) \phi, \phi' \rangle = \int_E f(\omega) d\langle M(\omega) \phi, \phi' \rangle,
\]

for every \( \phi \in X \) and \( \phi' \in X' \). Of course, we write

\[
\int_E f \, dM = \int_E f(\omega) \, dM(\omega) = M_E(f); \quad M(f) = \int f \, dM = \int f(\omega) \, dM(\omega) = M_\Omega(f).
\]

If \( f \) is an \( M \)-integrable function, then the integral \( M(f) \), is a continuous linear operator, an element of \( L(X) \). Essentially, only bounded functions are integrable with respect to many operator-valued measures of interest, such as spectral measures. The need often arises to extend the notion of integral to some functions which are not \( M \)-integrable in the strict sense. We have two alternatives.

For an \( S \)-measurable function \( f \) on \( \Omega \), we define \( M(f) \) to be the operator whose domain is the set of all vectors \( \phi \in X \) such that the function \( f \) is \((M\phi)\)-integrable (in the strict sense) and which is defined by the formula \( M(f)\phi = (M\phi)(f) \), for every vector \( \phi \) in the domain of \( M(f) \). Recall that \((M\phi)(f)\) represents the integral of the function \( f \) with respect to the vector-valued measure \( M\phi \).

The other alternative is to use integrability in the relaxed sense. As we shall see, this alternative can be more advantageous.

A scalar function \( f \) is said to be integrable in the relaxed sense with respect to an operator-valued measure \( M: S \rightarrow L(X) \) if \( f \) is \((M\phi)\)-integrable in the relaxed sense, for every \( \phi \in X \). Then, for every \( E \in S \), the symbol \( M_E(f) \) will denote a map from \( X \) into \( X^* \) defined by \( M_E(f) = (M\phi)_E(f) \), where \((M\phi)_E(f)\) is (a choice of)
the integral in the relaxed sense of the function \( f \) with respect to the \( X \)-valued measure \( M \) on the set \( E \), for every \( \phi \in X \). Of course, \( M(f) = M_{\Omega}(f) \).

So, if the function \( f \) is not \( M \)-integrable in the strict sense, the symbol \( M(f) \) is interpreted in two ways: either as an operator with values in \( X \) defined on a subset of \( X \), or as a map from the whole of \( X \) into \( X' \). However, no confusion should arise from this double interpretation. For every \( \phi \in X \) such that \( f \) is \( (M\phi) \)-integrable in the strict sense, the meanings of \( M(f)\phi \) are the same in both interpretations; and if \( f \) is not \( (M\phi) \)-integrable, in one interpretation \( M(f)\phi \) does not have a meaning at all, while in the other \( M(f)\phi \) is an element of \( X' \). Because it is more advantageous for our purposes, we shall adhere mostly to the later interpretation.

Perhaps, \( M(f) \) represents a useful concept only if the set of vectors \( \phi \in X \) such that \( M(f)\phi \) belongs to \( X \) is dense in the space \( X \). The following proposition says that this happens quite often.

**Proposition 1.** Let \( P: S + X \) be a spectral measure and let \( f \) be an \( S \)-measurable scalar-valued function defined \( P \)-almost everywhere on \( \Omega \). Then the function \( f \) is \( P \)-integrable in the relaxed sense and the set of vectors \( \phi \in X \), such that the function \( f \) is \( (P\phi) \)-integrable in the strict sense, is dense in \( X \).

**Proof.** Let \( E_n = \{ \omega: |f(\omega)| > n \}, n = 1, 2, \ldots \). Let \( Y_n = P(E_n)X, Z_n = P(\Omega \setminus E_n)(X), n = 1, 2, \ldots \). Let \( Y_{n_1} \) be the subspace of \( X \) which annihilates \( Y_{n_1} \), and let \( Y' \) be the union of the subspaces \( Y_{n_1} \), \( n = 1, 2, \ldots \). Because the intersection of the spaces \( Y_{n_k} \), \( n = 1, 2, \ldots \), is empty, \( Y' \) is a total subspace of \( X \). Given \( \phi' \in Y' \), there is an \( n \) such that, for every \( \phi \in X \), the measure \( \langle P\phi, \phi' \rangle \) vanishes on every measurable subset of \( E_n \). Hence, the function \( f \) is \( \langle P\phi, \phi' \rangle \)-integrable. By the definition, the function \( f \) is \( P \)-integrable in the relaxed sense.
The union, $Z$, of the subspaces $Z_n$, $n = 1, 2, \ldots$, is dense in $X$. If $\phi \in Z$, there is an $n$ such that the vector-valued measure $P\phi$ vanishes on every subset of $E_n$. Consequently, $f$ is $(P\phi)$-essentially bounded. It follows that $f$ is $(P\phi)$-integrable.

2. Fubini theorem

Let $S$ be a $\sigma$-algebra of sets in a space $\Omega$ and let $T$ be a $\sigma$-algebra of sets in a space $T$. Let $S \otimes T$ be the $\sigma$-algebra of sets in the space $\Omega \times T$ generated by the family of rectangles $E \times F$, for all $E \in S$ and $F \in T$. Let $m: S \rightarrow X$ be a vector measure and $\mu: T \rightarrow \mathbb{R}$ a finite real-valued measure.

Given a scalar $S \otimes T$-measurable function $f$ on $\Omega \times T$, we are looking for conditions guaranteeing that the equality

$$\int_{\Omega} \left( \int_{T} f(\omega, u) \, d\mu(u) \right) \, dm(\omega) = \int_{T} \left( \int_{\Omega} f(\omega, u) \, dm(\omega) \right) \, d\mu(u)$$

holds, including, of course, that the integrals involved exist at least in the relaxed sense. To be sure, the existence in a sense of the integral on the left-hand side means that, for $m$-almost every $\omega \in \Omega$, the function $u \mapsto f(\omega, u)$, $u \in T$, is $\mu$-integrable and the function

$$\omega \mapsto \int_{T} f(\omega, u) \, d\mu(u), \quad \omega \in \Omega,$$

which may be defined only for $m$-almost every $\omega \in \Omega$, is $m$-integrable in the respective sense. Similarly is interpreted the existence of the integral on the right-hand side.

The definitions of integrals (of a scalar function with respect to a vector measure and of a vector function with respect to a scalar measure), the strict integrability of every bounded measurable function and the classical Fubini theorem give
Proposition 2. If \( f \) is a bounded \( S \otimes T \)-measurable function then (1.2) holds and all integrals involved exist in the strict sense.

Now, for every set \( G \in S \otimes T \), we define

\[
(m \otimes \mu)(G) = \int_\Omega \left( \int_T \chi_G(\omega, u) \, d\mu(u) \right) \, dm(\omega),
\]

The dominated convergence theorem (Theorem II.4.2 in [2]) implies that \( m \otimes \mu: S \otimes T \to X \) is a vector measure. It will be called the tensor product of the measures \( m \) and \( \mu \). Proposition 2 implies that also

\[
(m \otimes \mu)(G) = \int_T \int_\Omega \chi_G(\omega, u) \, dm(\omega) \, d\mu(u),
\]

for every \( G \in S \otimes T \).

Proposition 3. \( \langle m \otimes \mu, \phi' \rangle = \langle m, \phi' \rangle \otimes \mu \), for every \( \phi' \in X' \).

When both measures \( m \) and \( \mu \) are scalar-valued, a sufficient condition for the validity of (2.1) is the integrability of the function \( f \) with respect to \( m \otimes \mu \). However, if \( m \) is a vector measure, the interior integral on the right-hand side of (2.1) may not exist in the strict sense for any \( \omega \in T \) even if \( f \) is strictly integrable with respect to \( m \otimes \mu \).

Example. Let \( X = L^2([0,1]) \). Let \( \Omega = [0,1] \) and let \( S \) be the \( \sigma \)-algebra of Borel sets in \([0,1]\). Let \( \alpha < 1 \). For every \( E \in S \), let \( \phi_E \) be the function on \([0,1]\) defined by

\[
\phi_E(x) = \int_E |\omega - x|^{-\alpha} \, d\omega, \quad x \in [0,1].
\]

For every \( x \in [0,1] \), the application \( E \mapsto \phi_E(x) \), \( E \in S \), is a measure on \( S \). Moreover, it can be shown that, for every \( E \in S \), the function \( \phi_E \) is continuous in \([0,1]\) and, if the Lebesgue measure of the set \( E \) is equal to \( c \), then
\[ 0 \leq \phi_E(x) \leq 2 \int_0^{1\varepsilon} \omega^{-\alpha} d\omega = \frac{2}{1-\alpha} (1\varepsilon)^{1-\alpha}. \]

Hence the measures \( E \rightarrow \phi_E(x), E \in S, \) are leb-absolutely continuous uniformly with respect to \( x \in [0,1]. \) Hence, if we let \( m(E) = \phi_E, \) for every \( E \in S, \) we obtain a vector measure \( m : S \rightarrow C([0,1]) \). Because every function in \( C([0,1]) \) represents also an element of the space \( L^2([0,1]) = X \) and the natural inclusion of \( C([0,1]) \) into \( X \) is continuous, we have a vector measure \( m : S \rightarrow X. \)

Let \( T = [0,1], \) and let \( T \) be, again, the \( \sigma \)-algebra of Borel sets in \([0,1], \) and let \( \mu = \text{leb}. \)

It is easy to show that, if \( \beta < 1, \) the function \( f \) defined on \( \Omega \times T \) by

\[ f(\omega,u) = |\omega-u|^{-\beta}, \ (\omega,u) \in \Omega \times T, \]

is \( m \otimes \mu \)-integrable. However, if \( \alpha \) and \( \beta \) are suitably chosen, there is no \( u \in T, \) such that the function \( \omega \mapsto f(\omega,u), \ \omega \in \Omega, \) is \( m \)-integrable.

Indeed, for a given \( u \in T \) and \( \gamma < \frac{1}{2}, \) let \( \psi(x) = |x-u|^{-\gamma}, \ x \in [0,1]. \) So, \( \psi \in X = X'. \) Then the measure \( \langle m,\psi \rangle \) is given by

\[ \langle m(E),\psi \rangle = \int_0^1 \left( \int_E |\omega-x|^{-\alpha} d\omega \right) |x-u|^{-\gamma} dx, \ E \in S. \]

Now, if \( \alpha + \beta + \gamma \geq 2, \) the integral

\[ \int_0^1 \int_0^1 |\omega-u|^{-\beta} |\omega-x|^{-\alpha} |x-u|^{-\gamma} dx d\omega \]

diverges. Hence, the function \( \omega \mapsto f(\omega,u), \ \omega \in \Omega, \) is not \( \langle m,\psi \rangle \)-integrable.

Consequently, it is not \( m \)-integrable in the strict sense.

If we interpret the vector measure \( m \) as \( C([0,1]) \)-valued, a simpler calculation shows that, with a choice \( \alpha + \beta \geq 1, \) the function \( \omega \mapsto f(\omega,u), \ \omega \in \Omega, \) is not \( m \)-integrable in the strict sense, for any \( u \in T. \)
The space $L^2([0,1])$ was chosen, even to the expense of more complicated calculations, to demonstrate that the difficulties are not due to the properties of the space $X$. These difficulties are observed even in the nicest spaces such as a Hilbert space.

Proposition 4. Let $f$ be a scalar $S \otimes T$-measurable and $m \otimes \mu$-integrable in the strict sense function on $\Omega \times T$.

For $m$-almost every $\omega \in \Omega$, the function $\nu \mapsto f(\omega, \nu)$, $\omega \in T$, is $\mu$-integrable, the function (2.2) is $m$-integrable in the strict sense, and

$$\int \limits_{\Omega \times T} f \, dm \otimes \mu = \int \limits_{\Omega} \left( \int \limits_{T} f(\omega, \nu) \, d\mu(\nu) \right) \, dm(\omega).$$

For $\mu$-almost every $\nu \in T$, the function $\omega \mapsto f(\omega, \nu)$, $\omega \in \Omega$, is $m$-integrable in the relaxed sense, the function

$$\nu \mapsto \int \limits_{\Omega} f(\omega, \nu) \, dm(\omega), \quad \nu \in T,$$

with values in $X'$, is $\mu$-integrable, and

$$\int \limits_{\Omega \times T} f \, dm \otimes \mu = \int \limits_{T} \left( \int \limits_{\Omega} f(\omega, \nu) \, dm(\omega) \right) \, d\mu(\nu).$$

Proof. If the set $E_0$ of the points $\omega \in \Omega$, for which

$$\int \limits_{T} |f(\omega, \nu)| \, d\mu(\nu) = \infty,$$

is not $m$-null, we can choose $\phi' \in X'$ such that the variation of the measure $<m, \phi'>$ on $E_0$ is not zero. It then follows that $f$ is not $<m, \phi'> \otimes \mu$-integrable over the set $E_0 \times T$ and, consequently, $f$ is not $m \otimes \mu$-integrable. So, the function (2.2) is defined for $m$-almost every $\omega \in \Omega$. By the classical Fubini theorem it is $<m, \phi'>$-integrable for every $\phi' \in X'$. It means that this function is $m$-integrable at least in the relaxed sense. But, using the integration with respect to the measures $<m, \phi'>$ and $<m \otimes \mu, \phi'> = <m, \phi'> \otimes \mu$, for every $\phi' \in X'$, and the classical
Fubini theorem, the integral of the function (2.2) on any set $E \in S$ is easily shown to be equal to $(m \otimes \mu)_{\mathcal{E} \times T}(\mathcal{f})$. Hence, the function (2.2) is integrable in the strict sense. For $E = \Omega$, we obtain (2.3).

For the proof of the second part, let $H$ be an algebraic (Hamel) base of the space $X'$. Let $\phi' \in H$ and let $\nu \in T$. If the function $\omega \mapsto f(\omega, \nu)$, $\omega \in \Omega$, is $\langle m, \phi' \rangle$-integrable, let

$$n(\nu, \phi'; E) = \int_{E} f(\omega, \nu) d\langle m(\omega), \phi' \rangle,$$

for every $E \in S$. If the function $\omega \mapsto f(\omega, \nu)$, $\omega \in \Omega$, is not $\langle m, \phi' \rangle$-integrable, let $n(\nu, \phi'; E) = 0$, for every $E \in S$. For every $\nu \in T$ and $E \in S$, let $n_{\nu}(E)$ be the element of $X'^{*}$ such that $\langle n_{\nu}(E), \phi' \rangle = n(\nu, \phi'; E)$, for every $\phi' \in H$. Because every element of $X'$ is a unique linear combination of a finite number of elements in $H$, this really defines $n_{\nu}(E)$ uniquely for every $\nu \in T$ and $E \in S$. Also, the application $E \mapsto \langle n_{\nu}(E), \phi' \rangle$, $E \in S$, is a $\sigma$-additive scalar measure for every $\phi' \in X'$, not only for $\phi' \in H$. Hence, $n_{\nu}: S \to X'^{*}$ is an $X'^{*}$-valued measure, for every $\nu \in T$, the space $X'^{*}$ being considered in the topology $\sigma(X'^{*}, X')$.

By the classical Fubini theorem,

$$\int_{G} f d\langle m \otimes \mu, \phi' \rangle = \int_{G} f d\langle m, \phi' \rangle \otimes \mu = \int_{\mathcal{E} \times T} \chi_{G} f d\langle m, \phi' \rangle \otimes \mu = \int_{T} (\int_{\Omega} \chi_{G}(\omega, \nu) f(\omega, \nu) d\langle m(\omega), \phi' \rangle) d\mu(\nu) = \int_{T} (\int_{\Omega} \chi_{G}(\omega, \nu) d\langle n_{\nu}(\omega), \phi' \rangle) d\mu(\nu),$$

for every $G \in S \otimes T$ and every $\phi' \in X'$, or

$$\int_{G} f d\mathcal{m} \otimes \mu = \int_{T} (\int_{\Omega} \chi_{G}(\omega, \nu) d\mathcal{n}_{\nu}(\omega)) d\mu(\nu),$$

for every $G \in S \otimes T$.

If we denote by $n$ the indefinite integral of the function $f$ with respect to the measure $m \otimes \mu$ and if we let $G_{\nu} = \{ \omega: (\omega, \nu) \in G \}$, for every $G \in S \otimes T$ and $\nu \in T$, the relation (2.6) can be written as
\[ n(G) = \int_T \mu(G) \, d\mu(u) \]

for every \( G \in S \otimes T \). That is, the family \( n_u, u \in T \), of measures represents a disintegration of the measure \( \mu \) with respect to \( \mu \). Hence, the measure \( n_u \) is uniquely determined for \( \mu \)-almost every \( u \in T \). So, for \( \mu \)-almost every \( u \in T \), the function \( \omega \mapsto f(\omega, u), \omega \in \Omega \), must be \(<m, \phi'>-integrable\) for every \( \phi' \) in a total set of functionals. This means, by the definition, that the function \( \omega \mapsto f(\omega, u), \omega \in \Omega \), is \( m \)-integrable in the relaxed sense, for \( \mu \)-almost every \( u \in T \).

The integrability of the function (2.4) and the equality (2.5) follow now by the application of linear functionals and the classical Fubini theorem and were, in fact, already established in the preceding reasonings.

It is sometimes desirable to have conditions for the validity of (2.1) expressed in terms of iterated integrals rather than the product measure. There is a variant of the Fubini theorem providing for such conditions in the case of two scalar measures. We reformulate it now for the case of one vector and one scalar measure.

**Proposition 5.** Let \( f \) be a scalar \( S \otimes T \)-measurable function on \( \Omega \times T \).

If, for \( \mu \)-almost every \( \omega \in \Omega \), the function \( u \mapsto f(\omega, u), u \in T \), is \( \mu \)-integrable and if the function

\begin{equation}
(2.7) \quad \omega \mapsto \int_T |f(\omega, u)| \, d|\mu|(u), \omega \in \Omega,
\end{equation}

is \( m \)-integrable in the strict sense, then the function \( f \) is \( m \otimes \mu \)-integrable in the strict sense.

If, for every \( G \in S \otimes T \), the function \( u \mapsto \chi_G(\omega, u)f(\omega, u), u \in T \), is \( \mu \)-integrable, for \( \mu \)-almost every \( \omega \in \Omega \), and the function
(2.8) \[ \omega \mapsto \int_T \chi_G(\omega, \nu) f(\omega, \nu) d\mu(\nu), \ \omega \in \Omega \]

is \( m \)-integrable in the strict sense, then the function \( f \) is \( m \otimes \mu \)-integrable in the strict sense.

If, for every \( G \in S \otimes T \), the function \( \omega \mapsto \chi_G(\omega, \nu) f(\omega, \nu), \ \omega \in \Omega \), is \( m \)-integrable in the relaxed sense, for \( \mu \)-almost every \( \nu \in T \), and the \( X' \)-valued function

\[ \nu \mapsto \int_\Omega \chi_G(\omega, \nu) f(\omega, \nu) d\mu(\omega), \ \nu \in T, \]

is \( \mu \)-integrable with the integral belonging to \( X \), then the function \( f \) is \( m \otimes \mu \)-integrable.

Proof. Using the decomposition of the function \( f \) on its positive and negative parts and of the measure \( \mu \) on its positive and negative variations, and the fact that a measurable function majored by an integrable one is integrable, we can show that the integrability of the function (2.7) is equivalent to the integrability of (2.8) for every \( G \in S \otimes T \). Now, under any of the conditions of the present Proposition, one can show, using Proposition 3 and the classical Fubini theorem, that, for any \( G \in S \otimes T \),

\[ \int_\Omega (\int_T \chi_G(\omega, \nu) f(\omega, \nu) d\mu(\nu)) d\mu(\omega) = \int_T (\int_\Omega \chi_G(\omega, \nu) f(\omega, \nu) d\mu(\omega)) d\mu(\nu) \]

is the vector in \( X \) equal to the integral of the function \( f \) with respect to the measure \( m \otimes \mu \).

3. **Operator Feynman-Kac Formula**

Let \( \Lambda \) be a locally compact Hausdorff space. By \( B(\Lambda) \) is denoted the family of all Baire sets in \( \Lambda \), the least \( \sigma \)-algebra with respect to which all bounded continuous functions on \( \Lambda \) are measurable.
Given $t \geq 0$, let $\Gamma_t$ be the set of all continuous functions $\gamma: [0,t] \to \Lambda$.

Let $n$ be a natural number and let $t_j \in [0,t]$, $j = 1,2,\ldots,n$, be numbers such that $t_{j-1} < t_j$, for $j = 2,\ldots,n$. For given $B_j \in \mathcal{B}(\Lambda)$, $j = 1,2,\ldots,n$, we denote

\[(3.1) \quad E_t(t_1,\ldots,t_n; B_1,\ldots,B_n) = \{ \gamma \in \Gamma_t, \gamma(t_j) \in B_j, j = 1,2,\ldots,n \} .\]

Let $S_t$ be the $\sigma$-algebra of sets in $\Gamma_t$ generated by the family of sets $E_t(t_1,\ldots,t_n; B_1,\ldots,B_n)$, for all choices of $B_j \in \mathcal{B}(\Lambda)$ and $t_j \in [0,t]$, $j = 1,2,\ldots,n$, such that $t_{j-1} < t_j$, for $j = 2,\ldots,n$, and for all choices of $n = 1,2,\ldots$.

The $\sigma$-algebras $S_t$ will be the domains of the operator-valued measures to be considered. The measures themselves will be constructed from two ingredients: a spectral measure on $\mathcal{B}(\Lambda)$ and a continuous semi-group of operators on $X$.

A continuous semi-group of operators acting on $X$ is a mapping $S: [0,\infty) \to L(X)$ such that $S(0) = I$; $S(t+s) = S(t)S(s)$, for every $t \geq 0$ and $s \geq 0$; and, for every $\phi \in X$, the mapping $t \mapsto S(t)\phi$, $t \in [0,\infty)$, from $[0,\infty)$ to $X$, is continuous.

If $A$ is the infinitesimal generator of the semi-group $S$, we write $S(t) = \exp(tA)$, for $t \geq 0$. To simplify the notation, the algebraic adjoint of the adjoint, $(S(t))^\dagger$, of the operator $S(t)$ is denoted by $S^\dagger(t)$, so that $S^\dagger(t) = (S(t))^\dagger$, for $t \geq 0$. The operator $S^\dagger(t)$ is the natural extension of the operator $S(t)$ onto the space $X^\dagger$.

Let $S: [0,\infty) \to L(X)$ be a continuous semi-group of operators, $P: \mathcal{B}(\Lambda) \to L(X)$ a spectral measure, and $t$ a non-negative number. The operator-valued measure $M_t: S_t \to L(X)$ such that

\[(3.2) \quad M_t(E) = S(t-t_n)P(B_n)S(t_{n-1}-t_{n-1})P(B_{n-1})\ldots P(B_2)S(t_2-t_1)P(B_1)S(t_1) , \]

whenever $E = E_t(t_1,\ldots,t_n; B_1,\ldots,B_n)$ for some $n = 1,2,\ldots$, some $B_j \in \mathcal{B}(\Lambda)$,
$t_j \in [0,t], \ j = 1,2,\ldots,n$, such that $t_{j-1} < t_j$, for $j = 2,\ldots,n$, will be termed the $(S,P,t)$-measure. It is an operator-valued analogue of the Wiener measure in the sense indicated in the introductory remarks.

The $(S,P,t)$-measure may not exist. However, if it does, then it is unique. Indeed, for every $\phi \in X$ and $\phi' \in X'$, the measure $\langle M_t \phi, \phi' \rangle$ is uniquely determined by its values on the family of sets of the form (3.1). Therefore, the operator-valued measure $M_t$ is determined uniquely by its values on these sets, that is, by the requirement (3.2).

The $(S,P,t)$-measures are the basis for a perturbation formula of Feynman-Kac type. We introduce now the last bit of notation needed for stating it.

Let $0 < s \leq \infty$. Given a function $V$ on $\Lambda \times [0,s)$ and a number $r \in [0,s)$, by $V_r$ is denoted the function on $\Lambda$ defined by $V_r(\lambda) = V(\lambda,r)$, for every $\lambda \in \Lambda$.

Theorem. Let $0 < s \leq \infty$. Assume that a continuous semi-group $S: [0,\infty) \rightarrow L(\Lambda)$ and a spectral measure $P: B(\Lambda) \rightarrow L(\chi)$ are given such that the $(S,P,t)$-measure $M_t: S_t \rightarrow L(\chi)$ exists for every $t \in [0,s)$.

If $\phi \in X$ and $V$ is a Baire function on $\Lambda \times [0,s]$ such that the function $f$ defined by

$$f(\gamma,q) = V(\gamma(q),q) \exp\left( \int_0^q V(\gamma(r),r)dr \right), \gamma \in \Gamma_t, \ q \in [0,t],$$

is $(M_t \phi \otimes 1_{\chi})$-integrable on $\Gamma_t \times [0,t]$, for every $t \in [0,s)$, then the integral

$$u(t) = \int_{\Gamma_t} \exp\left( \int_0^t V(\gamma(r),r)dr \right) d(M_t \phi)(\gamma)$$

exists in the strict sense and the equality

$$u(t) - \int_0^t S^t \phi(t-s)P(q)u(q)\,dq = S(t)\phi$$

holds for every $t \in [0,s)$.
Proof. Given \( t \in [0, s] \), we choose \( \Omega = \Gamma_t \), \( T = [0, t] \), \( S = S_t \), \( T = \mathcal{B}([0, t]) \), \( m = M_t \phi \), and \( \mu = \text{leb} \), in Proposition 4. It gives first that the integral (3.4) exists in the strict sense and

\[
\int_{\Gamma_t \times [0, t]} f (M_t \phi \cdot \text{leb}) = \int_{\Gamma_t} (\int_0^t f(\gamma, q) dq) dM_t(\gamma) \phi = \\
= \int_{\Gamma_t} (\exp(\int_0^t V(\gamma(r), r) dr) - 1) dM_t(\gamma) \phi = u(t) - S(t) \phi .
\]

Secondly, for almost every \( q \in [0, t] \),

\[
\int_{\Gamma_t} f(\gamma, q) dM_t(\gamma) \phi = S_1^*(t-q) P(V_q) u(q)
\]

with the integral existing, perhaps, in the relaxed sense only, and

\[
\int_{\Gamma_t \times [0, t]} f (M_t \phi \cdot \text{leb}) = \int_0^t S_1^*(t-q) P(V_q) u(q) dq .
\]

So, the equality (3.5) holds.

A few remarks are in place here.

Proposition 4 can often be used to guarantee the integrability of the function \( f \) with respect to \( M_t \phi \cdot \text{leb} \). For example, if the function \( V \) is real and of constant sign, then the existence of the integral (3.4) in the strict sense guarantees the integrability of \( f \). If the function \( V \) is bounded above, the Proposition 2 guarantees the integrability of \( f \) with respect to \( M_t \cdot \text{leb} \) or with respect to \( M_t \phi \cdot \text{leb} \), for every \( \phi \in X \).

If \( u(q) \) belongs to the domain of the operator \( P(V_q) \), that is, if \( P(V_q) u(q) \in X \), for almost every \( q \in [0, t] \), then the equation (3.5) can, of course, be written as

\[
(3.6) \quad u(t) = \int_0^t S(t-q) P(V_q) u(q) dq = S(t) \phi .
\]
If, moreover, \( u(t) \) belongs to the domain of the infinitesimal generator, \( A \), of the semigroup \( S \), then the validity of (3.6), for every \( t \in [0, s) \), implies that the function \( t \mapsto u(t), \ t \in (0, s) \), satisfies the initial-value problem

\[
(3.7) \quad u'(t) = Au(t) + P(V_t)u(t), \ t \in (0, s); \ \lim_{t \to 0^+} u(t) = u(0) = \phi.
\]

The fundamental solution of this problem is given by

\[
(3.8) \quad U(t) = \int_0^t \exp \left( \int_0^r V(\gamma(s), s) \, ds \right) dM_t(\gamma), \ t \in [0, s).
\]

The formula (3.4) or (3.8) gives a solution of a problem which may have a good meaning even if it cannot be obtained by solving (3.7). Such a problem can be described as follows.

The semi-group \( S \) is interpreted as a description of an evolution process in which any element \( \phi \) of the space \( X \) evolves during a time-interval of duration \( t \geq 0 \) into \( S(t)\phi \). Another process is described by the map \( T: [0, s) \to L(X) \), where

\[
(3.9) \quad T(t) = \exp( \int_0^t P(V_r) \, dr) = \int_\Lambda \exp( \int_0^t V(\lambda, r) \, dr) dP(\lambda), \ t \in [0, s).
\]

This is to say, we assume that the operators \( T(t) \) are well-defined by (3.9) and belong to \( L(X) \), for every \( t \in [0, s) \), and that any vector \( \phi \in X \) evolves in time \( t, \ t \in [0, s) \), by the action of \( T \) into the vector \( T(t)\phi \). If the operators \( P(V_r) \) do not depend on \( r \), that is, if \( V_r = V_0 \), for every \( r \), we can take \( s = \infty \), and \( T: [0, \infty) \to L(X) \) is a semi-group. More generally, if the operators (3.9) do not belong to \( L(X) \), we consider the evolution only of vectors \( \phi \in X \) such that \( T(t)\phi \in X, \ t \in [0, s) \).

The problem is to determine the vector of the space \( X \) into which a given vector \( \phi \in X \) evolves in a time \( t, \ t \in [0, s) \), if both processes go on simultaneously. In other words, we wish to construct a new process which is the superposition of the two processes.
If the function

\[(3.10)\] \[\gamma \mapsto \exp \left( \int_0^t V(\gamma(r), r) \, dr \right), \gamma \in \Gamma_t,\]

is $M_t$-integrable, for every $t \in [0, s)$, then the map $U: [0, s) \to L(X)$, where the operators $U(t)$, $t \in [0, s)$, are defined by (3.8), gives a meaningful description of the superposition of the processes $S$ and $T$ even if we cannot write

\[U(t) - \int_0^t S(t-q)P(V_q)U(q) \, dq = S(t), \quad t \in [0, s).\]

Or, given $\phi \in X$, the formula (3.4) gives the vector $u(t)$ into which $\phi$ evolves by the action of the combined process even if this vector cannot be obtained by solving the initial-value problem (3.7).

The difficulties with the formulation (3.7) may be caused by $u(t)$ not belonging to the domain of the infinitesimal generator $A$ or to the domain of $P(V_t)$. The difficulties with the domain of $A$ are removed in the formulation (3.6), but the difficulties with the domain $P(V_t)$ remain also in this formulation.

4. Diffusion

We return to the situation considered in the Introduction. We take $\Lambda = \mathbb{R}^\ell$. The space $X$ consists of all $\sigma$-additive scalar-valued measures on $B(\Lambda)$. The semi-group $S: [0, \infty) \to L(X)$ is defined by (0.2). For every $B \in B(\Lambda)$ and $\mu \in X$, the symbol $P(B)\mu$ will represent the measure such that $(P(B)\mu)(C) = \mu(B \cap C)$, for every $C \in B(\Lambda)$. This defines a spectral measure $P: B(\Lambda) \to L(X)$.

The integral, $P(V)$, of a Borel measurable function $V$, taken in the strict sense, is the operator assigning to every measure $\mu$ in its domain, the indefinite integral of $V$ with respect to $\mu$. On $L^1$ it acts as the operator of pointwise multiplication by $V$. Accepting the custom, we shall write simply $P(V) = V$.

Because we interpret the integral in the relaxed sense, $V_\mu$ represents an element of $X'^*$ for every $\mu \in X$ (Proposition 1).
For every $t \geq 0$, there exists the $(S,P,t)$-measure $M_t: S_t \to L(X)$. Indeed, the definition of $M_t(E)$, for $E = E_t(t_1, \ldots, t_n; B_1, \ldots, B_n)$ shows that $M_t(E)$ is the operator of convolution with a non-negative function belonging to $L^1(\mathbb{R}^n)$ which can be calculated in terms of the kernel $p$, the instants $t_1, \ldots, t_n$, and the sets $B_1, \ldots, B_n$. A calculation shows that the operator norm $\omega(E) = ||M_t(E)||$ is equal to the Wiener measure of the set $E$ corresponding to the Brownian motion with variance $2D$ starting at the origin. It then follows that this is the case also for every set in the algebra generated by the sets of the form $(3.1)$. Using now the continuity of $M_t$ with respect to the Wiener measure $\omega$ on this algebra, we deduce that $M_t$ can be extended uniquely onto the whole of the $\sigma$-algebra $S_t$, with the equality $\omega(E) = ||M_t(E)||$ remaining valid for every $E \in S_t$.

These reasonings imply that the measure $M_t: S_t \to L(X)$ is uniformly $\sigma$-additive, that is, $\sigma$-additive in the operator-norm topology, not merely strongly $\sigma$-additive.

Let $V$ be a measurable function on $\Lambda \times [0,\infty)$. Assume that the function $(3.3)$ is well-defined on $\Gamma_t \times [0,t]$, for every $t \geq 0$. If, for a given $\mu \in X$, the function $f$ is $M_t \mu \otimes \text{leb}$-integrable, for every $t \in [0,\infty)$, then we know, from Theorem 1, that

$$v_t = \int_{\Gamma_t} \exp(\int_0^t V(\gamma(r), r) dr) dM_t(\gamma) \mu$$

exists in the strict sense, so that $v_t \in X$, for every $t \in [0,\infty)$. Moreover, the equation

$$v_t - \int_0^t S^{t}(t-q)V_q v_q dq = S(t)\mu$$

is satisfied for every $t \in [0,\infty)$. If, for almost every $q \in [0,t]$, the indefinite integral $V_q v_q$ exists and belongs to $X$, then (4.2) is written as
(4.3) \[ \nu_t - \int_0^t S(t-q) V(q,q) \, dq = S(t) \mu \]

Let us observe that, no matter which measure \( \mu \in \mathcal{X} \) we take, the measure \( S(t) \mu \) is in \( L^1(\mathbb{R}^\mathbb{L}) \), for every \( t > 0 \). From this we deduce easily that \( M_t(g) \mu \) is in \( L^1(\mathbb{R}^\mathbb{L}) \) whenever \( t > 0 \) and \( g \) is an \( M_t \mu \)-integrable in the strict sense function on \( \Gamma_t \).

Hence, the measures given by (4.1) have densities for every \( t > 0 \). For any \( t > 0 \), let us denote the density of the measure \( \nu_t \) by \( u(x,t) \), \( x \in \mathbb{R}^\mathbb{L} \).

Then the equation (4.3) can be written as

\[ u(x,t) - \int_0^t \int_{\mathbb{R}^\mathbb{L}} p(x-y,t-q) V(y,q) \, dy \, dq = \int_{\mathbb{R}^\mathbb{L}} p(x-y,t) \, d\mu(y), \quad x \in \mathbb{R}^\mathbb{L}, \ t > 0. \]

This, in turn, is the problem (0.5) in the integral form.

The measures \( \nu_t \), defined by (4.1), represent a solution of a problem meaningfully stated either in probabilistic terms or in terms of diffusion or conduction of heat, also when the equation (4.3), let alone the problem (0.5), has no meaning.

If \( \mu \) is a probability measure, the measure \( S(t) \mu \) is a probability measure again, representing the probability distribution of the position at time \( t > 0 \) of a Brownian particle whose position at time 0 was distributed according to the law \( \mu \). Alternatively, if \( \mu \) represents the mass distribution of a substance in the space (filled by some solvent), then \( S(t) \mu \) represent the distribuiton of the substance after it was left to diffuse spontaneously for a time \( t > 0 \).

Assume that the substance is also created (say, by a reaction within the solvent) at a rate proportional to the amount already present, the coefficient of proportionality, \( V(x,t) \), depending on the place \( x \in \Lambda \) in the space and on time \( t \in [0,\infty) \). If the values of \( V \) are non-positive we should speak, more properly, about the destruction of the substance. It is easy to imagine such situation considering some radio-active substance, say.
In probabilistic terms, we assume that the Brownian particle is exposed to a risk of destruction characterized by a non-positive function \( V \) on \( \Lambda \times [0, \infty) \). The meaning of \( V \) is given by saying that, if the particle is at the point \( x \in \mathbb{R}^\mathcal{L} \) at the time \( t \geq 0 \), then the probability that it will survive there for a short period of time, \( \tau \), is equal to \( 1 + \tau V(x, t) + o(\tau), \tau \to 0 \).

It is clear that, if the function \( V \) has non-positive values, it should be possible to calculate, in principle, the distribution of the substance at any time \( t \geq 0 \), if the diffusion and the destruction processes are going on simultaneously. Or, for any \( t \geq 0 \) and \( B \in \mathcal{B}(\Lambda) \), there must be a number \( \nu_t(B) \) representing the probability that, at time \( t \), the particle will be found in the set \( B \), if it started a Brownian motion with an initial distribution \( \mu \) in the environment with the destruction risk characterized by a function \( V \). Indeed, the formula (4.1) gives the answer.

We may note that, for many functions \( V \) we have a perfectly sound solution \( \nu_t \) of the described problem, but it is not possible to find it by solving (0.5). If the function \( V_t \) is not integrable with respect to \( \nu_t \) for a non-negligible set of instants \( t \), as an alternative method to (4.1) for finding \( \nu_t \) we have to be content with (4.2). While it is possible, in this case, to replace (4.1) by integrals with respect to scalar measures (one for every \( B \in \mathcal{B}(\Lambda) \)) and retain a good probabilistic or physical meaning, the relation between (4.1) and (4.2) is best explained by operator-theoretic considerations.

5. Dynamical systems

Let first \( \mathcal{X} = \mathcal{L}^1_{\text{loc}}(\mathbb{R}^\mathcal{L}) \) be the space of (equivalence classes of) functions integrable with respect to the Lebesgue measure on every compact set in \( \mathbb{R}^\mathcal{L} \). The topology of \( \mathcal{X} \) is defined by declaring the family of sets

\[ \{ \phi \in \mathcal{X} : \int_B |\phi(x)| dx < \varepsilon \} , \]
for all compact sets $B \subset \mathbb{R}^L$ and all $\varepsilon > 0$, for a base of neighborhoods of 0 in $X$.

Let $t \mapsto \Sigma_t$, $t \in (-\infty, \infty)$, be a continuous group of autohomeomorphisms of the space $\mathbb{R}^L$. Define the semi-group $S: [0, \infty) \rightarrow L(X)$ by putting

\[(5.1) \quad S(t)\phi = \phi \circ \Sigma_t ,\]

for every $\phi \in X$ and every $t \in [0, \infty)$. Of course, we can define $S(t)$ by (5.1) also for $t < 0$ and obtain a group $S: (-\infty, \infty) \rightarrow L(X)$ of operators indexed by the whole of $(-\infty, \infty)$.

Let $\Lambda = \mathbb{R}^L$. For any set $B \in \mathcal{B}(\Lambda)$, let $P(B)$ be the operator of point-wise multiplication by the characteristic function of the set $B$. That is, $P(B)\phi = \chi_B\phi$, for every $\phi \in X$. So defined map $P: \mathcal{B}(\Lambda) \rightarrow L(X)$ is a spectral measure. The integral, $P(V)$, of a measurable function $V$ with respect to this spectral measure is the operator of point-wise multiplication by the function $V$. Therefore, we shall write simply $P(V) = V$.

Let $t \geq 0$. Given $\lambda \in \Lambda$, let

\[(5.2) \quad \gamma_\lambda(r) = \Sigma_{-r}\lambda, \quad r \in [0, t].\]

By the continuity of the group $\Sigma$, so defined map $\gamma_\lambda: [0, t] \rightarrow \Lambda$ is an element of $\Gamma_t$.

If $E \subset \Gamma_t$, let $B_E = \{\lambda: \gamma_\lambda \in E\}$ be the set of all points $\lambda \in \Lambda$ such that the path $\gamma_\lambda$ belongs to $E$.

For every $E \subset S_t$, the set $B_E$ belongs to $\mathcal{B}(\Lambda)$. Indeed, the family of sets $E \subset \Gamma_t$ such that $B_E \in \mathcal{B}(\Lambda)$ is a $\sigma$-algebra which contains every set of the form (3.1).

Now, let

$$M_t(E) = S(t)P(B_E) ,$$
for every set $E \in S_t$. It is a matter of easy checking that $M_t: S_t \to L(X)$ is the $(S,P,t)$-measure.

Let $V$ be a Borel measurable function on $\mathbb{R}^l \times [0,\infty)$. Let

$$W_q(x) = \exp \left( \int_0^q V(x,r)dr \right), \quad x \in \mathbb{R}^l, \quad q \in [0,t].$$

Let $\phi \in X$. If $W_q\phi \in X$, for every $q \in [0,t]$, then the function (3.10) is $M_t\phi$-integrable and

$$\int_0^t \exp( \int_0^q V(\gamma(r),r)dr) dM_t(\gamma)\phi = S(t)(W_t\phi) = (W_t\phi) \circ \Sigma_t.$$

Let us consider an interesting special case when $\Sigma$ is the fundamental solution of a dynamical system $\dot{x} = a(x)$, where $a: \mathbb{R}^l \to \mathbb{R}^l$ is a mapping with components $a_1, \ldots, a_l$. It means that, for every $\lambda \in \mathbb{R}^l$, the function $t \mapsto \Sigma_t\lambda$, $t \in (-\infty,\infty)$, is the solution of this system passing through $\lambda$ at $t = 0$. In this case, the infinitesimal generator of the semi-group $S$ is the differential operator

$$A = \sum_{j=1}^l a_j \frac{\partial}{\partial x_j}.$$

If the function $\phi$ is smooth enough and $u(x,t) = W_t(\Sigma_t x)\phi(\Sigma_t x)$, $x \in \mathbb{R}^l$, $t > 0$, then $u$ is the solution of the problem

$$\frac{\partial u}{\partial t} = \sum_{j=1}^l a_j \frac{\partial}{\partial x_j} \cdot Vu, \quad x \in \mathbb{R}^l, \quad t > 0; \quad \lim_{t \to 0^+} u(x,t) = u(x,0) = \phi(x), \quad x \in \mathbb{R}^l.$$

This example admits many variations. For example, the choice of the space $X$ is not at all crucial. The situation does not change much if, instead of $L^1_{loc}$, the space $L^2$, or $L^p$ for some $p \in [1,\infty)$, or even some space of measures on $S(\mathbb{R}^l)$, is taken. Other variations of importance can be obtained by considering, instead of $\mathbb{R}^l$, some other manifold with a group of automorphism acting on it.
The \((S,P,t)\)-measures, related to any of the suggested variations, can be considered rather 'degenerate'. However, using them as building blocks, a quite complex \((S,P,t)\)-measures can be constructed. Let us consider a sample example.

The definition of the space \(X\) is somewhat involved. Every vector of the space \(X\) is represented by a function \(\phi\) on \(\mathbb{R}^l \times [0,1]\) such that

(i) for every \(y \in [0,1]\), the function \(x \mapsto \phi(x,y), x \in \mathbb{R}^l\), belongs to \(L^1_{\text{loc}}(\mathbb{R}^l)\); and

(ii) for every compact set \(B \subset \mathbb{R}^l\), the function

\[
y \mapsto \int_B \phi(x,y) \, dx, \quad y \in [0,1],
\]

is Leb-integrable on \([0,1]\).

A function \(\phi\) on \(\mathbb{R}^l \times [0,1]\) with properties (i) and (ii) is said to represent the zero-vector of the space \(X\) if

\[
\int_0^1 (\int_B |\phi(x,y)| \, dx) \, dy = 0,
\]

for every compact set \(B \subset \mathbb{R}^l\).

The space \(X\) is the linear space, with respect to natural linear operations, of all function \(\phi\) on \(\mathbb{R}^l \times [0,1]\) with the properties (i) and (ii) modulo the subspace of functions representing the zero-vector. Following the usual conventions, it is not distinguished between a vector in \(X\) and a function \(\phi\) representing it.

The topology in the space \(X\) is defined by declaring the family of sets

\[
\{\phi \in X: \int_0^1 \int_B |\phi(x,y)| \, dx \, dy < \varepsilon\}
\]

for every compact \(B \subset \mathbb{R}^l\) and every \(\varepsilon > 0\), for a base of neighborhoods in the space \(X\).

The definition of \(X\) could have been simplified using the notion of the Bochner integral. This suggests possible variation and generalizations.
For every \( y \in [0,1] \), let \( t \mapsto \Sigma^y_t, t \in (-\infty, \infty) \), be a group of autohomeomorphisms of \( \mathbb{R}^l \). Assume that the map

\[
(x,y,t) \mapsto \Sigma^y_t x, \quad (x,y,t) \in \mathbb{R}^l \times [0,1] \times (-\infty, \infty),
\]
is continuous on the product-space \( \mathbb{R}^l \times [0,1] \times (-\infty, \infty) \).

Define the group \( S: (-\infty, \infty) \to L(X) \) by

\[
(S(t)\phi)(x,y) = \phi(\Sigma^y_t x, y), \quad t \in (-\infty, \infty), \quad \phi \in X, \quad x \in \mathbb{R}^l, y \in [0,1].
\]

Let \( \Lambda = \mathbb{R}^l \). Let \( P: \mathcal{B}(\Lambda) \to L(X) \) be the spectral measure defined by

\[
(P(B)\phi)(x,y) = \chi_B(x)\phi(x,y), \quad B \in \mathcal{B}(\Lambda), \quad \phi \in X, \quad x \in \mathbb{R}^l, y \in [0,1].
\]

Let \( t \geq 0 \). For any \( \lambda \in \Lambda \) and \( y \in [0,1] \), let

\[
\gamma^y_\lambda(r) = \Sigma^y_{-r}\lambda, \quad r \in [0,t].
\]

For \( E \subset \Gamma_t \), let \( B^y_E = \{ \lambda: \gamma^y_\lambda \in E \} \). For every \( y \in [0,1] \) and \( E \in S_t \), the set \( B^y_E \) belongs to \( \mathcal{B}(\Lambda) \).

Now, for every \( E \in S_t \) and \( \phi \in X \), let \( M_t(E)\phi \) be the vector such that

\[
(S(-t)M_t(E)\phi)(x,y) = \chi_{B^y_E}(x)\phi(x,y), \quad x \in \mathbb{R}^l, y \in [0,1].
\]

It is a matter of a direct calculation that this defines the \((S,P,t)\)-measure \( M_t: S_t \to L(X) \).
References

