ESTIMATING VARIANCE COMPONENTS
FOR THE NESTED MODEL, MINQUE
OR RESTRICTED MAXIMUM LIKELIHOOD

by

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ABSTRACT

This paper outlines an efficient method for computing MINQUE or restricted maximum likelihood estimates of variance components for the nested or hierarchical classification model. The method does not require storage of any large matrices and consequently does not impose limits on the size of the data set. Numerical examples indicate that the computations converge very rapidly.
Estimating Variance Components for the Nested Model,
MINQUE or Restricted Maximum Likelihood

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I. Introduction

In a series of papers Rao (1970, 1971a, 1971b, 1972) has developed the MINQUE (MINimum Norm Quadratic Unbiased Estimation) theory of estimating variance components. These papers provide a complete solution to the problem in the sense that if normality holds then the method will give locally best (minimum variance) quadratic unbiased estimates of the variance components. Unfortunately, the form of the estimation equations given is such as to tax even a very large computer if one has a reasonably large data set since they involve inversion of an \( n \times n \) matrix where \( n \) is the number of data points. Also the procedure requires prior information about the relative magnitudes of the variance components. The possibility of an iterative scheme seems reasonable, but places an even larger premium on efficient computations.

Hemmerle and Hartley (1973) develop a method for computing maximum likelihood estimates of the variance components. Corbeil and Searle (1976) modify the Hemmerle and Hartley algorithm to give restricted maximum likelihood estimates by partitioning the likelihood into two parts and then maximizing the part which is free of the fixed effects. However, both algorithms require the repeated inversion of a \( q \times q \) matrix, where \( q \) is the total number of random levels in the model. Typically \( q \) will be much smaller than \( n \), though for large data sets even that rapidly becomes unreasonable. Both algorithms are iterative and as such require only reasonable initial starting values for the unknown variance components rather than prior estimates.

The object of this paper is to show that for the hierarchal model the
MINQUE formulas can be re-written in a form that requires only that a $p \times p$ matrix be inverted, where $p$ is the number of variance components or levels of nesting in the model. It is shown that all necessary totals can be obtained by one pass through the data set. It is also shown that if the MINQUE process is iterated (with some provision to replace negative components with small positive values) and converges, then the final answers will also satisfy the equations obtained by applying the restricted maximum likelihood method.

II. MINQUE With Invariance and Normality

Following Rao (1971a, 1971b) let $Y$ be an $n$-vector of observations with structure

$$Y = X\beta + U_1\varepsilon_1 + \ldots + U_p\varepsilon_p$$

(1)

where $X$ is a known $n \times s$ matrix of rank $s$, $\beta$ is an $s$-vector of unknown parameters, $U_i$ is a known $n \times c_i$ matrix and $\varepsilon_i$ is a $c_i$-vector of normal random variables with zero means and dispersion matrix $\Sigma_i$ of appropriate dimension. It is also assumed that $\varepsilon_i$ and $\varepsilon_j$ are uncorrelated for all $i \neq j$. Clearly

$$E[Y] = X\beta$$

and

$$D[Y] = \sigma_1^2 V_1 + \ldots + \sigma_p^2 V_p$$

where

$$V_i = U_i U_i'$$

for $i = 1, \ldots, p$.

The quadratic, unbiased and invariant (invariant in the sense that the estimates are unchanged if we replace the vector of observations $Y$ by $Y + Xb$ for any $s$-vector $b$) estimate of

$$\sigma_1^2 V_1 + \ldots + \sigma_p^2 V_p$$

given by the MINQUE principle is

$$\lambda_1 Q_1 + \ldots + \lambda_p Q_p$$
where
\[ Q_1 = Y'RV_1RV, \]
\[ R = V^{*-1}(I-P), \]
\[ P = X(X'V^{*-1}X)^{-1}X'V^{*-1}, \]
\[ V^* = V_1 + \ldots + V_p \]

and the \( \{\lambda_i\} \) are suitable (derivable) constants for arbitrary \( \{p_i\} \).
Replacing \( V^* \) by
\[ V = \alpha_1V_1 + \ldots + \alpha_pV_p \]
amounts to selecting a different norm and disturbs neither the unbiasedness nor invariance. Further, if the \( \{\alpha_i\} \) happen to be proportional to the true but unknown \( \{\sigma_i^2\} \) then the estimates obtained will also be the minimum variance quadratic unbiased invariant estimates at that particular point in the parameter space.

If all variance components are to be estimated simultaneously, rather than only a particular linear function, then the \( \{\lambda_i\} \) need never be computed. The procedure is to compute the \( \{Q_i\} \), equate to their expectations and solve the resulting system of equations. This can be written as

\[
\begin{bmatrix}
S_{11} & S_{12} & \cdots & S_{1p} \\
S_{21} & S_{22} & \cdots & S_{2p} \\
\vdots & \vdots & \ddots & \vdots \\
S_{p1} & S_{p2} & \cdots & S_{pp}
\end{bmatrix}
\begin{bmatrix}
\sigma_1^2 \\
\sigma_2^2 \\
\vdots \\
\sigma_p^2
\end{bmatrix}
= \begin{bmatrix}
Q_1 \\
Q_2 \\
\vdots \\
Q_p
\end{bmatrix}
\tag{2}
\]

where \( S_{ij} = tr(RV_iRV_j) \).

It seems reasonable that if prior knowledge of the relative magnitudes of
the unknown variance components is available then this information should be used to define a set of \( \{ \alpha_i \} \) and the matrix \( V^* \) replaced by \( V \) in (2). Since (2) is a linear equation and the right hand sides are quadratic forms, the variance-covariance matrix of the estimated variance components follows immediately (in terms of the true variance components and \( \{ \alpha_i \} \)).

Formally this provides a complete solution for the variance component estimation problem. However the computational aspects of obtaining the elements of (2) directly are formidable. They involve inverting an \( n \times n \) matrix. Consequently the examination of special cases is in order. Of particular interest here is the nested analysis of variance model, since La Motte (1972) has obtained \( V^{-1} \) in a manageable form.

III. Connection with Restricted Maximum Likelihood

In order to obtain the maximum likelihood estimates of the variance components we proceed in two steps. The first is to assume \( \sigma_i^2 = \alpha_i \) for \( i = 1, \ldots, p \) and estimate \( \beta \). This yields

\[
\hat{\beta} = (X'V^{-1}X)^{-1}X'V^{-1}Y.
\]

The second step is to compute residuals

\[
Z = Y - X\hat{\beta} = Q_V Y
\]

and apply maximum likelihood to obtain estimates of \( \{ \sigma_i^2 \} \).

Define

\[
Z^* = T_V Y
\]

where \( T_V \) consists of \( n - s \) linearly independent rows of \( Q_V \). The likelihood can now be written as

\[
L = \frac{K}{|T_V VT_V'|^{\frac{p}{2}}} \exp\left\{ -\frac{1}{2} Y' T_V (T_V VT_V')^{-1} T_V Y \right\}.
\]

Treating \( T_V \) as a constant (the \( \{ \sigma_i^2 \} \) have been replaced by \( \{ \alpha_i \} \)) gives the
likelihood equations,
\[
\frac{d \ln f}{d \sigma_i^2} = -\frac{1}{2} \text{tr}[(T_v V_i T'_v)'^{-1} T_v V_i T'_v + \frac{1}{2} V'_v (T_v V_i T'_v)'^{-1} T_v V_i (T_v V_i T'_v)'^{-1} T_v T'_v]
\]
for \( i = 1, 2, \ldots, p \).

Equating the partial derivatives to zero and rearranging terms leads to
\[
\text{tr}[T'_v (T_v V_i T'_v)'^{-1} T_v V_i] = Y'_v (T_v V_i T'_v)'^{-1} T_v V_i (T_v V_i T'_v)'^{-1} T_v Y
\]
for \( i = 1, 2, \ldots, p \).

Khatri (1966) has shown that for any symmetric positive definite matrix \( C \),
\[
T'_c (T_c C T'_c)'^{-1} T_c = C^{-1} Q_c.
\]

Direct substitution into (3) gives
\[
\text{tr}(V^{-1} Q V_i) = Y'_v V^{-1} V_i V^{-1} Q_v Y
\]
for \( i = 1, 2, \ldots, p \). Also we can replace \( V \) by \( \hat{V} \) to obtain
\[
\text{tr}(\hat{V}^{-1} Q V_i) = Y'_v \hat{V}^{-1} V_i \hat{V}^{-1} Q_v Y
\]
for \( i = 1, 2, \ldots, p \).

The quadratic forms in (4) are similar to those required by MINQUE, with the only difference being the nature of \( V \) or \( \hat{V} \). In MINQUE theory \( V \) is a function of \( \{\alpha_i\} \) while in maximum likelihood estimation theory \( \hat{V} \) is a function of \( \{\sigma_i^2\} \). Also if the \( \{\alpha_i\} \) are chosen correctly i.e. agree with \( \{\sigma_i^2\} \) then the expected values of the quadratic forms obtained under the MINQUE principle can be written as
\[
E[Y'_v V^{-1} V_i V^{-1} Q_v Y] = \sum_j \text{tr}(Q'_v V^{-1} V_i V^{-1} Q_v V_j) \sigma_j^2
\]
\[
= \text{tr}(Q'_v V^{-1} V_i V^{-1} Q_v V)
\]
\[
= \text{tr}(V^{-1} Q_v V_i).
\]
It follows immediately that if one uses an iterative scheme for solving the maximum likelihood equations in which the \( \{ \alpha_i \} \) are updated to agree with the \( \{ \sigma_i^2 \} \) (with some provision to replace negative values with some small positive constants) the solution will agree exactly with that obtained from the MINQUE equations using the same rules for iteration.

IV. Estimation in the Nested Model

The development in sections II and III has been in terms of a relatively general linear model. For the remainder of this paper we will specialize to the 3-level nested model with only the one fixed parameter \( \mu \). The 3-level nested model was selected because it is sufficiently complex to display the general pattern of terms that arise when more levels are introduced without the notation becoming too complex. The extension to the case where the random effects are nested within levels of a fixed effect is obvious and will not be discussed. The 3-level nested model is given by

\[
y_{hijk} = \mu + a_h + b_{hi} + c_{hij} + e_{hijk} \tag{5}
\]

for \( k = 1, 2, \ldots, n_{hij} \)

\( j = 1, 2, \ldots, m_{hi} \)

\( i = 1, 2, \ldots, m_h \)

\( h = 1, 2, \ldots, m \)

and where

\( \{ y_{hijk} \} \) are the observations,

\( \mu \) is an unknown constant,

\( \{ a_h \} \) are independent \( N(0, \sigma_1^2) \),

\( \{ b_{hi} \} \) are independent \( N(0, \sigma_2^2) \),

\( \{ c_{hij} \} \) are independent \( N(0, \sigma_3^2) \),

\( \{ e_{hijk} \} \) are independent \( N(0, \sigma_4^2) \) and
the latter 4 sets are also mutually independent.

Since we will have occasion to refer to the value obtained by summation over certain subscripts, we will use the convention of indicating such a sum by replacing the subscript with the + symbol. For example

\[ n_{hi+} = \Sigma_{j} n_{hij} \]

and

\[ n_{h++} = \Sigma_{i} n_{hi+} \]

etc.

To establish the connection between (1) and (5), note that \( Y \) corresponds to the \( n_{++} \) elements \( \{y_{hijk}\} \) in dictionary order. \( X \) is a column of 1's and \( \beta \) the one element \( \gamma \). \( U_{1} \) corresponds to an \( n_{++} \times m \) matrix of 0's and 1's, \( U_{2} \) to an \( n_{++} \times m_{+} \) matrix of 0's and 1's, \( U_{3} \) to an \( n_{++} \times m_{+} \) matrix and \( U_{4} \) the \( n_{++} \times n_{++} \) identity matrix. The covariance between \( y_{hijk} \) and \( y_{hi’j’k’} \) can be written as

\[ \delta_{hh^{'}}(\sigma_{1}^{2} + \delta_{ii^{'}}(\sigma_{2}^{2} + \delta_{jj^{'}}(\sigma_{3}^{2} + \delta_{kk^{'}}(\sigma_{4}^{2}))) \]

where

\[ \delta_{\lambda m} = \begin{cases} 1 & \text{if } \lambda = m \\ 0 & \text{if } \lambda \neq m \end{cases} \]

The corresponding element of \( V \) can be written as

\[ v_{hijk, h'i'j'k'} = \delta_{hh^{'}}(\sigma_{1} + \delta_{ii^{'}}(\sigma_{2} + \delta_{jj^{'}}(\sigma_{3} + \delta_{kk^{'}}(\sigma_{4}))) \]

where \( \alpha_{i} \) represents the prior estimate for \( \sigma_{i}^{2} \). Since only the relative magnitudes of the \( \{\alpha_{i}\} \) are important they can be scaled so that one of them is 1. Also if one wishes the more primitive form of the MINQUE, Rao (1972), all can be set all equal to 1. The extension to more levels of nesting is immediate. The corresponding element of \( V^{-1} \) can be written as
\[
\delta_{hh} \delta_{ii} \delta_{jj} \delta_{kk} \alpha_{4}^{-1} - \delta_{hh} \delta_{ii} \delta_{jj} \omega_{hi} \omega_{h} \alpha_{4}^{-1}
\]
\[
- \delta_{hh} \delta_{ii} \omega_{hi} \omega_{hi} \alpha_{3} \omega_{h}^{-1}
\]
\[
- \delta_{hh} \omega_{hi} \omega_{hi} \omega_{i} \omega_{i} \alpha_{2} \omega_{h}^{-1}
\]

where

\[
\omega_{hi} = \alpha_{3} (\alpha_{3} \omega_{hi} + \alpha_{4})^{-1},
\]
\[
\omega_{hi} = \alpha_{2} (\alpha_{2} \omega_{hi} + \alpha_{3})^{-1},
\]
\[
\omega_{hi} = \alpha_{1} (\alpha_{1} \omega_{hi} + \alpha_{2})^{-1},
\]
\[
\omega = \left( \sum_{h} \omega_{hi} \omega_{h} \right)^{-1},
\]
\[
\omega_{hi} = \sum_{j} \omega_{hi} \omega_{hj},
\]

and

\[
\omega_{hi} = \sum_{i} \omega_{hi} \omega_{hi}.
\]

Summation over \(h', i', j', k'\) yields

\[
\alpha_{1}^{-1} \omega_{hi} \omega_{hi} \omega_{h}
\]

as the element of the vector \(V^{-1}X\) corresponding to \(y_{hijk}\) in \(Y\). Also note that the scalar \(X'V^{-1}X\) is \(\alpha_{1}^{-1} \omega_{hi}^{-1}\). The general element of \(R\) now follows as

\[
\delta_{hh} \delta_{ii} \delta_{jj} \delta_{kk} \alpha_{h}^{-1} - \delta_{hh} \delta_{ii} \delta_{jj} \omega_{hi} \omega_{hi} \alpha_{4}^{-1}
\]
\[
- \delta_{hh} \delta_{ii} \omega_{hi} \omega_{hi} \omega_{i} \omega_{i} \alpha_{3} \omega_{h}^{-1}
\]
\[
- \delta_{hh} \omega_{hi} \omega_{hi} \omega_{h} \omega_{h} \omega_{h} \omega_{i} \omega_{i} \omega_{i} \omega_{i} \alpha_{2} \omega_{h}^{-1}
\]
\[
- \omega_{hi} \omega_{hi} \omega_{h} \omega_{h} \omega_{h} \omega_{h} \omega_{i} \omega_{i} \omega_{i} \omega_{i} \alpha_{1}^{-1}.
\]

The pattern for the case with more levels is obvious.

The typical element of \(Z = RX\) is
\[ z_{hijk} = y_{hijk}^{-1} - w_{hij} y_{hij}^{-1} \]
\[ = w_{hij} w_{hi} (\Sigma_{j} w_{hij} y_{hij}^{-1})^{-1} \]
\[ - w_{hij} w_{hi} w_{h'ij} y_{hij}^{-1} \]
\[ - w_{hij} w_{hi} w_{h'ij} (\Sigma_{j} w_{hij} y_{hij}^{-1})^{-1} \]
\[ , \]

where \[ y_{hij} = \Sigma_{k} y_{hijk} . \]

The quadratic forms in (2) are now obtained as

\[ \Sigma_{h} \Sigma_{i} \Sigma_{j} \Sigma_{k} z_{hijk}^{2} , \quad \Sigma_{h} \Sigma_{i} z_{hij}^{2} , \]
\[ \Sigma_{h} z_{hi}^{2} \quad \text{and} \quad \Sigma_{h} z_{hi}^{2} , \]

where the + symbol indicates summation over the subscript before squaring. We note at this point that these quadratic forms are obtained by two passes through the data file and are not affected by size of data set.

In order to obtain expressions for the elements in \( RV_{1} \), \( RV_{2} \), \( RV_{3} \), \( RV_{4} \) and the various products in a systematic pattern it is convenient to define

\[ t_{h} = \alpha_{2}^{-1} + w_{h} \alpha_{1}^{-1} , \]
\[ t_{hi} = \alpha_{3}^{-1} + w_{hi} t_{h} \quad \text{and} \quad \]
\[ t_{hij} = \alpha_{4}^{-1} + w_{hij} t_{hi} . \]

The elements in \( RV_{1} \), \( RV_{2} \), \( RV_{3} \) and \( RV_{4} \) are given in Table 1 in a convenient form. Expressions for \( tr(RV_{i} RV_{j}) \) for \( l \geq i \geq j \geq 1 \) are given in Appendix A. These expressions can (and should) be condensed before being used for computations. However when this is done the pattern is not as clear. We also
notice that if one arrives at a point where the estimates obtained agree with the prior values then the variance-covariance matrix of the estimates can be approximated by twice the inverse of the left side of 2.
Table I. Elements in the $RV_1$, $RV_2$, $RV_3$, and $RV_4$ Matrices

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>$RV_4$</th>
<th>$RV_3$</th>
<th>$RV_2$</th>
<th>$RV_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta_{hh}^{\delta_{11}}\delta_{jj}^{\delta_{kk}}$</td>
<td>$\omega_{ij}^{-1}v_{ij}h_{ij}$</td>
<td>$v_{ij}h_{ij}(\sigma_1^{-1}n_{ij}v_{ij}h_{ij}h_{ij})$</td>
<td>$\omega_{ij}v_{ij}(\sigma_2^{-1}n_{ij}v_{ij}h_{ij})$</td>
<td>$\omega_{ij}v_{ij}(\sigma_1^{-1}n_{ij}v_{ij}w_{ij}^{-1})$</td>
</tr>
<tr>
<td>$\delta_{hh}^{\delta_{11}}\delta_{jj}^{(1-\delta_{kk})}$</td>
<td>$-\omega_{ij}v_{ij}h_{ij}$</td>
<td>$v_{ij}h_{ij}(\sigma_1^{-1}n_{ij}v_{ij}h_{ij}h_{ij})$</td>
<td>$\omega_{ij}v_{ij}(\sigma_2^{-1}n_{ij}v_{ij}h_{ij})$</td>
<td>$\omega_{ij}v_{ij}(\sigma_1^{-1}n_{ij}v_{ij}w_{ij}^{-1})$</td>
</tr>
<tr>
<td>$\delta_{hh}^{\delta_{11}}(1-\delta_{jj})$</td>
<td>$-\omega_{ij}v_{ij}v_{ij}h_{ij}h_{ij}h_{ij}$</td>
<td>$-\omega_{ij}v_{ij}v_{ij}h_{ij}h_{ij}h_{ij}h_{ij}$</td>
<td>$\omega_{ij}v_{ij}(\sigma_1^{-1}n_{ij}v_{ij}h_{ij})$</td>
<td>$\omega_{ij}v_{ij}(\sigma_1^{-1}n_{ij}v_{ij}w_{ij}^{-1})$</td>
</tr>
<tr>
<td>$\delta_{hh}(1-\delta_{11})$</td>
<td>$-\omega_{ij}v_{ij}v_{ij}v_{ij}h_{ij}h_{ij}h_{ij}$</td>
<td>$-\omega_{ij}v_{ij}v_{ij}v_{ij}h_{ij}h_{ij}h_{ij}h_{ij}h_{ij}$</td>
<td>$-\omega_{ij}v_{ij}v_{ij}v_{ij}h_{ij}h_{ij}h_{ij}h_{ij}h_{ij}$</td>
<td>$\omega_{ij}v_{ij}(\sigma_1^{-1}n_{ij}v_{ij}w_{ij}^{-1})$</td>
</tr>
<tr>
<td>$(1-\delta_{hh})$</td>
<td>$-\omega_{ij}v_{ij}v_{ij}v_{ij}v_{ij}v_{ij}h_{ij}h_{ij}h_{ij}h_{ij}h_{ij}$</td>
<td>$-\omega_{ij}v_{ij}v_{ij}v_{ij}v_{ij}v_{ij}h_{ij}h_{ij}h_{ij}h_{ij}h_{ij}h_{ij}$</td>
<td>$-\omega_{ij}v_{ij}v_{ij}v_{ij}v_{ij}v_{ij}h_{ij}h_{ij}h_{ij}h_{ij}h_{ij}h_{ij}h_{ij}$</td>
<td>$-\omega_{ij}v_{ij}v_{ij}v_{ij}v_{ij}v_{ij}h_{ij}h_{ij}h_{ij}h_{ij}h_{ij}h_{ij}h_{ij}h_{ij}$</td>
</tr>
</tbody>
</table>
V. Computational Aspects

An examination of the formulas developed in the previous section shows that a very efficient computing routine is possible. If the data are sorted or arranged according to the nesting structure then all totals required in equation (2) can be obtained in one pass. Alternatively, if one wishes to minimize the programming effort, all the accumulations can be obtained rather easily by using any one of a number of standard programs for computing means and totals of subsets of observations in a data set. Summaries need to be obtained at each level of nesting. These summaries are then merged and a final set of accumulations obtained.

In order to examine the behavior of the suggested iterative scheme several analyses were performed on synthetic, unbalanced data sets. The results of three typical cases are given in Table II. The striking feature of these computations is the very rapid convergence. In fact it appears that 2 or at most 3 cycles are sufficient. The extent of the unbalance in the data can be judged from Table III.

VI. Acknowledgement

The authors wish to thank T. M. Gerig for his valuable comments and suggestions in the preparation of this paper.
### TABLE II

Summary of Three Analyses

<table>
<thead>
<tr>
<th></th>
<th>$\sigma_1^2$</th>
<th>$\sigma_2^2$</th>
<th>$\sigma_3^2$</th>
<th>$\sigma_4^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>A. True parameter values</strong>&lt;sup&gt;*&lt;/sup&gt;</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Starting values</strong></td>
<td>2.</td>
<td>1.</td>
<td>5.</td>
<td>1.</td>
</tr>
<tr>
<td>Estimates first cycle</td>
<td>4.289</td>
<td>.006</td>
<td>.555</td>
<td>1.157</td>
</tr>
<tr>
<td>Estimates second cycle</td>
<td>4.146</td>
<td>-.015</td>
<td>.231</td>
<td>1.140</td>
</tr>
<tr>
<td>Estimates third cycle</td>
<td>4.133</td>
<td>-.011</td>
<td>.223</td>
<td>1.144</td>
</tr>
<tr>
<td>Estimates fourth cycle</td>
<td>4.134</td>
<td>-.012</td>
<td>.224</td>
<td>1.143</td>
</tr>
<tr>
<td>Estimates fifth cycle</td>
<td>4.134</td>
<td>-.012</td>
<td>.224</td>
<td>1.143</td>
</tr>
<tr>
<td>Estimates A.o.V.</td>
<td>4.398</td>
<td>-.061</td>
<td>.130</td>
<td>1.173</td>
</tr>
<tr>
<td><strong>B. True parameter values</strong>&lt;sup&gt;*&lt;/sup&gt;</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Starting values</strong></td>
<td>2.</td>
<td>1.</td>
<td>5.</td>
<td>1.</td>
</tr>
<tr>
<td>Estimates first cycle</td>
<td>5.</td>
<td>1.</td>
<td>1.</td>
<td>1.</td>
</tr>
<tr>
<td>Estimates second cycle</td>
<td>4.292</td>
<td>.039</td>
<td>.211</td>
<td>1.128</td>
</tr>
<tr>
<td>Estimates third cycle</td>
<td>4.149</td>
<td>-.006</td>
<td>.220</td>
<td>1.142</td>
</tr>
<tr>
<td>Estimates fourth cycle</td>
<td>4.135</td>
<td>-.012</td>
<td>.224</td>
<td>1.143</td>
</tr>
<tr>
<td>Estimates fifth cycle</td>
<td>4.134</td>
<td>-.012</td>
<td>.224</td>
<td>1.143</td>
</tr>
<tr>
<td><strong>C. True parameter values</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Starting values</strong></td>
<td>2.</td>
<td>1.</td>
<td>5.</td>
<td>1.</td>
</tr>
<tr>
<td>Estimates first cycle</td>
<td>3.187</td>
<td>.017</td>
<td>.354</td>
<td>1.135</td>
</tr>
<tr>
<td>Estimates second cycle</td>
<td>3.255</td>
<td>.003</td>
<td>.330</td>
<td>1.158</td>
</tr>
<tr>
<td>Estimates third cycle</td>
<td>3.260</td>
<td>-.005</td>
<td>.332</td>
<td>1.158</td>
</tr>
<tr>
<td>Estimates fourth cycle</td>
<td>3.260</td>
<td>-.005</td>
<td>.332</td>
<td>1.158</td>
</tr>
<tr>
<td>Estimates fifth cycle</td>
<td>3.260</td>
<td>-.005</td>
<td>.332</td>
<td>1.158</td>
</tr>
<tr>
<td>Estimates A.o.V.</td>
<td>3.102</td>
<td>.014</td>
<td>.318</td>
<td>1.158</td>
</tr>
</tbody>
</table>

* These two used a common data set.
<table>
<thead>
<tr>
<th>Sets</th>
<th>A &amp; B</th>
<th>C</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
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REFERENCES


APPENDIX

\[ \text{tr}(RV_4RV_4) = \sum \sum n_{hij}(\alpha_4^{-1} - w_{hij}t_{hij})^2 + \sum \sum n_{hij}(n_{hij} - 1)^2w_{hij}t_{hij} \]

\[ + \sum \sum \sum ((\sum n_{hij}w_{hij}^2 - n_{hij}w_{hij})n_{hij}w_{hij}w_{hij}^2t_{hij} \]

\[ + \sum \sum ((\sum n_{hij}w_{hij}^2 i_{hij} - \sum n_{hij}w_{hij}^2 i_{hij})(\sum n_{hij}w_{hij}^2 i_{hij})(\sum n_{hij}w_{hij}^2 i_{hij})w_{hij}^2w_{hij}^2t_{hij} \]

\[ + \sum ((\sum \sum n_{hij}w_{hij}^2 i_{hij} - \sum n_{hij}w_{hij}^2 i_{hij})(\sum n_{hij}w_{hij}^2 i_{hij})w_{hij}^2w_{hij}^2t_{hij} \]

\[ - (\sum n_{hij}w_{hij}^2 i_{hij} - \sum n_{hij}w_{hij}^2 i_{hij})(\sum n_{hij}w_{hij}^2 i_{hij})w_{hij}^2w_{hij}^2t_{hij} \]

\[ \text{tr}(RV_4RV_3) = \sum \sum n_{hij}w_{hij}(\alpha_4^{-1} - w_{hij}t_{hij})(\alpha_3^{-1} - n_{hij}w_{hij}w_{hij}t_{hij}) \]

\[ - \sum \sum n_{hij}(n_{hij} - 1)^2w_{hij}t_{hij} + \sum \sum \sum ((\sum n_{hij}w_{hij}^2 i_{hij} - \sum n_{hij}w_{hij}^2 i_{hij})(\sum n_{hij}w_{hij}^2 i_{hij})(\sum n_{hij}w_{hij}^2 i_{hij})w_{hij}^2w_{hij}^2t_{hij} \]

\[ + \sum \sum \sum ((\sum n_{hij}w_{hij}^2 i_{hij} - \sum n_{hij}w_{hij}^2 i_{hij})(\sum n_{hij}w_{hij}^2 i_{hij})(\sum n_{hij}w_{hij}^2 i_{hij})w_{hij}^2w_{hij}^2t_{hij} \]

\[ + \sum \sum \sum ((\sum n_{hij}w_{hij}^2 i_{hij} - \sum n_{hij}w_{hij}^2 i_{hij})(\sum n_{hij}w_{hij}^2 i_{hij})w_{hij}^2w_{hij}^2t_{hij} \]

\[ - (\sum n_{hij}w_{hij}^2 i_{hij} - \sum n_{hij}w_{hij}^2 i_{hij})(\sum n_{hij}w_{hij}^2 i_{hij})w_{hij}^2w_{hij}^2t_{hij} \]
\[
\text{tr}(R_{4}R_{V_{2}}) = \sum_{h} \sum_{i,j} n_{hij} w_{hij} w_{hi}(\alpha_{4}^{-1} - w_{hij} t_{hij})(\alpha_{2}^{-1} - n_{h} w_{hi} w_{t_{h}})
\]

\[
- \sum_{h} \sum_{i,j} n_{hij} (n_{hij} - 1) w_{hij}^{2} w_{hi} t_{hij}(\alpha_{2}^{-1} - n_{h} w_{hi} w_{t_{h}})
\]

\[
- \sum_{h} \sum_{i,j} ((\sum_{h} n_{hij} w_{hij}^{2}) - n_{hij} w_{hij}) n_{hij} w_{hi}^{2} t_{h}(\alpha_{2}^{-1} - n_{h} w_{hi} t_{h})
\]

\[
+ \sum_{h} \sum_{i,j} ((\sum_{h} n_{hij} w_{hij}^{2}) - (\sum_{i} n_{hij} w_{hij}^{2})) n_{hi} w_{i}^{2} w_{t_{h}}^{2}
\]

\[
+ \sum_{h} \sum_{i,j} ((\sum_{j} n_{hij} w_{hij}^{2}) - (\sum_{h} n_{hij} w_{hij}^{2})) n_{hi} w_{i}^{2} w_{h}^{2}
\]

\[
- (\sum_{h} n_{hij} w_{hij}^{2} w_{hi}^{2} w_{h})(\sum_{i} n_{hi} w_{hi}^{2} w_{h}^{2} w_{t_{h}}^{2})
\]

\[
\text{tr}(R_{4}R_{V_{1}}) = \sum_{h} \sum_{i,j} n_{hij} w_{hij} w_{hi} w_{h}(\alpha_{4}^{-1} - w_{hij} t_{hij})(\alpha_{1}^{-1} - n_{h} w_{hi} w_{\sigma_{1}}^{-1})
\]

\[
- \sum_{h} \sum_{i,j} n_{hij} (n_{hij} - 1) w_{hij}^{2} w_{hi} w_{t_{hij}}(\alpha_{1}^{-1} - n_{h} w_{hi} w_{\sigma_{1}}^{-1})
\]

\[
- \sum_{h} \sum_{i,j} ((\sum_{h} n_{hij} w_{hij})^{2})
\]

\[
- n_{hij}^{2} n_{hij}^{2} w_{hi}^{2} w_{h}^{2} t_{h}(\alpha_{1}^{-1} - n_{h} w_{hi} w_{\sigma_{1}}^{-1})
\]

\[
- \sum_{h} \sum_{i,j} ((\sum_{j} n_{hij} w_{hij}^{2}) - (\sum_{h} n_{hij} w_{hij}^{2})) n_{hi} w_{i}^{2} w_{h}^{2} t_{h}(\alpha_{1}^{-1} - n_{h} w_{hi} w_{\sigma_{1}}^{-1})
\]

\[
+ \sum_{h} \sum_{i,j} ((\sum_{j} n_{hij} w_{hij}^{2}) - (\sum_{h} n_{hij} w_{hij}^{2})) n_{hi} w_{i}^{2} w_{h}^{2} \sigma_{1}^{-2}
\]
\[
\text{tr}(R V_3 R V_3) = \sum_{h, i, j} n_{hij}^2 \frac{w_{hij}^2 + \alpha_3^{-1} n_{hij}^2 w_{hij}^2 t_{hij}^2}{\nu_{hij}^2 (\alpha_3^{-1} n_{hij}^2 w_{hij}^2 t_{hij})^2} \\
+ \sum_{h, i, j} n_{hij} (n_{hij} - 1) w_{hij}^2 (\alpha_3^{-1} n_{hij}^2 w_{hij}^2 t_{hij})^2 \\
+ \sum_{h, i, j} \frac{\sum_{j} n_{hij}^2 w_{hij}^2}{\nu_{hij}^2 (\alpha_3^{-1} n_{hij}^2 w_{hij}^2 t_{hij})^2} \\
+ \sum_{h, i, j} \left( \frac{\sum_{j} n_{hij}^2 w_{hij}^2}{\nu_{hij}^2 (\alpha_3^{-1} n_{hij}^2 w_{hij}^2 t_{hij})^2} \right) \left( \frac{\nu_{hij}^2}{\nu_{hij}^2 (\alpha_3^{-1} n_{hij}^2 w_{hij}^2 t_{hij})^2} \right) \\
+ \sum_{h, i, j} \frac{2 w_{hij}^2 w_{hij}^2}{w_{hij}^2 w_{hij}^2 \alpha_1^{-2}} \\
- \left( \frac{\sum_{i, j} n_{hij}^2 w_{hij}^2 w_{hij}^2}{\nu_{hij}^2 (\alpha_3^{-1} n_{hij}^2 w_{hij}^2 t_{hij})^2} \right) \left( \frac{\nu_{hij}^2}{\nu_{hij}^2 (\alpha_3^{-1} n_{hij}^2 w_{hij}^2 t_{hij})^2} \right) \alpha_1^{-2}.
\]

\[
\text{tr}(R V_3 R V_2) = \sum_{h, i, j} n_{hij}^2 \frac{w_{hij}^2 + \alpha_3^{-1} n_{hij}^2 w_{hij}^2 t_{hij}^2}{\nu_{hij}^2 (\alpha_3^{-1} n_{hij}^2 w_{hij}^2 t_{hij})^2} \\
+ \sum_{h, i, j} n_{hij} (n_{hij} - 1) w_{hij}^2 (\alpha_3^{-1} n_{hij}^2 w_{hij}^2 t_{hij})^2 \\
- \sum_{h, i, j} \sum_{j} n_{hij}^2 w_{hij}^2 n_{hij}^2 w_{hij}^2 t_{hij}^2 (\alpha_3^{-1} n_{hij}^2 w_{hij}^2 t_{hij})^2 \\
+ \sum_{h, i, j} \left( \frac{\sum_{j} n_{hij}^2 w_{hij}^2}{\nu_{hij}^2 (\alpha_3^{-1} n_{hij}^2 w_{hij}^2 t_{hij})^2} \right) \left( \frac{\nu_{hij}^2}{\nu_{hij}^2 (\alpha_3^{-1} n_{hij}^2 w_{hij}^2 t_{hij})^2} \right) \\
+ \sum_{h, i, j} \frac{2 w_{hij}^2 w_{hij}^2}{w_{hij}^2 w_{hij}^2 \alpha_1^{-2}} \\
- \left( \frac{\sum_{i, j} n_{hij}^2 w_{hij}^2 w_{hij}^2}{\nu_{hij}^2 (\alpha_3^{-1} n_{hij}^2 w_{hij}^2 t_{hij})^2} \right) \left( \frac{\nu_{hij}^2}{\nu_{hij}^2 (\alpha_3^{-1} n_{hij}^2 w_{hij}^2 t_{hij})^2} \right) \alpha_1^{-2}.
\]
\[ \text{tr}(\mathbf{R}_3 \mathbf{R}_1) = \sum \sum n_{hij} h_{hij} h_{hij} \cdot (\alpha_3^{-1} - n_{hij} h_{hij} h_{hij}) (\alpha_1^{-1} - n_{h} h_{h} \omega_1^{-1}) \]

\[ + \sum \sum n_{hij} (n_{hij} - 1) w_{hij} w_{hij} \cdot (\alpha_3^{-1} - n_{hij} h_{hij} h_{hij}) (\alpha_1^{-1} - n_{h} h_{h} \omega_1^{-1}) \]

\[ - \sum \sum (n_{hij} w_{hij}) (n_{hij} w_{hij}) n_{hij} h_{hij} h_{hij} h_{h} h_{hij} h_{hij} \cdot (\alpha_1^{-1} - n_{h} h_{h} \omega_1^{-1}) \]

\[ - \sum \sum (n_{hij} w_{hij}^2) - (n_{hij} w_{hij}) n_{hij} h_{hij} h_{hij} h_{h} h_{hij} h_{hij} \cdot (\alpha_1^{-1} - n_{h} h_{h} \omega_1^{-1}) \]

\[ + \sum (n_{hij} h_{hij} h_{hij} h_{hij} \cdot (\alpha_1^{-1} - n_{h} h_{h} \omega_1^{-1}) \]

\[ \text{tr}(\mathbf{R}_2 \mathbf{R}_2) = \sum \sum n_{hij} h_{hij} h_{hij} \cdot (\alpha_2^{-1} - n_{hij} h_{hij} h_{h}) \]

\[ + \sum \sum n_{hij} (n_{hij} - 1) w_{hij} w_{hij} \cdot (\alpha_2^{-1} - n_{hij} h_{hij} h_{h}) \]

\[ + \sum \sum (n_{hij} - n_{hij} h_{hij}) n_{hij} h_{hij} h_{hij} h_{h} h_{hij} h_{hij} \cdot (\alpha_2^{-1} - n_{hij} h_{hij} h_{h}) \]

\[ + \sum \sum (n_{hij} w_{hij}^2) - n_{hij} w_{hij}^2 n_{hij} h_{hij} h_{h} h_{hij} h_{hij} \cdot \]

\[ + \sum (n_{hij} w_{hij}^2) - (n_{hij} w_{hij}) n_{hij} h_{hij} h_{h} h_{hij} h_{hij} \cdot \]

\[ + \sum (n_{hij} w_{hij}^2) - (n_{hij} w_{hij}) n_{hij} h_{hij} h_{h} h_{hij} h_{hij} \cdot \]

\[ + \sum (n_{hij} w_{hij}^2) - (n_{hij} w_{hij}) n_{hij} h_{hij} h_{h} h_{hij} h_{hij} \cdot \]

\[ + \sum (n_{hij} w_{hij}^2) - (n_{hij} w_{hij}) n_{hij} h_{hij} h_{h} h_{hij} h_{hij} \cdot \]

\[ + \sum (n_{hij} w_{hij}^2) - (n_{hij} w_{hij}) n_{hij} h_{hij} h_{h} h_{hij} h_{hij} \cdot \]

\[ + \sum (n_{hij} w_{hij}^2) - (n_{hij} w_{hij}) n_{hij} h_{hij} h_{h} h_{hij} h_{hij} \cdot \]

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\[ + \sum (n_{hij} w_{hij}^2) - (n_{hij} w_{hij}) n_{hij} h_{hij} h_{h} h_{hij} h_{hij} \cdot \]

\[ + \sum (n_{hij} w_{hij}^2) - (n_{hij} w_{hij}) n_{hij} h_{hij} h_{h} h_{hij} h_{hij} \cdot \]

\[ + \sum (n_{hij} w_{hij}^2) - (n_{hij} w_{hij}) n_{hij} h_{hij} h_{h} h_{hij} h_{hij} \cdot \]
\[
\text{tr}(RV_2 RV_1) = \sum \sum n_{hij} \omega^2 v^2_{hi} (\alpha_2^{-1} - n_{h} \omega \omega_1^{-1}) \\
+ \sum \sum n_{hij} (n_{hij} - 1) \omega^2 v^2_{hi} (\alpha_2^{-1} - n_{h} \omega \omega_1^{-1}) \\
+ \sum \sum (n_{hi} - n_{hij} \omega_{ij}) \omega^2 v^2_{hi} (\alpha_2^{-1} - n_{h} \omega \omega_1^{-1}) \\
- \sum \sum \left( (\sum n_{hi}^2 \omega_{hi}^2) - (\sum n_{hi} \omega_{hi})^2 \right) n_{h} \omega \omega_1^{-1} \\
+ \sum \left( (\sum n_{h}^2 \omega_{h}^2) - (\sum n_{h} \omega_{h})^2 \right) n_{h} \omega \omega_1^{-1} \\.
\]

\[
\text{tr}(RV_1 RV_1) = \sum \sum n_{hij} \omega^2 v^2_{hi} \omega^2_{hi} (\alpha_1^{-1} - n_{h} \omega \omega_1^{-1})^2 \\
+ \sum \sum n_{hij} (n_{hij} - 1) \omega^2 v^2_{hi} \omega^2_{hi} (\alpha_1^{-1} - n_{h} \omega \omega_1^{-1})^2 \\
+ \sum \sum (n_{hi} - n_{hij} \omega_{ij}) \omega^2 v^2_{hi} \omega^2_{hi} (\alpha_1^{-1} - n_{h} \omega \omega_1^{-1})^2 \\
+ \sum \sum (n_{hi} \omega_{hi}) \omega^2 v^2_{hi} (\alpha_1^{-1} - n_{h} \omega \omega_1^{-1})^2 \\
+ \sum \left( (\sum n_{h}^2 \omega_{h}^2) - (\sum n_{h} \omega_{h})^2 \right) n_{h} \omega \omega_1^{-1} .
\]