ASYMPTOTIC THEORY OF SOME TESTS FOR A POSSIBLE CHANGE IN THE REGRESSION SLOPE OCCURRING AT AN UNKNOWN TIME POINT

by

Pranab Kumar Sen

Department of Biostatistics
University of North Carolina at Chapel Hill

Institute of Statistics Mimeo Series No. 1179

JUNE 1978
ASYMPTOTIC THEORY OF SOME TESTS FOR A POSSIBLE
CHANGE IN THE REGRESSION SLOPE OCCURRING AT AN UNKNOWN TIME POINT*

by
Pranab Kumar Sen
University of North Carolina, Chapel Hill

Abstract

Based on least squares estimators and aligned linear rank statistics, some
testing procedures for a possible change in the regression slope occurring at an
unknown time point are considered. The asymptotic theory of the proposed tests
rests on certain invariance principles pertaining to least squares estimators and
aligned rank order statistics and these are developed here.

AMS 1970 Classification Nos: 60F05, 62F05, 62G10

Key Words & Phrases: Asymptotic efficiency, asymptotic tests, Brownian bridge,
invariance principles, least squares estimators, rank order statistics, regression
coefficient, transition point.

*Work partially supported by the National Heart, Lung and Blood Institute, Contract
NIH-NHLBI-71-2243 from the National Institutes of Health.
1. Introduction.

Parametric as well as nonparametric testing procedures for possible shifts in the location of a distribution function (df) occurring at unknown time points between consecutively taken observations have been proposed and studied by Page (1955), Chernoff and Zacks (1964), Kander and Zacks (1966), Mustafi (1968), Bhattacharyya and Johnson (1968), Sen and Srivastava (1975) and Sen (1977), among others. The object of the present investigation is to consider a related problem of regression where a change in the regression coefficient may occur at a unknown time point and to develop suitable testing procedures.

Given the observations on independent random variables (rv) \( Y_i = Y(t_i), \) \( i = 1, \ldots, n, \) taken at time points \( t_1, \ldots, t_n \) (where \( t_1 \leq \ldots \leq t_n \) with at least one strict inequality sign), consider the following regression model:

\[
Y(t) = \alpha + I(t < \tau)\beta(t - \tau) + I(t \geq \tau)\gamma(t - \tau) + e(t),
\]

where \( \alpha, \beta, \gamma \) and \( \tau \) are unknown parameters \( (t_1 \leq \tau \leq t_n) \), I(A) stands for the indicator function of the set \( A \) and \( e(t) \) is a white noise i.e., for every real \( e \), the df

\[
F(e) = P\{e(t) \leq e\} \text{ does not depend on } t \ (t_1 \leq t \leq t_n).
\]

Note that if \( \tau = t_1 \) or \( t_n \), then (1.1) reduces to a simple regression model, while for \( t_1 < \tau < t_n \) and \( \beta \neq \gamma \), it relates to a segmented regression model with a common intercept \( (\alpha) \) at time point \( \tau \) and two different slopes \( \beta \) and \( \gamma \) for \( t < \tau \) and \( t \geq \tau \), respectively; \( \tau \) is termed a transition point. We assume that

\[
t_1 < \tau < t_n \quad \text{while } \beta \text{ and } \gamma \text{ may or may not be equal}.
\]

Then, under (1.3), (1.1) relates to a simple regression model only when \( \beta = \gamma \).
Such a segmented regression model is not very uncommon in practical problems. If we let the $t_i$ stand for the doses of a drug and the $Y_i$ for the responses, such a segmented dose-response regression also arises in some problems, where, at a higher dose, the regression pattern may differ from the one at a lower dose.

We desire to test for

$$H_0: \beta = \gamma \ vs. \ H_1: \beta > \gamma \ or \ H_2: \beta \neq \gamma,$$

treating $\alpha$, $\beta$ and $\tau$ as nuisance parameters. If $\tau$ were specified, one could have considered the two samples $\{Y_i: t_i < \tau\}$ and $\{Y_i: t_i \geq \tau\}$ (which are independent) and, bearing in mind the two simple regression models pertaining to these samples, one might have tested for the identity of the two slopes $\beta$ and $\gamma$. Since $\tau$ is not specified, our problem is somewhat more complicated. Moreover, we do not assume that $F$ in (1.2) is of a specified form (e.g., normal), and, for this reason, we take recourse to tests based on rank statistics and the classical least squares estimators.

In Section 2, along with the preliminary notions, the proposed test statistics are formulated. Some invariance principles for least squares estimators (LSE) are considered in Section 3 and also these are incorporated there in the study of the asymptotic properties of the tests based on the LSE. Similar invariance principles are developed for (aligned) linear rank statistics (LRS) in Section 4 and these are utilized then in the study of the asymptotic properties of the proposed rank tests. Section 5 deals with the asymptotic comparison of the procedures based on LSE and LRS.
2. Preliminary notions and the proposed tests.

Let us define

\( \bar{t}_k = \sum_{i=1}^{k-1} t_i \quad \text{and} \quad T_k^2 = \sum_{i=1}^{k} (t_i - \bar{t}_k)^2, \quad k \geq 1. \)

Note that \( T_k^2 \) is \( \uparrow \) in \( k(\geq 1) \). Under \( H_0 \) in (1.4), based on \( Y_1, \ldots, Y_k \), the LSE of \( \beta \) is

\( \hat{\beta}_k = T_k^{-2} \sum_{i=1}^{k} (t_i - \bar{t}_k) Y_k, \quad \text{for} \quad k = 2, \ldots, n; \hat{\beta}_1 = 0. \)

If we assume that \( F \) in (1.2) admits of a finite variance \( \sigma^2 \), then under \( H_0 \) in (1.4), \( \hat{\beta}_2, \ldots, \hat{\beta}_n \) are all unbiased estimators of \( \beta \) with variances \( \sigma^2 / T_2, \ldots, \sigma^2 / T_n \), respectively. On the other hand, if \( H_0 \) does not hold and \( t_m \leq \tau < t_{m+1} \) for some \( m: 1 \leq m \leq n-1 \), then

\[
E(\hat{\beta}_k | \beta, \gamma) = \begin{cases} 
\beta, & k \leq m, \\
\gamma + (\beta - \gamma) T_k^{-2} \sum_{i=1}^{m} (t_i - \bar{t}_k) (t_i - \bar{t}_k), & m+1 \leq k \leq n,
\end{cases}
\]

where the right hand side (rhs) of (2.3), for \( k > m \), differs from \( \beta \) (or \( \gamma \)).

Thus, the estimators cease to fluctuate around a common \( \beta \) when \( H_0 \) does not hold. We consider the residuals

\( \hat{\gamma}_i = Y_i - \hat{\beta}_n t_i, \quad i = 1, \ldots, n \)

and based on the partial set \( \{\hat{\gamma}_1, \ldots, \hat{\gamma}_k\} \), we compute

\( \hat{\beta}_k = T_k^{-2} \sum_{i=1}^{k} (t_i - \bar{t}_k) \hat{\gamma}_i = \hat{\beta}_k - \hat{\beta}_n, \quad 2 \leq k \leq n; \hat{\beta}_1 = 0. \)

Under \( H_0 \), \( \hat{\beta}_1, \ldots, \hat{\beta}_n \) al unbiasedly estimate \( 0 \), while under \( H_1 \) or \( H_2 \), they are not so. Our proposed test based on the LSE rests on the statistics

\( M^*_n = \max_{0 \leq s \leq n} S_{n,k} \quad \text{and} \quad M_n = \max_{0 \leq s \leq n} |S_{n,k}|, \)
where $S_{n,0} = S_{n,1} = 0$ and

$$S_{n,k} = T^{-1} n^{2k} = T^{-1} n^{2k} (\hat{\beta}_k - \hat{\beta}_n), \quad 2 \leq k \leq n,$$

so that $S_{n,n} = 0$. The test procedure will be formulated in Section 3.

For the rank tests (to follow), we do not need the existence of the second moment of $F$. However, to avoid ties among the observations $(Y_i)$, we assume that $F$ is continuous everywhere. Consider the usual LRS

$$L_k = \sum_{i=1}^{r_k} (t_i - \bar{t}_k) a_k (R_{ki}), \quad k = 1, \ldots, n,$$

where, for every $k(\geq 1)$, $R_{ki} = \text{rank of } Y_i$ among $Y_1, \ldots, Y_k$ for $1 \leq i \leq k$,

the scores $a_k(i)$ are defined by

$$a_k(i) = \mathbb{E}(U_{ki}), \quad i = 1, \ldots, k,$$

$U_{ki} < \ldots < U_{kk}$ are the ordered rv's of a sample of size $k$ from the rectangular $(0,1)$ df and the score generating function $\phi = \{\phi(u), \ 0 < u < 1\}$ is assumed to be square integrable inside $(0,1)$. Actually, bearing in mind, the elimination of the nuisance parameters through estimation, we assume that

$$\phi(u) \text{ is } \mathcal{C} \text{ in } u: 0 < u < 1.$$

For every real $b: -\infty < b < \infty$, let

$$L_k(b) = \sum_{i=1}^{r_k} (t_i - \bar{t}_k) a_k (R_{ki}(b)), \quad k \geq 1,$$

where $R_{ki}(b) = \text{rank of } Y_i - b t_i$ among $Y_1 - b t_1, \ldots, Y_k - b t_k$ for $1 \leq i \leq k$. Then [cf. Theorem 6.1 of Sen (1969)], under (2.10), for every $k(\geq 1)$,

$$L_k(b) \text{ is } \mathcal{C} \text{ in } b: -\infty < b < \infty.$$

Let then

$$\hat{\beta}^{*}_{k,1} = \sup \{b: L_k(b) > 0\}, \quad \hat{\beta}^{*}_{k,2} = \inf \{b: L_k(b) < 0\};$$
(2.14) \[ \beta_k^* = \frac{1}{2} (\beta_{k,1}^* + \beta_{k,2}^*), \quad k = 1, \ldots, n. \]

Under the null hypothesis in (2.14), \( \beta_k^* \) is a translation-invariant, robust and consistent estimator of \( \beta \) [viz., Adichie (1967)], for \( k \geq 1 \). As in (2.4), we consider the residuals

(2.15) \[ Y_i^* = Y_i - \beta_n^* \tilde{t}_i \quad \text{for} \quad i = 1, \ldots, n \]

and based on the partial set \( \{Y_1^*, \ldots, Y_k^*\} \), we define

(2.16) \[ L_{n,k}^* = L_k(\beta_n^*)/T_n \]

\[ = T_n^{-1} \gamma_k \sum_{i=1}^{n} (t_i - \bar{t}_k) a_k^*(R_{ki}^*), \quad k = 1, \ldots, n, \]

where \( R_{ki}^* = \text{rank of} \ Y_i^* \text{ among} \ Y_1^*, \ldots, Y_k^* \), for \( 1 \leq i \leq k; \ k \geq 1 \). Conventionally, we let \( L_{n,0}^* = 0 \). Then, parallel to (2.6), our proposed test statistics are

(2.17) \[ D_n^+ = A_n^{-1} \left\{ \max_{0 \leq k \leq n} L_{n,k}^* \right\} \quad \text{and} \quad D_n = A_n^{-1} \left\{ \max_{0 \leq k \leq n} |L_{n,k}^*| \right\} \]

where for \( n \geq 2 \),

(2.18) \[ A_n^2 = (n-1)^{-1} \sum_{i=1}^{n} [a_n(i) - \bar{a}_n]^2 \quad \text{and} \quad \bar{a}_n = \frac{1}{n} \sum_{i=1}^{n} a_n(i). \]

The test procedure will be formulated in Section 4.

3. Asymptotic properties of the tests based on \( M_n^+ \) and \( M_n^- \).

For the study of the asymptotic distribution theory of \( M_n^+ \) and \( M_n^- \) (under the null as well as local alternative hypothesis), we need to study first some invariance principles relating to the LSE. For this asymptotic study, we consider a triangular array \( \{t_{ni}, 1 \leq i \leq n; \ n \geq 1\} \) of time-variables and \( \{Y_i = Y(t_{ni}), 1 \leq i \leq n; \ n \geq 1\} \) are defined accordingly as in (1.1). We assume that for every \( \theta: 0 < \theta \leq 1 \),
\[ (3.1) \quad \lim_{n \to \infty} \frac{\tau}{n[n \theta]} = \lim_{n \to \infty} \frac{[n \theta]^{-1} \gamma^i_{n \theta}}{t_{ni}} = \mu(\theta) \text{ exists}, \]
\[ (3.2) \quad \lim_{n \to \infty} \frac{\tau^2}{n[n \theta]/n} = \lim_{n \to \infty} \left\{ \frac{1}{n} \sum_{i=1}^{n} \frac{1}{t_{ni}} \frac{t_{ni}}{\tau^2[n[n \theta]]} \right\} = \xi^2(\theta) \text{ exists} \]

and both \( \mu(\theta) \) and \( \xi(\theta) \) are continuous inside \([0,1]\). Note that (3.1) and (3.2) insure that
\[ (3.3) \quad \max \left\{ \frac{1}{n} \left| \frac{t_{ni}}{\tau^2[n[n \theta]]} \right| : 1 \leq i \leq k \leq n \right\} \to 0 \text{ as } n \to \infty. \]

Let then \( U_{n0} = U_{n1} = 0 \) and for \( k \geq 2 \)
\[ (3.4) \quad U_{nk} = \frac{k}{\sum_{i=1}^{k} \left( \frac{t_{ni}}{\tau} \right) Y_i} \]

and define a stochastic process \( W^{(1)}_n = \{ W_n^{(1)}(u), 0 \leq u \leq 1 \} \) be letting
\[ (3.5) \quad W_n^{(1)}(u) = U_{nk}(u)/(\sqrt{n} \sigma(1)), k_n(u) = \max \{ k : T^2_{nk} \leq u T^2_{nn} \}, 0 \leq u \leq 1. \]

Note that \( k_n(u) \) is a nondecreasing, right-continuous and integer-valued function of \( u \in [0,1] \) and \( W_n^{(1)} \) belongs to the \( D[0,1] \) space (having only jump discontinuities) endowed with the Skorokhod \( J_1 \)-topology. Let \( W = \{ W(u), 0 \leq u \leq 1 \} \) be a standard Wiener process on \([0,1]\). Then, we have the following.

**Theorem 3.1.** If \( \text{E}(t) = 0, \text{E}(t)^2 = \sigma^2 < \infty \) and (3.1)-(3.2) hold, then under \( H_0^\alpha : \alpha = \beta = \gamma = 0, \)
\[ (3.6) \quad W_n^{(1)} \overset{\mathcal{D}}{\to} W, \quad \text{in the } J_1 \text{-topology on } D[0,1]. \]

**Proof.** First, we establish the convergence of the finite-dimensional distributions (f.d.d.) of \( \{ W_n^{(1)} \} \) to those of \( W \). For every (fixed) \( m(\geq 1) \) and \( 0 \leq u_1 < \ldots < u_m \leq 1 \), let \( k_j = k_n(u_j), 1 \leq j \leq m \) and for an arbitrary \( d \neq 0 \), let
\[ (3.7) \quad Z_n = \sum_{j=1}^{m} d_j W_n^{(1)}(u_j) = (\sqrt{n} \sigma(1))^{-1} \sum_{j=1}^{m} d_j \sum_{i=1}^{k_j} \frac{1}{t_{ni}} \frac{t_{ni}}{\tau^2[n[n \theta]]} Y_i \]
\[ = (\sqrt{n} \sigma(1))^{-1} \sum_{i=1}^{n} d_i \sum_{i=1}^{n} Y_i, \]
where

\[(3.8) \quad f_{ni} = \sum_{j=1}^{d_j} \{t_{ni,t_{nk_j}} \} I(i \leq k_j), 1 \leq i \leq n.\]

It follows by some routine steps that \(z_i = 1 f_{ni} = 0\) and

\[(3.9) \quad \left(\sum_{i=1}^{n} f_{ni}^2 \right)/\left(n(\sigma^2 \xi^2(1)) = \sum_{j=1}^{d_j} d_j (u_j \wedge u_j);\right)\]

\[(3.10) \quad n^{-1/2} \max\{|f_{ni}|: 1 \leq i \leq n\} \to 0 \text{ as } n \to \infty.\]

Note that the rhs of (3.9) equals the variance of \(z_i = 1 d_j W(u_j).\) Further, under \(H_0^*,\) the \(Y_i\) are independent and identically distributed (i.i.d.) rv's with 0 mean and variance \(\sigma^2.\) Hence, using (3.10) and a special version of the central limit theorem in Hájek and Šidák (1967, p. 153), it follows that \(Z_n\) is asymptotically normally distributed. This establishes the convergence of the f.d.d.'s of \(\{W^{(1)}_n\}\) to those of \(W.\) It remains to show that \(\{W^{(1)}_n\}\) is tight. Since by definition, \(W^{(1)}_n(0) = 0,\) with probability 1, it suffices to show that for every \(\varepsilon > 0\) and \(\eta > 0,\) there exist a \(\delta: 0 < \delta < 1\) and an \(n_0,\) such that for \(n \geq n_0\) and every \(k: 0 \leq k \leq n-[n\delta], q = k+[n\delta],\)

\[(3.11) \quad P\left\{\max_{k \leq m \leq q} |U_{m,k} - U_{k,k}| > \varepsilon \sigma \xi(1)/\sqrt{n}\right\} < \eta \delta ;\]

see Theorem 8.3 of Billingsley (1968, p. 56). For this note that for every \(k,q:\)

\[0 \leq k < q \leq n,\]

\[\max_{k \leq m \leq q} n^{-1/2} |U_{m,k} - U_{k,k}| \leq |\tilde{t}_{nk} - \tilde{t}_{nk}| n^{-1/2} \left\{ \max_{k \leq m \leq q} \left| \sum_{i=k+1}^{m} Y_{i} \right| \right\}
\]

\[+ n^{-1/2} \left\{ \max_{k \leq m \leq q} |\tilde{t}_{nk} - \tilde{t}_{nm}| |\sum_{i=1}^{n} Y_{i} | \right\} + n^{-1/2} \left\{ \max_{k \leq m \leq q} \left| \sum_{i=k+1}^{m} (t_{nk} - \tilde{t}_{nk}) Y_{i} \right| \right\}\]

\[(3.12) \quad = C_{k,q}^{(1)} + C_{k,q}^{(2)} + C_{k,q}^{(3)} , \text{ say.}\]

By the nondecreasing nature of \(t_{ni}\) (in \(i),\)
\[
\max_{k \leq m \leq q} |\bar{\tau}_{nm} - \bar{\tau}_{nk}| = |\bar{\tau}_{nq} - \bar{\tau}_{nk}|,
\]
where by (3.1) and the continuity of \( \mu(\theta), \theta \in [0,1] \),

\[
\lim_{n \to \infty} \left| \frac{1}{n} \sum_{i=n}^{n+\delta} Y_i \right| = 0 \quad \text{as} \quad \delta \to 0, \quad \forall \ s \in [0,1).
\]

Further, the \( Y_i \) are i.i.d. r.v.'s with 0 mean and variance \( \sigma^2 \), so that by the

Donsker Theorem [cf. Billingsley (1968)], for every \( \varepsilon' > 0 \) and \( \eta' > 0 \), there

exist a \( \delta: 0 < \delta < 1 \) and an \( n_0 \), such that for \( n \geq n_0 \) and every \( k: 0 \leq k \leq n - [n\delta], \)

\[
q = k + [n\delta],
\]

\[
P\left( n^{-k} \max_{k \leq m \leq q} |\sum_{i=k+1}^{m} Y_i| > \varepsilon \right) < \eta'\delta,
\]

\[
P\left( n^{-k} \max_{1 \leq i \leq n} |\sum_{i=1}^{k} Y_i| > K_{\eta'\delta} \right) < \eta'\delta,
\]

where \( K_{\eta'\delta} \) is a positive number \( (< \infty) \), depending on \( \eta'\delta \). Thus, it follows

from (3.12) through (3.16) that for every \( \varepsilon > 0 \) and \( \eta > 0 \), there exist

\( \varepsilon' > 0, \eta' > 0 \) and \( \delta: 0 < \delta < 1 \), such that for \( q - k = [\delta n], n \geq n_0, \)

\[
P\left( C_{kq}^{(1)} + C_{kq}^{(2)} > \frac{1}{2} \varepsilon \right) < \frac{1}{2} \eta\delta; \quad \eta' = \frac{1}{4} \eta, \quad \varepsilon' = \varepsilon/4
\]

where \( \delta(>0) \) is so small that \( \lim_{n \to \infty} |\bar{\tau}_{nq} - \bar{\tau}_{nk}| K_{\eta'\delta} < \varepsilon/4 \). Hence, to prove (3.11),

it suffices to show that for every \( \varepsilon' > 0 \) and \( \eta' > 0 \), there exist a \( \delta: 0 < \delta < 1 \)

and an \( n_0 \), such that for \( n \geq n_0 \) and every \( k: 0 \leq k \leq n - [n\delta], q = k + [n\delta], \)

\[
P\left( C_{kq}^{(3)} > \varepsilon' \right) < \eta'\delta.
\]

If we let \( V_{ni} = (t_{ni} - \bar{t}_{nq}) Y_i, k \leq i \leq q \), then the \( V_{ni} \) are independent, \( EV_{ni} = 0, \)

\[
\sum_{i=k+1}^{q} E_{ni}^2 = \sigma^2 \sum_{i=k+1}^{q} (t_{ni} - \bar{t}_{nq})^2 = \sigma^2 (\bar{t}_{nq}^2 - \sum_{i=1}^{k} (t_{ni} - \bar{t}_{nq})^2) = \sigma^2 (\bar{t}_{nq}^2 - \bar{t}_{nk}^2 - k(\bar{t}_{nk} - \bar{t}_{nq})^2) \leq \sigma^2 (\bar{t}_{nq}^2 - \bar{t}_{nk}^2) \sim n \sigma^2 \left[ \frac{\xi^2(a)}{n} - \xi^2(\xi^2(k)) \right], \quad \left\{ \frac{\xi^2}{n} \right\}
\]

is a martingale and, finally,

\[
\left( \sqrt{n}\sigma \right)^{-1} \left\{ \sum_{s=k+1}^{q} V_{ns} \right\}
\]

is asymptotically normally distributed with 0 mean and variance.
\[
\left\{ \xi^2_{n} - \xi^2_{k} - (k/n) \left[ \mu_{n}(u) - \mu_{k}(u) \right]^2 \right\} \leq \left[ \xi^2_{q/n} - \xi^2_{k/n} \right].
\]
Hence, (3.18) follows by using Lemma 4 of Brown (1961) and proceeding as in the proof of Theorem 3. Q.E.D.

Let us now consider the process \( W_n^{(2)} = \{W_n^{(2)}(u), 0 \leq u \leq 1\} \) where

(3.19) \[ W_n^{(2)}(u) = S_{n,k}(u)/\sigma, \ 0 \leq u \leq 1, \]

\( k_n(u) \) is defined by (3.5) and the \( S_{n,k} \) by (2.7). Note that in (2.7) and, elsewhere, we replace the \( t_i \) by \( t_{ni} \) and \( T_k^2 \) by \( T_{nk}^2, 1 \leq k \leq n. \) Also, let \( W_0 = \{W_0(t) = W(t) - tW(1), 0 \leq t \leq 1\} \) be a standard Brownian bridge on \([q, 1]\). Note that by definition in (2.5), the \( \tilde{\beta}_k \) are invariant under shift and regression i.e.,

if we work with \( Y_i - a - bt_i, 1 \leq i \leq n, \) then the resulting \( \tilde{\beta}_k \) will be the same as the ones in (2.5) for every real \((a, b)\). Hence, the distribution of \( W_n^{(2)} \) under \( H_0 \) in (1.4) will be the same as under \( H_0^* \). Further, by definition,

(3.20) \[ W_n^{(2)}(u) = (T_{nn}/\sigma)^{-1}[U_{nk}(u) - T_{nn}^{-2}T_{nk}(u)U_{n,n}] \]

\[ = (\sqrt{n}\xi(1)/T_{nn})[W_n^{(1)}(u) - T_{nn}^{-2}T_{nk}(u)W_n^{(1)}(1)], \ 0 \leq u \leq 1. \]

Also, note that \( T_{n0}^2 = T_{n1}^2 = 0, \) while for \( k \geq 1, \)

(3.21) \[ (T_{nk+1} - T_{nk})^2/T_{nn} = \frac{[(k+1)/k][t_{nk+1} - \overline{t}_{nk+1}]^2/T_n^2}{T_n^2} + 0, \]

by (3.3), while by (3.2), \( \sqrt{n}\xi(1)/T_n \to 1 \) as \( n \to \infty. \) Hence,

(3.22) \[ W_n^{(2)}(u) = (\sqrt{n}\xi(1)/T_{nn})[W_n^{(1)}(u) - uW_n^{(1)}(1)] + R_n(u), \]

where \( \sup\{|R_n(u)|: 0 \leq u \leq 1\} \to 0. \) From Theorem 3.1 and (3.22), we arrive at the following.
Theorem 3.2. If \( Ee(t) = 0, \) \( Ee^2(t) = \sigma^2 < \infty \) and (3.1)-(3.2) hold, then under 
\[ H_0 \] 
in (1.4),

\[ W_n^{(2)} \xrightarrow{\mathcal{L}} W^0, \] 
in the \( J_1 \)-topology on \( D[0,1] \).

For the standard Brownian bridge \( W^0 \), it is well known [viz., Billingsley (1968, p.85)] that for every \( t \geq 0 \)

\[ P\left\{ \sup_{0 \leq u \leq 1} W^0(u) \leq t \right\} = 1 - \exp(-2t^2), \tag{3.24} \]

\[ P\left\{ \sup_{0 \leq u \leq 1} |W^0(u)| \leq t \right\} = 1 - 2 \sum_{k=1}^{\infty} (-1)^k \exp(-2k^2t^2), \tag{3.25} \]

where the rhs of (3.25) is bounded from below by \( 1 - 2 \exp(-2t^2) \) and is practically equal to this lower bound when \( t \) is not very small. On equating the rhs of (3.24) and (3.25) to \( 1 - \varepsilon \), where \( \varepsilon \) is the desired level of significance \( (0 < \varepsilon < 1) \), we denote the solutions by \( \Delta_\varepsilon^+ \) and \( \Delta_\varepsilon^- \) respectively. Then, by Theorem 3.2, we have on noting that

\[ M_n^+ / \sigma = \sup_{0 \leq u \leq 1} W_n^{(2)}(u) \quad \text{and} \quad M_n^- / \sigma = \sup_{0 \leq u \leq 1} |W_n^{(2)}(u)|, \tag{3.26} \]

\[ P\{M_n^+ > \sigma \Delta_\varepsilon^+ | H_0^+ \} + \varepsilon \quad \text{and} \quad P\{M_n^- > \sigma \Delta_\varepsilon^- | H_0^- \} + \varepsilon, \tag{3.27} \]

and hence, the asymptotic critical values of \( M_n^+ \) and \( M_n^- \) (at the desired level of significance \( \varepsilon: 0 < \varepsilon < 1 \)) are \( \sigma \Delta_\varepsilon^+ \) and \( \sigma \Delta_\varepsilon^- \), respectively. Thus, if \( \sigma \) is specified, we have the following test procedure:

Compute the \( S_{n,k}, k \leq n, \) defined by (2.7). If, for at least one \( k: \)

\[ 1 \leq k \leq n-1, \quad S_{n,k} \quad \text{(or} \quad |S_{n,k}| \text{)} \quad \text{exceeds} \quad \sigma \Delta_\varepsilon^+ \quad \text{(or} \quad \sigma \Delta_\varepsilon^- \text{), reject} \]

\( H_0 \) in (1.4). If, no such \( k \) exists, accept \( H_0. \)
In practice, mostly, $\sigma$ is not specified. We consider the estimator

$$\hat{\sigma}^2_n = n^{-1} \sum_{i=1}^{n} (Y_i - \bar{Y}_n - \hat{\beta}_n (t_i - \bar{t}_n))^2 \left( \text{where } \bar{Y}_n = \frac{1}{n} \sum_{i=1}^{n} Y_i \right).$$

Under $H_0$ in (1.4), we have a simple regression model, and hence [viz., Sen and Puri (1970)], $\hat{\sigma}_n \overset{D}{=} \sigma$. As such, we may proceed as in (3.28) where we replace $\sigma$ by $\hat{\sigma}_n$; the asymptotic level of significance remains equal to $\epsilon$.

Let us now consider the behavior of these tests when $H_0$ in (1.4) does not hold. Suppose that (1.1) holds with

$$\tau = \frac{t}{m} \text{ for some } m; 1 < m \leq n - 1 \text{ and } \beta - \gamma = \delta (\neq 0).$$

Let $Y_i^0 = Y_i + \delta (t_i - \bar{t}_m) I(t_i \geq \bar{t}_m)$, $i = 1, \ldots, n$ and in (2.2), (2.4), (2.5) and (2.7), we replace the $Y_i$ by $Y_i^0$ and denote the resulting quantities by $\hat{\beta}_k^0, \hat{Y}_i^0, \tilde{\beta}_k^0$ and $S_{n,k}^0$, respectively. Then, we have by some direct computations that

$$\hat{\beta}_k^0 = \begin{cases} \hat{\beta}_k, & k \leq m, \\ \hat{\beta}_k + \delta \bar{T}_k^2 \{ \bar{T}_k^2 - \bar{T}_m^2 - m (t_m - \bar{t}_m) (\bar{t}_k - \bar{t}_m) \}, & m < k \leq n, \end{cases}$$

where $\bar{t}_k = \frac{1}{k} \sum_{i=1}^{k} t_i$ is $\Rightarrow$ in $k(1 \leq k \leq n)$, and

$$S_{n,k}^0 = \begin{cases} S_{n,k} - \delta (\bar{T}_k^2 / T_n) \{ 1 - \bar{T}_m^2 / T_n - m (t_m - \bar{t}_m) (\bar{t}_k - \bar{t}_m) / T_n^2 \}, & k \leq m, \\ S_{n,k} - \delta (\bar{T}_k^2 / T_n) \left\{ \left[ \frac{T_m^2}{T_n} - \frac{T_k^2}{T_n} \right] - m (t_m - \bar{t}_m) \left[ \frac{T_n^2 - \bar{T}_k^2}{T_n} \right] \right\}, & m < k \leq n. \end{cases}$$

As such, if $m/n$ is bounded away from 0 and 1, then under (3.1), (3.2) and (3.30), $\max \{ S_{n,k}^0 : 1 \leq k \leq n \} \rightarrow \infty$ as $n \rightarrow \infty$ when $\delta > 0$ and $\max \{ S_{n,k} - S_{n,k}^0 : 1 \leq k \leq n \} \rightarrow \infty$ as $n \rightarrow \infty$ when $\delta \neq 0$. On the other hand, for the $Y_i^0$, the simple regression model holds, so that Theorem 3.2 applies to the $S_{n,k}^0$, and hence,
max\{S_{n,k}^0; 1 \leq k \leq n\} \text{ or } max\{|S_{n,k}^0|; 1 \leq k \leq n\} \text{ is } O_p(1). \text{ Thus, } \max\{S_{n,k}; 1 \leq k \leq n\} \rightarrow \infty, \text{ in probability, if } \delta > 0 \text{ and } \max\{|S_{n,k}|; 1 \leq k \leq n\} \rightarrow \infty, \text{ in probability, if } \delta \neq 0. \text{ Hence, the tests based on (3.28) are consistent.}

In view of the consistency of the tests, for the study of the asymptotic power properties, we confine ourselves to a suitable sequence \{H_n\} of alternative hypotheses for which the asymptotic power is different from 1. Keeping (3.1)-(3.2) in mind, we assume that

\[(3.33) \quad \tau = t_m \text{ where } m/n + \nu: 0 < \nu < 1, T_m^2/T_n^2 \rightarrow \rho: 0 < \rho < 1.\]

Actually, if we let

\[(3.34) \quad h(u) = \inf\{t: \xi_2(t)/\xi_2(1) \geq u\}, 0 \leq u \leq 1,\]

then \(\rho = \xi_2(\nu)/\xi_2(1)\) and \(\mu(\nu) = \mu(h(\rho))\). We consider then

\[(3.35) \quad H_n: \beta = \gamma + T_n^{-1}\delta\text{ for some real } \delta, \text{ where (3.3) holds}.

Let us then define \(a_\delta = \{a_\delta(t), 0 \leq t \leq 1\}\) by letting

\(a_\delta(t) = \begin{cases} \delta t [1 - \rho - \nu (\tau - \mu(\nu)) (\mu(1) - \mu(\nu))/\xi_2^2(1)]/\sigma, 0 \leq t \leq \rho, \\ \frac{1}{\delta} [\rho (1-t) - \nu (\tau - \mu(\nu)) ((1-t)\mu(\nu) - \mu(h(\rho) + t\mu(1)))/\xi_2^2(1)], \rho < t \leq 1. \end{cases}\)

In (3.32), replacing \(\delta\) by \(\delta/T_n\) and then using Theorem 3.2 (for the \(S_{n,k}^0\)) along with (3.5) and (3.33)-(3.36), we arrive at the following.

**Theorem 3.3.** If \(Ee(t) = 0, Ee^2(t) = \sigma^2 < \infty\) and (3.1), (3.2) and (3.33) holds, then under \(\{H_n\}\) in (3.35),

\[(3.37) \quad W_n^{(2)} + W^0 + a_\delta^n, \text{ in the } J_1\text{-topology on } D[0,1].\]

By virtue of Theorem 3.3, the asymptotic power of the test based on \(M_n^+\) under \(\{H_n\}\) in (3.35), is given by
\begin{equation}
(3.38) \quad P\{W^0(t) + a_{\delta}(t) \geq \Delta^*, \text{ for some } t \in (0,1)\},
\end{equation}

and for the test based on $M_n$, the corresponding expression is

\begin{equation}
(3.39) \quad P\{|W^0(t) + a_{\delta}(t)| \geq \Delta^*, \text{ for some } t \in (0,1)\}.
\end{equation}

Since $a_{\delta}$ is not a linear (in $t$) boundary (in general), closed expression for (3.38) and (3.39) are not generally available.

4. Asymptotic properties of the tests based on $D^+_n$ and $D^-_n$.

We need to develop some invariance principles for aligned LRS for the study of the distribution theory of $D^+_n$ and $D^-_n$. Consider first the case of $\{L_k\}$, defined by (2.8), and define $W^{(3)}_n = \{W^{(3)}(u), 0 \leq u \leq 1\}$ by letting

\begin{equation}
(4.1) \quad W^{(3)}_n(u) = T^{-1}_n A^{-1}_n L^*_n(u), \quad 0 \leq u \leq 1;
\end{equation}

\begin{equation}
(4.2) \quad k^*_n(u) = \max(k: T^2_n A^2_n \leq u T^2_n A^2_n), \quad 0 \leq u \leq 1,
\end{equation}

where $T^2_n$ and $A^2_n$ are defined by (2.1) and (2.18), respectively. Here also, $W^{(3)}_n$ belongs to the $D[0,1]$ space. Then, we have the following.

**Theorem 4.1.** For scores defined by (2.9) with nondecreasing and square integrable $\phi$, under (3.3) and $H^*_0: \beta = \gamma = 0$ (refer to (1.1)),

\begin{equation}
(4.3) \quad W^{(3)}_n \xrightarrow{L^*} W, \text{ in the } J_1\text{-topology on } D[0,1].
\end{equation}

**Proof:** Since under $H^*_0$, the $Y_i$ are i.i.d. rv with a continuous df $F(x-\alpha)$, (4.3) directly follows from Theorem 2.2 of Sen (1975).

We proceed on to the case where $H^*_0$ may not hold. Here, we assume, that

(i) the df $F$ admits of an absolutely continuous probability density function
(pdf) \( f \) with a finite Fisher information

\[
I(f) = \int_{-\infty}^{\infty} \left[ \frac{f'(x)}{f(x)} \right]^2 dF(x) < \infty.
\]

Also, let

\[
\psi(u) = -f'(F^{-1}(u)/f(F^{-1}(u)), 0 < u < 1,
\]

\[
A^2 = \int_0^1 \phi^2(u) du - \left( \int_0^1 \phi(u) du \right)^2 \quad \text{and} \quad \lambda(\phi, \psi) = \left( \int_0^1 \phi(u)\psi(u) du \right) / AI^2(\phi).
\]

We assume that

\[
\lambda^* = I^2(\phi)\lambda(\phi, \psi) > 0.
\]

Further, we assume that (3.1), (3.2) and (3.33) hold and consider a sequence \( \{k^*_n, b\} \) of alternative hypotheses where

\[
k^*_n, b: (1.1) \text{ holds with } \beta = T^{-1}_n b_1, \gamma = T^{-1}_n b_2; \ b = (b_1, b_2)
\]

and \( b_1, b_2 \) are real numbers. Then, we have the following.

**Theorem 4.2.** Under \( \{k^*_n, b\} \) and the assumptions made above,

\[
\mathcal{N}_n^{(3)} \overset{I}{\rightarrow} \mathcal{N} + \omega, \ \text{in the } J_1 \text{-topology on } D[0,1],
\]

where \( \omega = \{\omega(u), 0 \leq u \leq 1\} \) is given by

\[
\omega(u) = \begin{cases}
\lambda^* b_1 u, & 0 \leq u \leq \rho \\
\lambda^* \{b_2 u + (b_1 - b_2)[\rho + \nu(\tau - \mu(\nu))(\mu(h(u)) - \nu(\nu))/\xi^2(1)]\}, & \rho \leq u \leq 1,
\end{cases}
\]

where \( \nu, \rho, \xi(1) \) and \( h(u) \) are defined as in (3.1), (3.2), (3.33) and (3.34).

**Proof.** Let us denote the joint distribution of \( (Y_1, \ldots, Y_n) \) under \( k^*_n, b \) by \( P_{n,b} \) so that \( P_{n,0} \) stands for the null hypothesis \( (H_0^*) \) case. Then, by an
appeal to the basic results in Hájek and Šidák (1967, Ch. VI), we conclude that under the hypothesis of this theorem, \( \{P_{n, k}\} \) is contiguous to \( \{P_{n, 0}\} \). Also, (4.3) insures the tightness of \( W_n^{(3)} \) under \( \{P_{n, 0}\} \). Hence, proceeding as in the proof of Theorem 2 of Sen (1976), we conclude that \( W_n^{(3)} \) remains tight under \( \{k^*_n, b\} \) as well. Hence, to prove (4.9), it suffices to prove the convergence of f.d.d.'s of \( \{W_n^{(3)}\} \) to those of \( W + \omega \), when \( \{k^*_n, b\} \) holds.

Towards this, we define

\[
S_{n, k}^* = T_{nn}^{-1} \sum_{i=1}^{k} (t_{ni} - \bar{t}_{nk}) \phi(F(Y_i, -\alpha)), 1 \leq k \leq n.
\]

Then, for every (fixed) \( m(\geq 1) \) and \( 0 \leq u_1 < \ldots < u_m \leq 1 \), defining \( k_j = k^*_n(u_j) \) by (4.2), \( 1 \leq j \leq m \), it follows by an appeal to Hájek (1961) that under \( H_0^* \):

\[
\beta = \gamma = 0, T_{nn}^{-1} k_j = W_n^{(3)}(u_j)
\]

is equivalent in quadratic mean to \( S_{n, k_j}^* \), \( \forall 1 \leq j \leq m \).

By contiguity of \( \{P_{n, b}\} \) to \( \{P_{n, 0}\} \), \( |W_n^{(3)}(u_j) - S_{n, k_j}^*| = 0, \forall 1 \leq j \leq m \) under \( \{k^*_n, b\} \) as well. Finally, under \( \{k^*_n, b\} \), the asymptotic joint normality of \( (S_{n, k_1}^*, \ldots, S_{n, k_m}^*) \) can be derived by an appeal to a theorem on page 216 of Hájek and Šidák (1967). In view of the similarity of this proof with that of Theorem 3.1 of Sen (1977), the details are omitted.

For every real \( d \); let us define

\[
\bar{L}_{n, k}(d) = L_k(d/T_n), 0 \leq k \leq n,
\]

where the \( L_k(b) \) are defined by (2.11). Also, in (4.1), replacing the \( L_{k^n}(u) \) by \( \bar{L}_{k^n}(u)(d) \), we define the corresponding stochastic process by \( \bar{W}_n^{(3)} = \{\bar{W}_n^{(3)}(u), 0 \leq u \leq 1\} \). Thus, \( W_n^{(3)} = \bar{W}_n^{(3)} \), \( n, 0 \).

**Theorem 4.3.** Under \( \{k^*_n, b\} \) in (4.8) and the hypothesis of Theorem 4.2, for every (fixed) \( d; -\infty < d < \infty \),

\[
\bar{W}_n^{(3)}_{n, d} \rightarrow \{W(t) + \omega(t) - d\lambda^* t, 0 \leq t \leq 1\}
\]

where \( \omega(t) \) is defined by (4.10).
Proof. Let \[ \Pi_{n, \mathbf{b}}^d \] be the distribution of \( \overline{\mathbf{W}}_n^{(3)} \) under \( \{K_{n, \mathbf{b}}^*\} \), defined for Borel subsets \( \mathcal{D} \) of \( \mathbb{D}[0,1] \). Note that by denoting by \( R_{nk}^*(d) = \text{rank of } Y_i - dt_i/T_n \) among \( Y_i - dt_1/T_n, \ldots, Y_i - dt_n/T_n \), for \( i = 1, \ldots, n \), \( R_{nk}^*(d) = (R_{n1}^*(d), \ldots, R_{nn}^*(d))^t \), it follows that
\[
[R_{nk}^*(d), \text{ under } K_{n, \mathbf{b}}^*] \overset{d}{=} [R_{nk}^*(0), \text{ under } K_{n, \mathbf{b}}^*],
\]
and hence, for every \( D \in \mathcal{D} \),
\[
\Pi_{n, \mathbf{b}}^d(D) = P_{n, \mathbf{b}}^d \left( \overline{\mathbf{W}}_n^{(3)} \in D \right| K_{n, \mathbf{b}}^* \}
\]
\[
= P_{n, 0} \left( \overline{\mathbf{W}}_n^{(3)} \in D \right| K_{n, \mathbf{b}}^* \}
\]
\[
= \Pi_{n, \mathbf{b}}^0(D),
\]
which insures that \( \{\overline{\mathbf{W}}_n^{(3)}, \text{ under } K_{n, \mathbf{b}}^*\} \overset{d}{=} \{\overline{\mathbf{W}}_n^{(3)}, \text{ under } K_{n, \mathbf{b}}^*\} \), and hence, (4.13) follows from Theorem 4.2. Q.E.D.

Theorem 4.4. Under \( \{K_{n, \mathbf{b}}^*\} \) in (4.8) and the hypothesis of Theorem 4.2, for every fixed \( \mathbf{b} \) and \( K(< \infty) \),
\[
\sup \{ |\overline{\mathbf{W}}_n^{(3)}(t) - \mathbf{W}_n^{(3)}(t) + \lambda^* dt| : 0 \leq t \leq 1, |\lambda| \leq K \} \overset{D}{=} 0.
\]

Proof. We may virtually repeat the proof of Theorem 3.3 of Sen (1977). We note that (3.24) and (3.25) of Sen (1977) hold here, by virtue of the basic result of Jurecková (1969) and our Theorem 4.3 here. Hence, the details are omitted.

We are now in a position to study the invariance principles for our aligned LRS\( \{L_{n,k}^* ; 0 \leq k \leq n\} \). Consider a sequence \( \{K_{n, \mathbf{b}}^*\} \) of alternative hypotheses, where
\[
K_{n, \mathbf{b}}^*: (1.1) \text{ holds with } \beta = \theta + T_{n}^{-1} b_1, \gamma = \theta + T_{n}^{-1} b_2,
\]
\( \mathbf{b} = (b_1, b_2) \neq 0 \). Further, in (4.1), we replace the \( L_{nk}^*(u) \) by \( L_{n,k}^*(u) \), \( 0 \leq u \leq 1 \) and denote the resulting process by \( \mathbf{W}_n^{(4)} = \{\mathbf{W}_n^{(4)}(u), 0 \leq u \leq 1\} \).
Note that \( \{L_k(\theta + a), 1 \leq k \leq n\} \) under \( K_n^* \) as the same distribution as \( \{L_k(a), 1 \leq k \leq n\} \) under \( K_n^* \). Hence, for the study of the distribution of \( W_n^{(4)} \) under \( \{K_n^*, n\} \), we may, without any loss of generality, set \( \theta = 0 \), i.e., under \( \{K_n^*, n\} \) in (4.8). By virtue of (2.13), (2.14), Theorem 4.4 and the above discussion, it follows by some standard steps that under \( \{K_n^*, n\} \),

\[
\lambda_n T_n \beta_n \times (1) + o_p(1),
\]

so that by Theorem 4.2,

\[
|T_n \beta_n^*| = o_p(1), \text{ under } \{K_n^*, n\}.
\]

By Theorem 4.4, (4.18) and (4.19), we obtain that under \( \{K_n^*, n\} \) (or equivalently, under \( \{K_n, n\} \) as \( W_n^{(4)} \) remains invariant if the \( Y_i \) are replaced by \( Y_i - \theta_i \), \( 1 \leq i \leq n \), for any real \( \theta \)),

\[
\sup\{|W_n^{(4)}(u) - W_n^{(3)}(u) + uW_n^{(3)}(1)|: u \in [0,1]\} \overset{p}{\rightarrow} 0,
\]

so that, if we define \( W_n^{(5)} = \{W_n^{(5)}(u) = W_n^{(3)}(u) - uW_n^{(3)}(1), 0 \leq u \leq 1\} \), then under \( \{K_n, n\} \),

\[
W_n^{(4)} \text{ and } W_n^{(5)} \text{ are convergent equivalent.}
\]

Finally, by an appeal to Theorem 4.2, we conclude that under \( \{K_n^*, n\} \),

\[
W_n^{(5)} \overset{L}{\underset{\r}{\rightarrow}} W^0 + \omega^0, \text{ in the } J_1\text{-topology on } D[0,1],
\]

where \( W^0 \) is standard Brownian bridge on \([0,1]\) and

\[
\omega^0(t) = \left\{ \begin{array}{ll}
\lambda^* (b_1 - b_2) t [1 - \rho - \nu(\mu(1) - \mu(\nu)) (t - \mu(\nu)) / \xi^2(1)], & 0 \leq t \leq \rho, \\
\lambda^* (b_1 - b_2) [\rho (1 - t) - \nu (t - \mu(\nu)) ((1 - t) \mu(\nu) - \mu(\mu(t)) + t \mu(1)) / \xi^2(1)], & \rho \leq t \leq 1,
\end{array} \right.
\]

\[
\lambda^* (b_1 - b_2) t [1 - \rho - \nu(\mu(1) - \mu(\nu)) (t - \mu(\nu)) / \xi^2(1)], & 0 \leq t \leq \rho, \\
\lambda^* (b_1 - b_2) [\rho (1 - t) - \nu (t - \mu(\nu)) ((1 - t) \mu(\nu) - \mu(\mu(t)) + t \mu(1)) / \xi^2(1)], & \rho \leq t \leq 1,
\end{array} \right.
\]
where \( \mu, \xi^2, \tau, \rho \) and \( h(t) \) are defined by (3.1), (3.2), (3.33) and (3.34).

This leads us to the main theorem of this section.

**Theorem 4.5.** Under \( \{K_n,b\} \) and the hypothesis of Theorem 4.2,

\[
W_n^{(4)} \to W^0 + \omega^0, \quad \text{in the } J_1\text{-topology on } D[0,1]
\]

where \( \omega^0 = \{\omega^0(t), 0 \leq t \leq 1\} \) is given by (4.23).

We notice that by definition in (2.17),

\[
D_n^+ = \sup_{0 \leq t \leq 1} W_n^{(4)}(t) \quad \text{and} \quad D_n = \sup_{0 \leq t \leq 1} |W_n^{(4)}(t)|.
\]

Hence, by Theorem 4.5 and (4.26), under \( H_0: \beta = \gamma \) i.e., \( K_n, Q, D_n^+ \) and \( D_n \) have the asymptotic distributions given by (3.24) and (3.25), respectively.

This leads us to the following test procedure:

Compute the \( L_{n,k}^* \), defined by (2.16). If, for at least one \( k: 1 \leq k \leq n-1, \)

\[
L_{n,k}^* \text{ (or } |L_{n,k}^*| \text{) exceeds } A_n^+ \text{ (or } A_n \epsilon \text{), reject } H_0
\]

in (1.4). If, no such \( k \) exists, accept \( H_0 \).

As in Section 3, we confine ourselves to local alternatives for the study of the asymptotic power properties of the tests based on (4.26). We assume that the same sequence \( \{H_n\} \) of alternative hypotheses in (3.35) holds. Then, by an appeal to Theorem 4.5, we conclude that under the regularity conditions of Theorem 4.5,

\[
\lim_{n \to \infty} P\{D_n^+ > \Delta_n^+ | H_n\} = P\{W^0(t) = \omega^0(t) > \Delta_n^+ \text{ for some } t \in [0,1]\},
\]

and

\[
\lim_{n \to \infty} P\{D_n > \Delta_n \text{ } |H_n\} = P\{|W^0(t) + \omega^0(t)| > \Delta_n \text{ for some } t \in [0,1]\},
\]

where
\[
\omega_0^0(t) = \begin{cases} 
\lambda^* \delta t [1-\rho-\nu(\mu(1)-\mu(\nu))(\tau-\mu(\nu))/\xi^2(1)], & 0 \leq t \leq \rho \\
\lambda^* \delta [\rho(1-t)-\nu(\tau-\mu(\nu))(1-t)\mu(\nu)-\mu(h(t))]
\end{cases}
\]
\[
t\mu(1)/\xi^2(1)], & \rho \leq t \leq 1.
\]

Note that \(\omega_0^0(t)\) is linear in \(t\) for \(0 \leq t \leq \rho\), but, in general, for \(\rho \leq t \leq 1\), \(\omega_0^0(t)\) is not linear in \(t\) [as \(\mu(h(t))\) need not be linear in \(t\)].

5. Asymptotic comparison of the LSE and LRS procedure.

It may be remarked that both \(M_n^+\) and \(D_n^+\) (or \(M_n/\sigma\) and \(D_n\)) have the same limiting null distribution (3.24) [or (3.25)]. If, we let

\[
b(t) = \begin{cases} 
(t+1)[1-\rho-\nu(\tau-\mu(\nu))(\mu(1)-\mu(\nu))/\xi^2(1)], & 0 \leq t \leq \rho/(1-\rho) \\
\rho-\nu(\tau-\mu(\nu))(\nu(\mu)+t\mu(1)-(t+1)\mu(h(t/(t+1))))/\xi^2(1)], & t \geq \rho/(1-\rho)
\end{cases}
\]

and note that \(\{(t+1)W^0_0\left(\frac{t}{t+1}\right), \ 0 \leq t < \infty\} = \{X(t), \ 0 \leq t < \infty\}\), where \(\{X(t), \ t \geq 0\}\) is a standard Wiener process on \(\mathbb{R}^+\), we obtain then from (3.35), (3.36) and (3.38) that the asymptotic power of \(M_n\) is equal to

\[
P\{W^0_0(t) \geq \Delta^+_\varepsilon - a_0(t) \text{ for some } t \in [0,1]\}
\]
\[
= P\left\{(t+1)W^0_0\left(\frac{t}{t+1}\right) \geq (t+1)\Delta^+_\varepsilon - (t-1)a_0\left(\frac{t}{t+1}\right) \text{ for some } t \in \mathbb{R}^+\right\}
\]
\[
= P\left\{X(t) \geq (t+1)\Delta^+_\varepsilon - \frac{\delta}{\sigma} b(t), \text{ for some } t \in \mathbb{R}^+\right\}
\]

Similarly, for \(M_n\), we have the asymptotic power is equal to

\[
P\left\{|X(t) + \frac{\delta}{\sigma} b(t)| \geq (t+1)\Delta^+_\varepsilon \text{ for some } t \in \mathbb{R}^+\right\}.
\]
Likewise, from (4.27)-(4.29) and (5.1), we have the asymptotic power of the $D_n^+$ test given by

$$P\{X(t) \geq (t+1)\Delta^+_c - \lambda^* \delta b(t) \text{ for some } t \in R^+\}$$

while for the $D_n^-$ test, it is equal to

$$P\{|X(t) + \lambda^* \delta b(t)| \geq (t+1)\Delta^-_c \text{ for some } t \in R^+\}.$$ 

Note that (5.2) though (5.5) are computed for the common sequence $\{H_n\}$ of alternatives [in (3.35)] when all the statistics are based on the same sample size $n$. Suppose now that $M_n^+$ and $M_n$ are based on sample size $\{n\}$ while $D_n^+$ or $D_n^-$ are based on $\{N=N(n)\}$, where

$$\lim_{n \to \infty} N(n)/n = e^{-1} \text{ for some } 0 < e < \infty.$$ 

For $D_n^+$ and $D_n^-$ statistics with $N=N(n)$ satisfying (5.6), we may proceed as in Section 4 where we replace $W_n^{(3)}$, $W_n^{(4)}$ and $W_n^{(5)}$ by $W_N^{(3)}$, $W_N^{(4)}$ and $W_N^{(5)}$, respectively, defined for $n=N$, while we stick to the same alternatives $K_n, b, K^*_n, b^*$ and $H_n$. Since, by (3.2) and (5.6),

$$\lim_{n \to \infty} T_n^2/N^2 = \lim_{n \to \infty} [(N^{-1}T_n^2/(n^{-1}T_n^2))\{N/n\}]$$

$$= \lim_{n \to \infty} [N(n)/n] = e^{-1},$$ 

in (4.10), (4.23) and (4.29), $\omega(u)$, $\omega^0(u)$ and $\omega^0_0(u)$ are to be replaced by $e^{-\lambda^* \omega(u)}$, $e^{-\lambda^* \omega^0_0(u)}$ and $e^{-\lambda^* \omega^0_0(u)}$, respectively. Thus, the asymptotic power of the test based on $\{D_n^+\}$ under $\{H_n\}$ in (3.33), when (5.6) holds, is given by

$$P\{X(t) \geq (t+1)\Delta^+_c - e^{-\frac{1}{2}\lambda^*} \delta b(t) \text{ for some } t \in R^+\}.$$ 

Consequently, if we choose $e$ by letting

$$e^{-\frac{1}{2}\lambda^*} = \sigma^{-1} \text{ i.e., } e = \sigma^2(\lambda^*)^2,$$ 

(5.8)
then (5.2) and (5.7) are equal (for any \( \delta \)). Hence, \( \theta \) in the usual Pitman-

sense, the asymptotic relative efficiency (A.R.E.) of the LRS procedure with
respect to the LSE procedure is

\[
(5.9) \quad e = \lim_{n \to \infty} \left[ \frac{n}{N(n)} \right] = \sigma^2(\lambda^*)^2 = \sigma^2 I(f) \left( \int_0^1 \phi(u) \psi(u) \, du \right)^2.
\]

The same efficiency result holds for \( D_n \) relative to \( M_n \).

Now (5.9) agrees with the classical Pitman-efficiency of the two-sample
rank order test (for location) relative to the Student's t-test. Thus, we may
conclude that if \( \phi(u) \equiv u; 0 < u < 1 \), then the corresponding rank procedure has
an A.R.E. with respect to the LSE procedure equal to \( 3/\pi \) when \( F \) is normal,
is bounded from below by 0.864 for all continuous \( F \) and is usually \( \geq 1 \) when
\( F \) has heavier (than normal) tails. Also, if \( \phi(u) = \Phi^{-1}(u), 0 < u < 1 \) (i.e.,
normal scores), then (5.9) is bounded from below by 1 where the lower bound
is attained only when \( F \) is normal. Thus, from the A.R.E. point of view, the
LRS procedures are attractive, they do not require the estimation of \( \sigma^2 \) (as
is need for the LSE procedure) and they are expected to be robust.

We conclude this section with the remark that \( b(t) \) in (5.1) is linear
in \( t \) for \( t \leq \rho/(1-\rho) \), while for \( t > \rho/(1-\rho) \), in general, it is not so.

Since \( \mu(s) \) is \( \nearrow \) in \( s \in [0,1] \), we have for \( t > \rho/(1-\rho) \)

\[
(5.10) \quad \rho - \rho(\tau - \mu(\nu)) t (\mu(1) - \mu(\nu)) \leq b(t) \leq \rho + \nu(\tau - \mu(\nu)) (\mu(1) - \mu(\nu)).
\]

Hence, for (5.2) and (5.4), bounds for the asymptotic power can be obtained
by using the segmented linear boundaries in (5.1) and (5.10) and then using
the results in Anderson (1960, Section 6). In general, these are quite com-
plicated to be expressible in closed form.
REFERENCES


