A NOTE ON THE P_N METHOD WITH MARK BOUNDARY CONDITIONS

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Abstract—In this short note, it is shown that the P_N solution for a half-space with isotropic scattering subject to an isotropic incident distribution yields, in any order, the exact scalar flux at the boundary when boundary conditions of the Mark type are used.

I. INTRODUCTION

Recently,[1] during a study that compared different types of P_N boundary conditions for a few basic transport problems, we found numerical evidence that the P_N method with boundary conditions of the Mark type yields, in any order, the exact scalar flux at the boundary of an isotropically scattering half-space subject to an isotropic incident distribution. Since we are not aware of any previous mention of this result, we report here a proof that confirms our numerical observations.

II. THE PROOF

We start with the transport equation, for $x > 0$ and $-1 \le \mu \le 1$,

$$\mu \frac{\partial}{\partial x} \Psi(x, \mu) + \Psi(x, \mu) = \frac{c}{2} \int_{-1}^{1} \Psi(x, \mu') \mathrm{d}\mu' \tag{1}$$

and the boundary conditions, for $\mu > 0$,

$$\Psi(0, \mu) = 1 \tag{2a}$$

and

$$\lim_{x \to \infty} \Psi(x, -\mu) = 0, \tag{2b}$$

where $\Psi(x, \mu)$ denotes the particle distribution function at position x and angle $\cos^{-1} \mu$, and where $c \in (0, 1)$ denotes the mean number of secondary particles emitted per collision.

As is well known,[2] the first $N + 1$ moments of Eq. (1) are satisfied by the P_N approximation (with N odd)

$$\Psi(x, \mu) = \sum_{n=0}^{N} \left(\frac{2n+1}{2} \right) \psi_n(x) P_n(\mu), \tag{3}$$

where, considering Eq. (2b),

$$\psi_n(x) = \sum_{j=1}^{J} A_j e^{-x/\xi_j} g_n(\xi_j). \tag{4}$$

Here $J = (N+1)/2$, $g_n(\xi)$ is the Chandrasekhar polynomial of order n, the eigenvalue ξ_j is the j-th positive zero of $g_{N+1}(\xi)$ and $\{A_j\}$ are coefficients to be determined.

If we now use the Mark prescription for boundary conditions,[2,3] i.e. if we require that Eq. (2a) be satisfied at μ_i, $i = 1, 2, \ldots, J$, the positive zeros of the Legendre polynomial $P_{N+1}(\mu)$, we obtain, for $i = 1, 2, \ldots, J$,

$$\sum_{j=1}^{J} A_j \sum_{n=0}^{N} \left(\frac{2n+1}{2} \right) P_n(\mu_i) g_n(\xi_j) = 1, \tag{5}$$

a system of linear algebraic equations to be solved in order to determine the coefficients A_j, $j = 1, 2, \ldots, J$. We now wish to show that we can derive an analytical expression for these coefficients. First, we specialize to isotropic scattering a more general relation reported by İnönü[4] to obtain

$$(\mu - \xi) \sum_{l=0}^{k} (2l+1) P_l(\mu) g_l(\xi) = (k+1) \left[P_{k+1}(\mu) g_k(\xi) - P_k(\mu) g_{k+1}(\xi) \right] - c\xi. \tag{6}$$

Using this result, we can rewrite Eq. (5) for $i = 1, 2, \ldots, J$ as

$$\left(\frac{c}{2} \right) \sum_{j=1}^{J} A_j \left(\frac{\xi_j}{\xi_j - \mu_i} \right) = 1. \tag{7}$$

We now define, in the manner of Chandrasekhar,[5]

$$H(-z) = \Gamma \frac{C(z)}{D(z)}, \tag{8}$$

where

$$C(z) = \prod_{k=1}^{J} (z - \mu_k), \tag{9a}$$

$$D(z) = \prod_{k=1}^{J} (z - \xi_k) \tag{9b}$$

and

$$\Gamma = \prod_{k=1}^{J} \left(\frac{\xi_k}{\mu_k} \right). \tag{10}$$

In regard to $H(-z)$, we note that by using Cauchy's theorem we can develop the alternative representation

$$H(-z) = \Gamma \left[1 + \sum_{k=1}^{J} \frac{C(\xi_k)}{(\xi_k - z)D'(\xi_k)} \right]. \tag{11}$$

In addition, since

$$P_{N+1}(z) = (-1)^J \frac{(2N+1)!!}{(N+1)!} C(z)C(-z) \tag{12a}$$

and, for isotropic scattering,

$$g_{N+1}(z) = (-1)^J (1-c) \frac{(2N+1)!!}{(N+1)!} D(z)D(-z), \tag{12b}$$

we can show that the constant Γ can be written as

$$\Gamma = (1-c)^{-1/2}. \tag{13}$$

Proceeding with our proof, we use Eq. (11) for $z = \mu_i$, $i = 1, 2, \dots, J$, to obtain

$$\sum_{j=1}^{J} \frac{C(\xi_j)}{(\xi_j - \mu_i)D'(\xi_j)} = -1. \tag{14}$$

By comparing this result to Eq. (7), we conclude that

$$A_j = -\left(\frac{2}{c}\right) \left[\frac{C(\xi_j)}{\xi_j D'(\xi_j)} \right], \tag{15}$$

for $j = 1, 2, \dots, J$. Having found the required coefficients, we can now compute the scalar flux at the boundary

$$\phi(0) = \int_{-1}^{1} \Psi(0, \mu) \mathrm{d}\mu. \tag{16}$$

With the help of Eqs. (3), (4) and (15), we find

$$\phi(0) = -\left(\frac{2}{c}\right) \sum_{j=1}^{J} \frac{C(\xi_j)}{\xi_j D'(\xi_j)}. \tag{17}$$

This result can be simplified further if we set $z = 0$ in Eq. (11), use the resulting equation in Eq. (17) and note from Eqs. (8) to (10) that $H(0) = 1$. We find

$$\phi(0) = \left(\frac{2}{c}\right)\left[1 - (1-c)^{1/2}\right]. \tag{18}$$

Finally, we can easily integrate Chandrasekhar's H-function solution to the half-space albedo problem[5] to verify that Eq. (18) is indeed the exact result for $\phi(0)$ in our problem.

III. CONCLUSION

We believe that the result just proved in Sec. II can be useful to users of the P_N method with boundary conditions of the Mark type as a simple check of the correctness of their numerical implementations, for isotropic scattering.

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