Title: Vibration Suppression with Approximate Finite Dimensional Compensators for Distributed Systems: Computational Methods and Experimental Results

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ABSTRACT

Based on a distributed parameter model for vibrations, an approximate finite dimensional dynamic compensator is designed to suppress vibrations (multiple modes with a broad band of frequencies) of a circular plate with Kelvin-Voigt damping and clamped boundary conditions. The control is realized via piezoceramic patches bonded to the plate and is calculated from information available from several pointwise observed state variables. Examples from computational studies as well as use in laboratory experiments are presented to demonstrate the effectiveness of this design.

1. INTRODUCTION

In recent years, a great deal of research has been carried out on the development and derivation of control designs from an infinite dimensional state-space approach; however we are unaware of any implementation based on such designs being reported in the literature. We have presented briefly in [1] some preliminary experimental results on implementation of an output feedback control which was designed based on an infinite dimensional (or distributed) system. In this paper we attempt to summarize the design methodology and give further discussion of the implementation.

The feedback control system was implemented on a circular plate with a piezoceramic patch as actuator. This choice of structure was motivated by the fact that it is an isolated component from the structural acoustic system described in [7, 8]. The structure in that system is made up of a hardwalled cylinder with a clamped circular plate at one end and the control problem consists of using piezoceramic patches on the plate to reduce the interior structure-born sound pressure levels which result when the plate is subjected to a strong exterior acoustic field. The partial differential equation (PDE) system which describes the dynamics of this circular plate is presented in

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Section 3 below.

In our control design, three primary concerns are: 1) presence of disturbance in both input and output of the system; 2) robustness of control; 3) lack of full state measurement. Those concerns lead to a design problem involving dynamic compensators for distributed parameter systems. A great deal of recent research has been carried on the individual or combined topics of our concerns here. For example, see [2, 4, 13, 15, 16, 17, 18, 20, 21], and the references therein. This problem involves difficult issues and many theoretical and computational questions remain to be resolved. The purpose of this paper is to demonstrate how finite dimensional control theory together with approximation theory for certain optimal control problems can be used to successfully design and implement feedback controllers for flexible structures. A finite dimensional dynamic compensator design is outlined in Section 2. The approximation scheme which leads to a finite dimensional control problem will be discussed in Section 4. A numerical example is given in Section 5 along with experimental results to provide preliminary validation regarding the implementation of a PDE-based (distributed parameter or infinite dimensional system based) method for reducing structural vibrations.

2. CONTROL PROBLEM FORMULATION

We first consider an n-dimensional system

$$\begin{aligned} \dot{y}(t) &= Ay(t) + Bu(t), \quad y(0) = y_0, \\ y_{ob}(t) &= Cy(t), \\ z(t) &= Hy(t) + Gu(t), \end{aligned} \tag{1}$$

where the state variable y is in $\mathbb{R}^n$, the control u is in $\mathbb{R}^m$, the measurement y_{ob} is in $\mathbb{R}^p$ and the controlled output z is in $\mathbb{R}^r$ for some finite positive integers n, m, p, and r. The coefficients A, B, C, H, and G are time invariant matrices. The performance index (or cost function) is given by

$$\begin{aligned} J(u) &= \int_0^\infty \left|z(t)\right|^2 dt \\ &= \int_0^\infty \left\{ \left\langle Qy(t), y(t) \right\rangle + \left\langle Ru(t), u(t) \right\rangle \right\} dt, \end{aligned} \tag{2}$$

subject to (1). In (2), $Q = H'H$, $R = G'G$, where we assume $H'G = 0$. The control problem is to find a controller $u \in L_2(0, \infty; \mathbb{R}^m)$ which minimizes the cost function (2).

We are interested in the case when, as in most practical situations, measurement of the full state is not available ($p < n$). (We note that when $y_{ob}(t) = y(t)$ the solution can be obtained by applying the well known linear quadratic regulator (LQR) optimal control theory.) One possible approach is to build a state estimator or observer to reconstruct the state from the measured partial state. One can then feed back this reconstructed state. In this paper, we consider a full order observer for simplicity (it is adequate for the plate experiments described below). Design of low order observers is important when the dimension of the control system is large, and we refer to [10, 15] for details on reduced order observers.

Let the reconstructed state be denoted by $y_c(t)$. We consider the standard compensator (of Luenberger type [19]) for system (1) given by

$$\begin{aligned} \dot{y}_c(t) &= A_c y_c(t) + F y_{ob}(t), \\ A_c &= A - FC - BK, \\ u(t) &= -K y_c(t), \end{aligned} \tag{3}$$

for a properly chosen feedback gain K and observer gain F so that the reconstruction error $|y_c(t) - y(t)| \to 0$ as $t \to \infty$ and the closed loop system

$$\begin{bmatrix} \dot{y}(t) \\ y_c(t) \end{bmatrix} = \begin{bmatrix} A & -BK \\ FC & A_c \end{bmatrix} \begin{bmatrix} y(t) \\ y_c(t) \end{bmatrix}$$

is exponentially stable. Intuitively, we would like to choose F such that the observer poles of $A - FC$ are deep in the left half complex plane to obtain fast convergence of the reconstruction error. This must be done with care since an observer so constructed is very sensitive to any observation noise that may exist.

Among several compensator designs, we first consider the so called optimal compensator. Suppose that the matrix Q is nonnegative-definite, R is positive-definite, the pair (A, B) is stabilizable, (A, C) is detectable, (A, G) is controllable, and (A, H) is observable. Then there exist unique (minimal) optimal feedback gain K and observer gain F given by

$$K = R^{-1} B' \Pi \tag{4}$$

$$F = P C' \tilde{R}^{-1} \tag{5}$$

where Π and P are unique nonnegative-definite solutions to the following regulator and observer algebraic Riccati equations

$$\Pi A + A' \Pi - \Pi B R^{-1} B' \Pi + Q = 0, \tag{6}$$

$$P A' + A P - P C' \tilde{R}^{-1} C P + \tilde{Q} = 0, \tag{7}$$

respectively. Thus the optimal estimator is obtained and given by (3)-(5). In (7), $\tilde{Q}$ is a nonnegative symmetric matrix and $\tilde{R}$ is a positive symmetric matrix. The matrices Q, R, $\tilde{Q}$, and $\tilde{R}$ are determined by some design criteria for the specific control problems. We point out that this "optimal" observer can be defined without depending on the (traditional) stochastic formulation. The name "optimal" is derived from the stochastic interpretation of the above design (see [13, 19] for further discussions). Briefly, the above described observer

$$\dot{y}_c(t) = \Big(A - B R^{-1} B' \Pi\Big) y_c(t) + P C' \tilde{R}^{-1} C \Big(y(t) - y_c(t)\Big) \tag{8}$$

is simply the *Kalman-Bucy filter* if we consider the system (1) disturbed by the uncorrelated stationary Gaussian white noise $v_1(t)$ and $v_2(t)$:

$$\begin{aligned} \dot{y}(t) &= Ay(t) + Bu(t) + v_1(t), \quad y(0) = y_0, \\ y_{ob}(t) &= Cy(t) + v_2(t), \\ z(t) &= Hy(t) + Gu(t), \end{aligned} \tag{9}$$

where

$$\begin{aligned} &\mathcal{E}\{v_1(t)\} = 0, \quad \mathcal{E}\{v_1(t)\, v_1'(\tau)\} = \tilde{Q}\, \delta(t-\tau) \\ &\mathcal{E}\{v_2(t)\} = 0, \quad \mathcal{E}\{v_2(t)\, v_2'(\tau)\} = \tilde{R}\, \delta(t-\tau). \end{aligned}$$

Here $\mathcal{E}\{\ \}$ is the expected value. The observer (8) is optimal in the sense that the limit of the mean square reconstructed error

$$\lim_{t\to\infty} \mathcal{E}\Big\{\Big(y(t) - y_c(t)\Big)' W \,\Big(y(t) - y_c(t)\Big)\Big\}$$

(W is a weighting matrix) is minimal with respect to all other observers (e.g., see [19]).

Even though the optimal compensator provides us with the desired performance, it is well-known that it may lack robustness. To design a robust dynamic compensator, let us consider the system (1) with input and output disturbance $w(t)$

$$\begin{aligned} \dot{y}(t) &= Ay(t) + Bu(t) + Dw(t), \quad y(0) = y_0, \\ y_{ob}(t) &= Cy(t) + Ew(t), \\ z(t) &= Hy(t) + Gu(t), \end{aligned} \tag{10}$$

where the disturbance vector $w(t)$ is in $\mathbb{R}^q$ for some finite positive integer q. The coefficients D and E are time invariant matrices. Furthermore, we will restrict ourselves to matrices H, G, D, and E such that $H'H = Q \geq 0$, $G'G = R > 0$, $H'G = 0$, $DD' = \tilde{Q} \geq 0$, $EE' = \tilde{R} > 0$, $DE' = 0$. The more general case where the cross product terms $H'G$ and DE' are not zero can be dealt with in a similar manner with slight modifications (see [9]).

Our objective is to design a robust controller that provides acceptable performance with disturbed incomplete state measurements. One such design technique is the so-called $H_\infty/MinMax$ compensator given in [9]. One formulates H_∞-control problems in the time domain and obtains a soft-constrained dynamic game associated with the disturbance attenuation problem. The control problem is formulated as a form of optimization of a performance index (or cost function). For this purpose, we introduce the extended performance index:

$$
\begin{aligned}
J_\gamma(u,w) &= \int_0^\infty \left\{ \left|z(t)\right|^2 - \gamma^2 \left|w(t)\right|^2 \right\} dt \\
&= \int_0^\infty \left\{ \left\langle Qy(t), y(t) \right\rangle + \left\langle Ru(t), u(t) \right\rangle - \gamma^2 \left\langle w(t), w(t) \right\rangle \right\} dt
\end{aligned} \tag{11}
$$

subject to (10). The optimization problem is to find a controller $u^* \in U \equiv L_2(0,\infty; I\!R^m)$ and disturbance $w^* \in W \equiv L_2(0,\infty; I\!R^q)$ such that

$$J_\gamma^* = \inf_{u \in U} \sup_{w \in W} J_\gamma(u,w) = J_\gamma(u^*, w^*).$$

One seeks necessary and sufficient conditions on γ so that quantity J_γ^* is finite. The lower bound of γ for which $J_\gamma^* < \infty$ is the optimal minimax attenuation level and is denoted by γ^*, i.e.

$$\gamma^* = \inf\{\gamma : J_\gamma^* < \infty\}.$$

The first part of the optimization problem formulates the soft-constrained game and the second is a disturbance attenuation problem (γ is the attenuation level). It can be shown that the results of this optimization problem yields a bound for the H_∞-norm of the transfer function from disturbance $w(t)$ to the controlled output $z(t)$.

To be more precise, the central results for this control problem can be summarized as following. Let the pair (A,B) be stabilizable, (A,C) be detectable, (A,G) be controllable, and (A,H) be observable. For a given attenuation $\gamma > 0$, there exist (minimal) positive definite solutions Π and P to the following two algebraic Riccati equations

$$\Pi A + A'\Pi - \Pi(BR^{-1}B' - \gamma^{-2}\tilde{Q})\Pi + Q = 0, \tag{12}$$

$$PA' + AP - P(C'\tilde{R}^{-1}C - \gamma^{-2}Q)P + \tilde{Q} = 0, \tag{13}$$

respectively. Moreover, if the spectral radius ρ of $P\Pi$ satisfies the condition

$$\rho(P\Pi) < \gamma^2, \quad \text{or} \quad \Pi - \gamma^2 P^{-1} < 0, \tag{14}$$

then there exists a unique optimal controller

$$u^*(t) = -R^{-1}B'\Pi y_c(t), \tag{15}$$

and the state estimator $y_c(t) \in I\!R^n$ satisfies

$$
\begin{aligned}
\dot{y}_c(t) &= A_c y_c(t) + F y_{ob}(t), \\
y_c(0) &= y_{c0},
\end{aligned} \tag{16}
$$

where

$$
\begin{aligned}
A_c &= A - BK - FC + \gamma^{-2}\tilde{Q}\Pi \\
F &= (I - \gamma^{-2}P\Pi)^{-1}PC'\tilde{R}^{-1}.
\end{aligned}
$$

In addition, we have $\gamma \geq \gamma^*$.

The resulting closed-loop system

$$\begin{bmatrix} \dot{y}(t) \\ y_c(t) \end{bmatrix} = \begin{bmatrix} A & -BK \\ FC & A_c \end{bmatrix} \begin{bmatrix} y(t) \\ y_c(t) \end{bmatrix} \tag{17}$$

with the controlled output

$$z(t) = \begin{bmatrix} H & 0 \\ 0 & -GK \end{bmatrix} \begin{bmatrix} y(t) \\ y_c(t) \end{bmatrix} \tag{18}$$

is stable.

Finally, if we let $\hat{z}(s)$ and $\hat{w}(s)$ denote the Laplace transform of $z(t)$ and $w(t)$ respectively, then the transfer function from the disturbance $w(t)$ to the controlled output $z(t)$ is expressed by

$$T(s) = \frac{\hat{z}(s)}{\hat{w}(s)} = \begin{bmatrix} H & 0 \\ 0 & -GK \end{bmatrix} \left[sI - \begin{pmatrix} A & -BK \\ FC & A_c \end{pmatrix} \right]^{-1} \begin{bmatrix} D \\ FE \end{bmatrix}. \tag{19}$$

Furthermore, the H_∞ norm of the transfer function (19) is bounded by

$$\|T(\cdot)\|_\infty \leq \gamma.$$

Thus, if we follow this procedure we obtain a dynamic compensator which not only stabilizes the system with imperfect state measurements, but also provides robustness.

3. STRUCTURAL MODEL

In this section, the mathematical model used to describe the experimental setup is given. To reduce computational complexity, in our initial experiments the structure is axisymmetriclly configured. We point out that all of the results and techniques presented here can be extended directly to the more general case of nonaxisymmetric configurations. The structure under study is a fixed-edge circular plate with a centrally placed circular shaped piezoelectric ceramic patch for actuation and sensing. The equations of motion will be formulated in polar coordinates (r, θ). Under the Love-Kirchhoff plate theory with Kelvin-Voigt (or strain rate) damping, the transverse vibrations $w(t, r, \theta)$ of a plate of radius a subject to an axisymmetric external force $g(t, r, \theta)$ are described by the system

$$\begin{aligned} &\tilde{\rho}(r,\theta)\frac{\partial^2 w}{\partial t^2} + c_a \frac{\partial w}{\partial t} + \frac{\partial^2 M_r}{\partial r^2} + \frac{2}{r}\frac{\partial M_r}{\partial r} - \frac{1}{r}\frac{\partial M_\theta}{\partial r} = \nabla^2 M_{pe} + g, \quad t > 0,\ 0 < r < a, \\ &w(t, a, \theta) = 0, \qquad \frac{\partial w}{\partial t}(t, a, \theta) = 0, \end{aligned} \tag{20}$$

where the internal bending moments are

$$
\begin{aligned}
M_r &= D(r,\theta)\left(\frac{\partial^2 w}{\partial r^2} + \frac{\nu(r,\theta)}{r}\frac{\partial w}{\partial r}\right) + c_D(r,\theta)\left(\frac{\partial^3 w}{\partial r^2 \partial t} + \frac{\nu(r,\theta)}{r}\frac{\partial^2 w}{\partial r \partial t}\right) \\
M_\theta &= D(r,\theta)\left(\frac{1}{r}\frac{\partial w}{\partial r} + \nu(r,\theta)\frac{\partial^2 w}{\partial r^2}\right) + c_D(r,\theta)\left(\frac{1}{r}\frac{\partial^2 w}{\partial r \partial t} + \nu(r,\theta)\frac{\partial^3 w}{\partial r^2 \partial t}\right),
\end{aligned}
$$

and the piezoceramic patch generated excitation moment is $M_{pe} = \mathcal{K}_B \chi_{pe} u(t)$. Here c_a is the viscous (air) damping coefficient, and $u(t)$ is the voltage applied to the patch. With E denoting the Young's modulus, the spatial variables $D = \frac{Eh^3}{12(1-\nu^2)}$, ν, $\tilde{\rho}$ and c_D represent the flexural rigidity, Poisson's ratio, the mass density per area, and Kelvin-Voigt damping for the plate/patch structure. The constant $\mathcal{K}_B$ is a piezoelectric parameter depending on the material piezoelectric properties as well as geometry, and the characteristic function χ_{pe} is given by $\chi_{pe}(r) = 1$, for $r < a_{pe}$, $\chi_{pe}(r) = 0$, for $r \geq a_{pe}$, for a patch of radius a_{pe}. The term $\nabla^2 M_{pe}$ in (20) is an unbounded operator involving Dirac delta function derivatives.

Let Ω denote the region occupied by the plate and Ω_{pe} that of the patch. The energy (weak or variational) form of system (20) is

$$
\begin{aligned}
\int_\Omega \tilde{\rho}\,\frac{\partial^2 w}{\partial r^2}\,\overline{\eta}\,d\omega + \int_\Omega c_a\,\frac{\partial w}{\partial t}\,\overline{\eta}\,d\omega + \int_\Omega M_r\,\overline{\frac{\partial^2 \eta}{\partial r^2}}\,d\omega + \int_\Omega \frac{1}{r}\,M_\theta\,\overline{\frac{\partial \eta}{\partial r}}\,d\omega \\
= \int_{\Omega_{pe}} \mathcal{K}_B\,u(t)\,\overline{\nabla^2 \eta}\,d\omega + \int_\Omega g\,\overline{\eta}\,d\omega
\end{aligned}
\tag{21}
$$

for a class of test functions η (see [8] for details).

4. PLATE VIBRATION CONTROL

For the control examples discussed here, we will concentrate on the situation where the plate starts with a given initial displacement and velocity and is then allowed to vibrate. It is also assumed that there are no external forces applied, i.e. $g(t,r,\theta) = 0$. The goal in the control problem is to determine a voltage $u(t)$ which, when applied to the piezoceramic patch, leads to a significantly reduced level of vibration. For our discussions in this note, the control problem formulation (1)-(2) is used and we implement the optimal observer of (3)-(8). Robustness of the controller was not considered in our first attempt of implementation of the control design. However, it is the subject of our current efforts and will be reported on elsewhere.

The system describing the dynamics of the plate is infinite dimensional. To approximate the plate dynamics, a Fourier-Galerkin scheme is used to discretize the infinite dimensional system (21). Following the ideas detailed in [8], the plate displacement is approximated by

$$
w^N(t,r,\theta) = \sum_{n=1}^{N} w_n^N(t)\,B_n(r,\theta) \tag{22}
$$

where $\{B_n(r,\theta)\}_{n=1}^N$ are cubic spline/Fourier basis functions. The substitution of the expansion (22) into (21) yields the $2N \times 2N$ matrix system

$$\begin{aligned} \dot{y}^N(t) &= A^N y^N(t) + B^N u(t), \\ y^N(0) &= y_0^N, \end{aligned} \tag{23}$$

where $y^N(t) = [w_1^N(t), \cdots, w_N^N(t), \dot{w}_1^N(t), \cdots, \dot{w}_N^N(t)]$ denotes the $2N \times 1$ vector containing the generalized Fourier coefficients for the approximate displacement and velocity (see [8] for details concerning the discretization of the circular plate equation and formulation of the matrices A^N and B^N). To simplify the notation, the superscript N (which is fixed) will be dropped hereafter in this note. The systems in what follows are understood to be finite dimensional.

It has been shown in [5, 6, 12] that the approximation scheme is well defined in the sense that solution to the finite dimensional system (23) converges to the solution to the original infinite dimensional system (21).

For the finite dimensional approximate system, the problem of determining a controlling voltage can be posed as the problem of finding $u(t)$ which minimizes the cost function (2) where $y(t)$ is the solution to (23). From the control design results in Section 2, the optimal controller and observer are easily obtained from (3)–(7).

We report here on a finite dimensional compensator for the approximate system. A natural question is whether this compensator will stabilize the infinite dimensional system. For bounded input and bounded output operator systems, we refer to [13, 15] for detailed discussions on this issue. For the systems with unbounded input and output operators, additional results and conditions under which well-posedness and convergence are assured can be found in [4, 16, 17, 18, 20] as well as other references.

5. NUMERICAL EXAMPLES

It is known that the PDE-based control design introduced in preceding sections requires accurate knowledge of system parameters $\tilde{\rho}$, D, ν, c_D, c_a and $\mathcal{K}_B$. Even though material handbooks may provide partial information, the damping coefficients are always unknown and the piezoelectric material constant is given only up to certain range of values. Before the feedback control law can be designed and implemented, significant parameter identification efforts must be carried out. The methodology and results for theoretical issues for this parameter identification is reported in [3, 8]. Specific parameter identification results using experimental data which were obtained from our circular plate were reported in [1]. The same experimental setup was later used in control law implementation. The dimension of the aluminum plate and piezoceramic patch are summarized in Table I. The table also contains "handbook" values for the Young's modulus, Poisson ratio and density of the plate and patch. The estimated parameters via fitting model response to the experimental data are summarized in Table II. As explained in [12], the parameters $\tilde{\rho}$, D, c_D and ν have discontinuities (at the patch boundary $r = a_p$) which must be estimated.

TABLE I: PLATE AND PZT PROPERTIES.

	Plate Properties	Patch Properties
Radius	$a = .2286\,(m)$	$rad = .01905\,(m)$
Thickness	$h = .00127\,(m)$	$T = .0001778\,(m)$
Young's modulus	$E = 7.1 \times 10^{10}\,(N/m^2)$	$E_{pe} = 6.3 \times 10^{10}\,(N/m^2)$
Density	$\rho = 2700\,(kg/m^3)$	$\rho_{pe} = 7600\,(kg/m^3)$
Poisson ratio	$\nu = .33$	$\nu_{pe} = .31$
Strain coefficient		$d_{31} = 190 \times 10^{-12}\,(m/V)$

TABLE II: ANALYTICAL AND ESTIMATED VALUES OF THE PHYSICAL PARAMETERS.

	$\tilde{\rho}\,(kg/m^2)$		$D\,(N \cdot m)$		$c_D\ (N \cdot m \cdot s)$ $\times 10^{-4}$		ν		c_a $(s \cdot N/m)$	$\mathcal{K}_B$ (N/V)
	beam	b+P	beam	b+P	beam	b+P	beam	b+P		
Ana.	3.429		13.601				.33			.013369
Est.	3.157	3.123	11.017	11.178	2.158	2.210	.3304	.3271	15.566	.015288

Using these estimated values of the physical parameters, simulation studies were carried out. To closely resemble the experimental setup, we assumed that a single point observation, velocity at center, is available. The nonnegative $2N \times 2N$ matrix Q was chosen by taking energy into consideration and weighted as explained in [3], and the positive matrix R is just a positive constant which penalizes unrealistically large voltages. The matrices $\tilde{Q}$, $\tilde{R}$ are chosen to be $2N \times 2N$ and $p \times p$ identity matrices, respectively, where p is the number of observations ($p = 1$ in our simulation study). The simulation was carried out in two steps. First, the PDE system with an external excitation force and without control ($u(t) = 0$) was solved for the time period of $[0, t_1]$. The excitation force was cut off before time t_1. The solutions at t_1 were then used as initial conditions (displacement and velocity) in solving the system with control. A recorded impact hammer hit was used as the excitation force. The simulation result is depicted in Figure 1. In this figure, plot (a) is a time history of uncontrolled versus controlled velocities, and plot (b) is the control voltage fed back to the piezoceramic patch. The maximum voltage reflected the choice of weights $d = 1$ and $R_e = 10^{-7}$ for the design parameters Q and R respectively. It was observed that it is the ratio R_e/d which influences the amplitude of the controlling voltage. The sampling time was set to 1/12000 Hz, and dimension of the approximation was set to $N = 16$ under the criteria that solution to (23) does not vary significantly if the dimension was larger than 16.