NONPARAMETRIC ESTIMATORS OF AVAILABILITY UNDER
PROVISIONS OF SPARE AND REPAIR

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Nonparametric (as well as jackknifed) estimators are developed for the 'availability' of an equipment supported by a single spare and a repair facility, where down-time occurs whenever no spare/repaird unit is available at the point of failure of an operating unit. Asymptotic properties of the natural and the jackknifed estimators of availability are studied (without assuming the stochastic independence of life and repair times). Fixed sample size as well as sequential procedures are considered, and progressively censored schemes are also introduced in this context.

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1. INTRODUCTION

To introduce the basic model, we consider a single-unit system supported by a repair facility and a single spare. At the time when the operating unit fails, simultaneously, the spare takes over (instantaneously) as the new operating unit and the failed unit is sent for repair at the same instant which is a regeneration point. This system fails when the unit, currently operating, fails before the repair of the currently failed unit is completed. Otherwise, a failed unit on completion of repair assumes the role of a spare in cold stand-by attitude. Usually, it is assumed that the repair of a failed unit
restores it to its new condition, and also that the life distributions of the
original initial spare and the initial operating unit are the same, say F, defined on \( R^+ = (0, \infty) \). If G be the distribution of the repair time of the
failed unit (also defined on \( R^+ \)), and if \( \mu_F \) and \( \mu_G \) denote respectively the
means of the life time \( X \) and repair time \( Y \) having the distributions F and G,
then the \textit{limiting average availability} (i.e., the limiting expected
proportion of \textit{system up-times}) is defined as
\[
A_{FG} = \frac{E0}{[E0 + ED]},
\]
(1.1)
where EO is the mean time until system failure, measuring from the regen-
eration point, and ED is the mean \textit{system down time}. If we denote by
\[
\alpha_{FG} = P\{ X < Y \},
\]
(1.2)
then, we have
\[
E0 = \{1 - \alpha_{FG}\}^{-1} \mu_F \quad \text{and} \quad ED = \{1 - \alpha_{FG}\}^{-1} E\{(Y-X)I(Y > X)\},
\]
(1.3)
where \( I(A) \) stands for the indicator function of the set A. The renewal theorem
based results in (1.3) assume that for different \( i \), the vectors \( (X_i, Y_i) \) are
stochastically independent, though for each \( i \), \( X_i \) and \( Y_i \) may not be indepen-
dent. Assuming the operating periods \( (X_i) \) and repair periods \( (Y_i) \) to be
mutually independent, a simplified version of (1.1) is given in Barlow and
Proschan (1975, Ch.7), where other parametric models have also been briefly
discussed. When both F and G are exponential d.f.'s, this reduces to
\[
\mu_F (\mu_F + \mu_G) (\mu_F^2 + \mu_F \mu_G + \mu_G^2)^{-1} = (1+\rho)/(1+\rho^2),
\]
(1.4)
where \( \rho = \mu_G/\mu_F \). But, in general, \( A_{FG} \) in (1.1) depends on F and G in a more
involved manner (and not just on their means). In practice, however, F and G
are generally not known, and hence, for a desirable maintenance of the system,
one may like to estimate \( A_{FG} \) in a nonparametric way (and also to provide some
sharp bounds on \( A_{FG} \)). Recently, Bhattacharjee and Kandar (1983) have
considered some useful bounds for \( A_{FG} \) which are computable with only a limited
knowledge about a few parameters of the unknown life and repair distributions.
One of the problems with such parametric procedures is their lack of robustness for departures from the assumed model (e.g., F actually Weibull against the assumed exponential form). Even for a small departure, there may be considerable loss of efficiency of the parametric procedure, while for major departures, they may even turn out to be inefficient or inconsistent. The main objective of the current study is to focus on some nonparametric developments both in the fixed sample size and sequential sampling schemes. Progressively censored schemes (PCS) based on the (regeneration) cycle times \( \{T_i\} \) are also considered in this context.

Looking at (1.1), (1.2) and (1.3), we may gather that \( A_{FG} \) is a function of \( u_F, \alpha_{FG} \) and ED, each one of which being a regular functional of the d.f. \( F \) (and \( G \)). Thus, these estimable parameters (i.e., \( u_F, \alpha_{FG} \) and ED) can be estimated in a nonparametric fashion (under quite general regularity conditions), wherein the validity and reliability of the estimates will be ensured for a broader class of \( F \) and \( G \). Or, in other words, these nonparametric procedures will be robust and have broader scopes. However, in view of the fact that \( A_{FG} \) is not a linear function of these parameters, the resulting estimates may not be unbiased, and hence, some resampling schemes may be incorporated to reduce the effective bias, and jackknifing techniques are therefore adopted to achieve this goal.

In Section 2, without assuming the independence of the failure and repair times, a reformulation of \( A_{FG} \) in terms of some other functionals is provided, and this enables us to consider a very natural estimator. Fixed sample size estimation procedures are then introduced in Section 3. Section 4 is devoted to the development of sequential procedures pertaining to the confidence interval problem as well as the sequential testing problem. The concluding section deals briefly with PCS and some of the useful applications in the current context.
2. A REFORMULATION OF \( A_{FG} \)

Note that the \( i \)th cycle involves the life time \( X_i \) and repair time \( Y_i \), and we do not necessarily assume that \( X_i \) and \( Y_i \) are mutually independent, although it is assumed that \( \{(X_i, Y_i), i \geq 1\} \) form a sequence of independent and identically distributed (i.i.d.) random vectors (r.v.) with a (bivariate) distribution function (d.f.) \( H(x, y) \), defined on \([0, \infty)^2\). Thus, \( F(x) = H(x, \infty) \) and \( G(y) = H(\infty, y) \) are the two marginal d.f.'s. We also denote by \( Z_i = X_i - Y_i \) and denote its d.f. by \( P(z), -\infty < z < \infty \). Then, note that parallel to (1.2),

\[
\alpha_{FG} = P\{X > Y\} = P\{Z < 0\} = P(0) = \alpha_p,
\]

which may not be a sole functional of the marginal d.f.'s, but is simply expressible in terms of \( P(0) \). Thus, starting from the regeneration point, \( E_0 \), the mean time up to the system failure is given by

\[
E_0 = \mu_F / (1 - \alpha_p) = (1 - \alpha_p)^{-1} \int_0^\infty x dF(x).
\]

Note that if the system fails (i.e., encounters down time) in the \( m \)th cycle for the first time, for some \( m \geq 1 \), then \( X_i > Y_i \) for every \( i \leq m-1 \) and \( Y_m > X_m \). Therefore for the stopping number \( N \), we have

\[
P\{N = m\} = [P(0)]^{m-1}[1 - P(0)], \text{ for } m = 1, 2, \ldots \text{ ad inf.} \tag{2.3}
\]

Also, note that \( D \), the system down time, is given by \( Y_N - X_N \), where the stopping number \( N \) is defined earlier. Hence, we obtain that

\[
ED = \sum_{m \geq 1} E\{ (Y_m - X_m) I( N = m ) \}
= \sum_{m \geq 1} E\{ (Y_m - X_m) I( Y_i < X_i, i \leq m-1, Y_m > X_m ) \}
= \sum_{m \geq 1} [P(0)]^{m-1}[1 - P(0)] E\{ (Y_m - X_m ) | Y_m > X_m \}
= [1 - P(0)] \sum_{m \geq 1} [P(0)]^{m-1} \int_0^\infty z dP(z) / [1 - P(0)] \\
= \{\int_0^\infty z dP(z) \} \sum_{m \geq 1} [P(0)]^{m-1} \\
= \{1 - P(0)\}^{-1} \{\int_0^\infty z dP(z) \}
= \{1 - P(0)\}^{-1} E\{(Y - X) I(Y > X)\} \tag{2.4}
\]

Hence, from (1.1), (2.2) and (2.4), we obtain that
\[ A_{FG} = A_H = \frac{(EX)(EX + E((Y-X)I(Y > X)))}{(EX)(E(XI(X > Y)) + E(YI(Y > X)))} \]
\[ = \frac{(EX)(E[X \lor Y])}{(E[X \lor Y])}, \text{ where } a \lor b = \max(a, b). \] (2.5)

This gives a clear picture of the availability in terms of the life and repair times. In the special case where \( X \) and \( Y \) are independent with d.f. \( F \) and \( G \), respectively, (2.5) reduces to

\[ A_{FG} = \{\int_0^\infty (1-F(x))dx \} / \{\int_0^\infty (1-F(x)G(x))dx \}. \] (2.6)

In the sequel, we shall find these expressions very useful.

3. ESTIMATION OF \( A_H \) : NON-SEQUENTIAL CASE.

Note that for a single copy of the system, we have the set of observations \( X_i, Y_i, i = 1,\ldots,N \), where the stopping number \( N \) is defined by \( \min\{k \geq 1: X_k < Y_k\} \).
There is a down time \( (N-N) \), and following which, we have a regeneration point \( X_1 + \ldots + X_{N-1} + Y_N \) where the repaired unit resumes the role of the operating one and the unit failing at time point \( X_1 + \ldots + X_N \) (and remaining unattended for the down time period) is put to the repairing facility. Thus, from one regeneration point to the next one, we have a cycle of time length \( T = X_1 + \ldots + X_{N-1} + Y_N = \sum_{i=1}^{N}(X_i \lor Y_i) \).

Now, for the \( i \)th cycle, we denote the associated r.v.'s by \( X_{ij}, Y_{ij}, j = 1,\ldots,N_i \), \( N_i \) and \( T_i \), so that \( N_i = \min\{k \geq 1: X_{ik} < Y_{ik}\} \) and \( T_i = \sum_{j=1}^{N_i}(X_{ij} \lor Y_{ij}) \), for \( i = 1,2,\ldots \). In a non-sequential setup, we are given these observable r.v.'s for \( m(\geq 1) \) cycles, and based on these, we desire to provide a non-parametric estimate of \( A_H \) in (2.5). For latter use, we also define \( O_i = \sum_{j=1}^{N_i} X_{ij} \) (the \( i \)th cycle on time), for \( i = 1,\ldots,m \). Since the \( N_i \) are stopping times, we have

\[ E(O_i) = EX_{i1}.EN_i = \mu_F \{\sum_{m=1}^{\infty} m[P(0)]^{m-1}[1-P(0)] \} \]
\[ = \mu_F/[1-P(0)]. \] (3.1)

Similarly, by (2.4) and (3.1),

\[ E(T_i) = E(O_i) + E(Y_{iN_i} - X_{iN_i}) = [1-P(0)]^{-1}\{ \mu_F + E((Y-X)I(Y > X)) \}. \] (3.2)

Thus, we may be tempted in estimating \( A_H \) by
\[ \hat{A}_m = m^{-1} \sum_{i=1}^{m} (O_i / T_i) . \] (3.3)

However, this estimator may have a basic undesirable property. Note that the \( O_i / T_i \) are i.i.d.r.v. (in fact, bounded between 0 and 1), so that \( \hat{A}_m \) converges almost surely (a.s.) to \( E(O_i / T_i) \). But, \( E(O_i / T_i) \) need not be equal to \( A_H \); the amount of bias, i.e., \( E(O_i / T_i) - A_H \), depends on the unknown d.f. \( H \). Even when \( H = F_G \), \( O_i / T_i \) may not unbiasedly estimate \( A_H \), so that even if \( m \) is large, \( \hat{A}_m \) may not converge to \( A_H \) (but to \( E(O_i / T_i) \) which may differ from \( A_H \)). For this reason, we consider the alternative estimator

\[ A_m^* = \bar{O}_m / \bar{T}_m \] \[ \text{where } \bar{O}_m = m^{-1} \sum_{i=1}^{m} O_i \text{ and } \bar{T}_m = m^{-1} \sum_{i=1}^{m} T_i . \] (3.4)

Note that by the Khintchine strong law of large numbers, as \( m \to \infty \),

\[ \bar{O}_m \to E(O_i) = \mu_F / (1-P(0)) \text{ a.s.,} \] \[ \bar{T}_m \to E(T_i) = (1-P(0))^{-1} \{ \mu_F + E[(Y-X)(Y > X)] \} \text{ a.s.,} \] (3.5)

so that

\[ A_m^* \to A_H \text{ a.s., as } m \to \infty . \] (3.6)

Moreover, \( A_m^* \) is a bounded valued random variable (\( 0 \leq \hat{A}_m \leq 1 \)), so that (3.6) also ensures that \( E(A_m^*) \) converges to \( A_H \) as \( m \to \infty \). Thus, \( A_m^* \) is asymptotically unbiased for \( A_H \). We denote the dispersion matrix of \( (O_i, T_i) \) by

\[ \Sigma = \begin{pmatrix} \sigma_{OO} & \sigma_{OT} \\ \sigma_{TO} & \sigma_{TT} \end{pmatrix} . \] (3.7)

Then, by the classical central limit theorem (on the \( (O_i, T_i) \)), as \( m \to \infty \),

\[ m^{1/2} \left( \bar{O}_m - E(O_i), \bar{T}_m - E(T_i) \right) \to \mathcal{N}_2(0, \Sigma) . \] (3.8)

Therefore, it follows by some standard steps that as \( m \to \infty \),

\[ m^{1/2}(A_m^* - A_H) \to \mathcal{N}(0, \{E(T_i)\}^{-2} \{ \sigma_{OO} - 2A_H \sigma_{OT} + \sigma_H^2 \sigma_{TT} \}) . \] (3.9)

Natural estimates of the parameters appearing on the right hand side of (3.9) may be used to test for a suitable hypothesis on \( A_H \) or to attach a confidence interval on \( A_H \). In the absence of the knowledge of these parameters, the sample size \( m \) may not be fixed in a manner that the test has a specified power against a specified alternative or the width of the confidence interval is bounded by
some prefixed positive number \(2d\). For either of these problems, we may need a sequential procedure, and we shall consider the same in the next section.

4. SEQUENTIAL METHODS

We have already noticed in Section 3 that \(A_m^*\) is generally not unbiased for \(A_H\). Moreover, we need to estimate the parameters in the asymptotic distribution in (3.9). For this purpose, we employ jackknifing, by which we are able to reduce the bias and to estimate the asymptotic variance as well.

For every \(m \geq 2\), we define the \(\bar{O}_m, \bar{T}_m\) and \(A_m^*\), as in (3.4). Also, let \(\bar{O}_{m-1}^{(i)}\) and \(\bar{T}_{m-1}^{(i)}\) be defined as in (3.4), but based on a sample of size \(m-1\) obtained from the original sample (of size \(m\)) by omitting \((O_i, T_i)\), for \(i = 1, \ldots, m\). For every \(i = 1, \ldots, m\), we define

\[
A_{m-1}^*(i) = \frac{\bar{O}_{m-1}^{(i)}}{\bar{T}_{m-1}^{(i)}}, \quad A_{m, i}^* = mA_m^* - (m-1)A_{m-1}^*(i). \tag{4.1}
\]

Then, the jackknifed estimator of \(A_H\) is defined by

\[
A_m^{**} = m^{-1} \sum_{i=1}^{m} A_{m, i}^* = A_m^* + \frac{(m-1)/m}{\sum_{i=1}^{m} A_{m, i}^* - \sum_{j=1}^{m-1} A_{m-1}^*(j)}. \tag{4.2}
\]

Side by side, we introduce the jackknifed variance estimator

\[
s_m^2 = (m-1)^{-1} \sum_{i=1}^{m} (A_{m, i}^* - A_m^{**})^2 = (m-1) \left[ \sum_{i=1}^{m} (A_{m, i}^* - m \sum_{j=1}^{m-1} A_{m-1}^*(j))^2 \right]. \tag{4.3}
\]

Note that in the current situation, we have a function

\[
g(a, b) = a/b, \text{where } 0 < a < b < \infty, \tag{4.4}
\]

and moreover, in (3.4) or (4.1), the estimators \(\bar{O}_m\) and \(\bar{T}_m\) (or \(\bar{O}_{m-1}^{(i)}\) and \(\bar{T}_{m-1}^{(i)}\)) are sample means (and hence, U-statistics of degree 1). Note further that

\[
\frac{\partial g}{\partial a} = a/b, \quad \frac{\partial g}{\partial b} = -a/b^2, \quad \frac{\partial^2 g}{\partial a^2} = 0, \quad \frac{\partial^2 g}{\partial a \partial b} = -b^{-2}, \quad \text{and } \frac{\partial^2 g}{\partial b^2} = 2a/b^3. \tag{4.5}
\]

Therefore, we may formally write

\[
g(a, b) = g(\alpha, \beta) + (a-\alpha)/\beta - \alpha(b-\beta)/\beta^2 - (a-\alpha)(b-\beta)/\beta^2 + (b-\beta)^2 a^0/b^3, \tag{4.6}
\]

where \(a^0\) lies between \(a\) and \(\alpha\), and \(b^0\) lies between \(b\) and \(\beta\). The second and third terms on the right hand side of (4.6) are the so called linear terms, while the last two terms are the quadratic ones.
By virtue of (4.6), for every \(1 \leq i \leq m\) and \(m > 1\), we have
\[
A_{m-1}^* = A_m^* + (\overline{O}_{m-1} - \overline{O}_m)/\overline{T}_m - (\overline{O}_{m-1} - \overline{O}_m)(\overline{T}_{m-1} - \overline{T}_m)/\overline{T}_m^2 + \frac{(\overline{T}_{m-1} - \overline{T}_m)^2}{\overline{T}_m} \frac{\overline{O}_{m-1} - \overline{O}_m}{\overline{T}_m} \frac{\overline{T}_{m-1} - \overline{T}_m}{\overline{T}_m} + \frac{\overline{T}_{m-1} - \overline{T}_m}{\overline{T}_m} \frac{\overline{O}_{m-1} - \overline{O}_m}{\overline{T}_m} \frac{\overline{T}_{m-1} - \overline{T}_m}{\overline{T}_m},
\]
where \(\overline{O}_{m-1} \in (\overline{O}_{m-1}, \overline{O}_m)\) and \(\overline{T}_{m-1} \in (\overline{T}_{m-1}, \overline{T}_m)\).

Therefore, we have
\[
A_{m,i}^* = A_m^* + (\overline{O}_{m-1} - \overline{O}_m)/\overline{T}_m - (\overline{T}_{m-1} - \overline{T}_m)/\overline{T}_m^2 + (m-1)(\overline{O}_{m-1} - \overline{O}_m)(\overline{T}_{m-1} - \overline{T}_m)/\overline{T}_m^2,
\]
\[
- (m-1)(\overline{T}_{m-1} - \overline{T}_m)^2 \overline{O}_{m-1} / \overline{T}_m \overline{T}_m, i = 1, \ldots, m.
\]

By (4.2) and (4.8), we obtain that
\[
A_{m}^{**} = A_m^* + \frac{m-1}{m} \sum_{i=1}^{m-1} (\overline{O}_{m-1} - \overline{O}_m)(\overline{T}_{m-1} - \overline{T}_m)/\overline{T}_m^2 \overline{O}_{m-1} / \overline{T}_m \overline{T}_m - \frac{m-1}{m} \sum_{i=1}^{m-1} (\overline{T}_{m-1} - \overline{T}_m)^2 \overline{O}_{m-1} / \overline{T}_m \overline{T}_m,
\]
(4.9)
Since \(\overline{O}_m\) and \(\overline{T}_m\) are both U-statistics (of degree 1), we may now appeal to the proof of Theorem 3.1 of Sen (1977) and conclude that as \(m \to \infty\),
\[
\max \{ (\overline{O}_{m-1} - \overline{O}_m)^2 : 1 \leq i \leq m \} = O(m^{-1}) \text{ a.s.,}
\]
(4.10)
\[
\max \{ (\overline{T}_{m-1} - \overline{T}_m)^2 : 1 \leq i \leq m \} = O(m^{-1}) \text{ a.s.,}
\]
(4.11)
\[
(m-1) \sum_{i=1}^{m-1} (\overline{O}_{m-1} - \overline{O}_m)^2 + \sigma_{oo} \text{ a.s.,}
\]
(4.12)
\[
(m-1) \sum_{i=1}^{m-1} (\overline{T}_{m-1} - \overline{T}_m)^2 + \sigma_{tt} \text{ a.s.,}
\]
(4.13)
and (3.5) holds. As such, \(\max \{ \overline{T}_{m,i} : 1 \leq i \leq m \}\) and \(\max \{ \overline{O}_{m,i} / \overline{T}_m : 1 \leq i \leq m \}\) can both be bounded a.s., by some positive, finite numbers, as \(m \to \infty\). Therefore, from (4.9) through (4.13), we conclude that
\[
|A_{m}^{**} - A_m^*| = O(m^{-1}) \text{ a.s., as } m \to \infty.
\]
(4.14)
Moreover, by (4.8), (4.14) and the a.s. orders in (4.10) through (4.13), we conclude that as \(m \to \infty\),
\[
\max \{ |(A_{m,i}^* - A_m^{**} - (\overline{O}_{m-1} - \overline{O}_m)/\overline{T}_m + \overline{O}_m(\overline{T}_{m-1} - \overline{T}_m)/\overline{T}_m^2 | : 1 \leq i \leq m \} = O(m^{-1}) \text{ a.s.,}
\]
(4.15)
so that by (4.3) and (4.15), we conclude that as \(m \to \infty\),
\[
|s_m^2 - s_{m,02}^2| = O(m^{-1}) \text{ a.s.,}
\]
(4.16)
where
\[
s_{m,02}^2 = \frac{\overline{T}_m - 2}{m-1}(m-1)^{-1} \sum_{i=1}^{m} (\overline{O}_{m-1} - \overline{O}_m)^2 - 2 \overline{O}_m \overline{T}_m - 3 (m-1)^{-1} \sum_{i=1}^{m} (\overline{O}_i - \overline{O}_m)(\overline{T}_i - \overline{T}_m)
\]
\[
+ \overline{O}_m \overline{T}_m - 4 (m-1)^{-1} \sum_{i=1}^{m} (\overline{T}_i - \overline{T}_m)^2.
\]
(4.17)
By the strong convergence of U-statistics, we conclude that as \(m \to \infty\),
\[
(m-1)^{-1} \sum_{i=1}^{m} (O_i - \bar{O}_m)^2 \rightarrow \sigma_{OO} \text{ a.s., (} m-1)^{-1}\sum_{i=1}^{m} (O_i - \bar{O}_m)(T_i - \bar{T}_m) \rightarrow \sigma_{OT} \text{ a.s.,}
\]
\[
(m-1)^{-1}\sum_{i=1}^{m} (T_i - \bar{T}_m)^2 \rightarrow \sigma_{TT} \text{ a.s.,}
\]
(4.18)

and for these, the existence of the second order moments suffices. Thus, by (3.5), (4.17) and (4.18), we conclude that whenever the d.f. \( H \) has finite second order moments, as \( m \rightarrow \infty \),
\[
s_m^2 \rightarrow \sigma^2_A \left( = \left\{ \mathbb{E}(T_1) \right\}^{-2} \left\{ \sigma_{OO} - 2A_H \sigma_{OT} + A_H^2 \sigma_{TT} \right\} \right) \text{ a.s.} \quad (4.19)
\]
Therefore, from (4.16) and (4.19), we obtain that
\[
s_m^2 \rightarrow \sigma^2_A \quad \text{a.s., as } m \rightarrow \infty. \quad (4.20)
\]

Having obtained the a.s. convergence results in (3.6), (4.14) and (4.20), we are in a position to present the sequential estimation and testing procedures.

4.1. Bounded-width confidence interval for \( A_H \). The underlying d.f. \( H \), and hence, \( A_H \) being unknown, we intend to provide a confidence interval for \( A_H \), such that the confidence coefficient is \( 1 - \alpha \), for some preassigned \( \alpha \) \((0 < \alpha < 1)\), and the width of this interval is bounded from above by \( 2d \), for some preassigned \( d( > 0) \). For every \( m \geq 1 \), and \( d > 0 \), let
\[
I_m(d) = \{ t : (A_m^{**} - d) \nabla 0 \leq t \leq (A_m^{**} + d) \land 1 \}. \quad (4.21)
\]
Also, let \( \tau_\alpha \) be the upper 100\( \alpha \)% point of the standard normal d.f., and let \( m_o \) \((= m_o(d))\) be an initial sample size, usually greater than 2. Then, we may consider a stopping number \( M (= M(d)) \), defined by
\[
M(d) = \min \{ m \geq m_o : s_m^2 \leq m d^2 / \tau_{\alpha/2}^2 \} \quad (4.22)
\]
if no such \( m \) exists, we let \( M(d) = \infty \). Whenever \( M < \infty \), the confidence interval for \( A_H \) is taken as \( I_M(d) \), defined as in (4.21), for \( M=m \). Since \( M(d) \) is a stopping number, we can verify that for every (fixed) \( d > 0 \), \( M(d) < \infty \), with probability 1. Now, by definition in (4.21), \( I_M(d) \) has a width \( \leq 2d \), so that we need to verify that \( I_M(d) \) has the coverage probability \( 1 - \alpha \). Towards this, we proceed as in Chow and Robbins (1965) and work out the properties of this procedure in the asymptotic case where \( d \) is made to converge to 0.

We may note that by virtue of the law of iterated logarithm,
\begin{align*}
\lim \{ (m/\log log m)^{1/2} \mid \overline{\sigma}_m - \text{EO}_1 \} &= \{ 2 \sigma_{00}^{1/2} \} \text{ a.s.,} \\
\lim \{ (m/\log log m)^{1/2} \mid \overline{T}_m - \text{ET}_1 \} &= \{ 2 \sigma_{TT}^{1/2} \} \text{ a.s.,}
\end{align*}

so that by (3.4), (4.6), (4.14) and (4.23)-(4.24), we conclude that as \( m \to \infty \),

\begin{align*}
A_m^{**} - A_H &= (\overline{O}_m - \text{EO}_1)/\text{ET}_1 - (\text{EO}_1)(\overline{T}_m - \text{ET}_1)/(\text{ET}_1)^2 + O(m^{-1}\log log m) \text{ a.s.} \\
&= m^{-1} \Sigma_{i=1}^{m} \text{ET}_1^2 \{ (O_i - \text{EO}_1) \text{ET}_1 - (T_i - \text{ET}_1) \text{EO}_1 \} + O(m^{-1}\log log m) \text{ a.s.} \\
&= m^{-1} \Sigma_{i=1}^{m} W_i + O(m^{-1}\log log m) \text{ a.s.}
\end{align*}

where the \( W_i \) are i.i.d.r.v. with mean 0 and a finite positive variance \( \sigma_A^2 \),
defined by (4.19). This basic representation provides the basic tools for our subsequent analysis. For every \( m \), we introduce a stochastic process \( Q_m = \{ Q_m(t), \ t \in [0,1] \} \) by letting \( Q_m(t) = m^{1/2} [mt]( A_{[mt]}^{**} - A_H )/\sigma_A^2, \ t \in [0,1] \). Also, let \( Q = \{ Q(t), \ t \in [0,1] \} \) be a Wiener process on the unit interval \([0,1]\). Then,

by an appeal to weak invariance principles for sums of i.i.d.r.v.'s [see for example, Sen(1981, Ch.2)], we conclude that under the assumed regularity conditions,

\[ Q_m \xrightarrow{D} Q, \ \text{in the J}_1\text{-topology on D}[0,1], \ \text{as} \ m \to \infty, \quad (4.26) \]

and further, the compactness of \( \{ Q_m \} \) remains in tacit with respect to the uniform topology as well. A direct consequence of this weak invariance principle is that for every \( \varepsilon > 0 \) and \( \eta > 0 \), there exist a \( \delta : 0 < \delta < 1 \) and an \( m_o \), such that

\[ P\left\{ \sup_{0 \leq s < t \leq s + \delta} \mid Q_n(t) - Q_n(s) \mid > \varepsilon \right\} < \eta, \quad \forall \ m \geq m_o, \quad (4.27) \]

so that if \( \{ T_m \} \) is any sequence of positive r.v. , such that as \( m \to \infty \),

\[ T_m \xrightarrow{P} t, \ \text{in probability, for some fixed} \ t \in [0,1], \quad (4.28) \]

then, by (4.26) and (4.27),

\[ Q_m(T_m) \text{ is asymptotically normal with 0 mean and variance } t. \quad (4.29) \]

Now, for every \( d(> 0) \), we define

\[ m(d) = \min \{ m \geq m_o : \sigma_A^2 m^2 \leq d^2/\alpha^2 \} \].

Then, by virtue of (3.9) and (4.14), along with the definition of \( \sigma_A^2 \) in (4.19),

\[ \lim_{d \to 0} P\{ A_H \in I_{m(d)}(d) \} = 1 - \alpha. \quad (4.30) \]

On the other hand, by (4.20), (4.22) and (4.30), as \( d \to 0 \),

\[ M(d)/m(d) \to 1 \text{ a.s.} \quad (4.31) \]
Note that if we define \( T_m(d) = M(d)/m(d) \), \( d > 0 \), then, by (4.31), \( T_m(d) \xrightarrow{P} 1 \), as \( d \downarrow 0 \), so that by (4.28) and (4.29) (with \( t = 1 \)), we obtain that as \( d \downarrow 0 \),

\[
\{M(d)\}^{1/2} \left( A_{M(d)}^{**} - A_{H} \right)/\sigma_A \sim \mathcal{N}(0,1),
\tag{4.32}
\]

and combining (4.32) with (4.20), we have by the Slutsky theorem, as \( d \downarrow 0 \),

\[
\{M(d)\}^{1/2} \left( A_{M(d)}^{**} - A_{H} \right)/s_{M(d)} \sim \mathcal{N}(0,1).
\tag{4.33}
\]

By (4.22) and (4.33), we conclude that

\[
\lim_{d \downarrow 0} P\{ A_{H} \in I_{M(d)}(d) \} = 1 - \alpha.
\tag{4.34}
\]

In the literature, (4.34) is termed the asymptotic consistency of the sequential procedure. In a sense, (4.31) reflects the asymptotic efficiency of the sequential procedure; however, one usually defines this in terms of the following:

\[
\lim_{d \downarrow 0} EM(d)/m(d) = 1,
\tag{4.35}
\]

which would fit with the Chow-Robbins(1965) definition. Verification of (4.35) naturally requires the convergence results on the moments of \( M(d) \), and, this in turn, requires the study of the moment convergence properties of \( s_m^2 \). Note that unlike the case in Chow and Robbins(1965), we do not have here a reversed (sub-)martingale property of \( \{s_m^2, m \geq m_0\} \), so that the computation of the first order moment of \( \sup_m s_m^2 \), needed for the purpose, requires more elaborate analysis and more stringent regularity conditions. We note in this context that by (4.1), for every \( i : 1 \leq i \leq m, m > 1, 0 \leq A_{m-1}^{(i)} \leq 1 \), so that by (4.3)

\[
s_m^2 \leq (m-1) \Sigma_{i=1}^m (A_{m-1}^{(i)})^2 \leq m(m-1), \quad \text{with probability 1.}
\tag{4.36}
\]

Moreover, using the inequality that \( \Sigma_{i=1}^m (O_i - \bar{O}_m)^2 \leq \Sigma_{i=1}^m O_i^2 \leq (\Sigma_{i=1}^m O_i)^2 \) along with the same for the \( T_i \), we obtain from (4.17) that

\[
s_m^{O_2} \leq 4m^2/(m-1), \quad \text{with probability 1 (} \forall m \geq 2 \).
\tag{4.37}
\]

Therefore, from (4.36) and (4.37), we obtain that

\[
\left| s_m^2 - s_m^{O_2} \right| \leq 5m^2, \quad \text{with probability 1, for every } m \geq 2.
\tag{4.38}
\]

Now, we assume that for some \( r > 8 \), \( E|O_{1, i}|^r < \infty \) and \( E|T_{1, i}|^r < \infty \). Then, in (4.10)-(4.11), in the right hand side, we may replace 'a.s.' by 'with a probability greater than 1 - 0(m^{-r/2+1})', while in (4.12)-(4.13), we have with a probability
greater than $1-0(m^{-r/2})$. As such, using (4.8)-(4.9), (4.38), and the modified steps in (4.10)-(4.13), we obtain that

$$E|s_m^2 - s_m^{o2}| = O(m^{-r/2 + 3}), \text{ for every } m \geq 2,$$

(4.39)

and this ensures that

$$E(\sup_{m \geq 2} |s_m^2 - s_m^{o2}|) \leq \sum_{m \geq 2} E|s_m^2 - s_m^{o2}| < \infty, \text{ for } r > 8,$$

(4.40)

Moreover, we may rewrite

$$\sup_m s_m^2 \leq \sigma_A^2 + \sup_m |s_m^{o2} - \sigma_A^2| + \sup_m |s_m^2 - s_m^{o2}|,$$

(4.41)

so that, for our purpose, it suffices to show that for some $q > 1$,

$$E(\sup_m |s_m^{o2} - \sigma_A^2|^q) \leq \sum_{m \geq m_0} E|s_m^{o2} - \sigma_A^2|^q \text{ converges.}$$

(4.42)

By virtue of (4.37), we may now virtually repeat the proof of a theorem due to Cramér(1946, pp. 353-356) and conclude that for $q = r/2$, $r > 8$,

$$E|s_m^{o2} - \sigma_A^2|^q = O(m^{-q/2}) + O(m^{-q/2-1/2}),$$

(4.43)

so that (4.42) holds by noting that $q/2 = r/4 > 2$.

Note that by (4.22), for every $d > 0$,

$$s_{M(d)}^2 \leq \left(d^2/\tau_{\alpha/2}^2\right)M(d) \text{ and } s_{M(d)-1}^2 > \left(d^2/\tau_{\alpha/2}^2\right)(M(d)-1).$$

(4.44)

Combining (4.44) with (4.20), (4.30) and (4.39)-(4.42), we conclude that (4.35) holds. However, this result is not really needed from the practical application point of view, and (4.31) requiring only the finiteness of the second moments would suffice. We may notice that in view of (4.16), (4.17), (4.18), (4.44) and the (joint) asymptotic normality of Hoeffding's (1948) U-statistics, weak invariance principles are applicable for the $\{m(d)\}^{-1/2}[s_{m(d)}^2 - \sigma_A^2]$, as $d \downarrow 0$,

so that we have under the finiteness of the fourth order moments,

$$\{m(d)\}^{-1/2} (M(d) - m(d)) \sim \chi(0, \nu^2),$$

(4.45)

for some finite $\nu$, and this provides the asymptotic normality of the stopping time. The asymptotic normality result in (4.45) requires less stringent regularity conditions than the asymptotic efficiency result in (4.35). (4.45) also casts light on the variability of $M(d)$ around $m(d)$, the optimal sample size if $\sigma_A$ were known.
4.2. Sequential tests for Availability. Consider the hypothesis testing problem:

\[ H_0^* : A_H = \theta_0 \quad \text{vs.} \quad H_1^* : A_H = \theta_1 = \theta_0 + \Delta, \quad (4.46) \]

where \( \theta_0 \) and \( \Delta \) are specified, and we like the test to have the prescribed strength \((\alpha, \beta)\), where we restrict ourselves to positive \( \alpha, \beta \) for which \( \alpha < \frac{1}{2} \) and \( \beta < \frac{1}{2} \). Since the underlying d.f. \( H \) is not known, no fixed sample size test may meet the desired goal, and hence, we take recourse to a sequential procedure. In this context, \((4.20)\) and \((4.25)\) play the vital role.

We define two positive numbers \((A,B)\), such that \( \beta/(1-\alpha) \leq B < 1 < A < \alpha/(1-\beta) < \infty \), and let \( a = \log A \) and \( b = \log B \), so that \( b < 0 < a \). Typically, for small values of \( \Delta \), the lower (and upper) bounds for \( B \) (and \( A \)) are taken to be equal.

Further, for every \( \Delta \), we define a positive integer \( m_0(\Delta) \), such that as \( \Delta \to 0 \), \( m_0(\Delta) \) goes to \( \infty \) but \( \Delta^2 m_0(\Delta) \) converges to \( 0 \). Further, for every \( m > 1 \), we define the jackknifed estimator \( A_m^{**} \) as in \((4.2)\) and the variance estimator \( s_m^2 \) as in \((4.3)\). Then, starting with the initial sample size \( m_0(\Delta) \), we continue drawing observations one by one so long as

\[ \frac{bs_m^2}{m} < m\Delta[ A_m^{**} - (\theta_0 + \theta_1)/2 ] < \frac{as_m^2}{m}, \quad m \geq m_0(\Delta); \quad (4.47) \]

if, for the first time, \((4.47)\) is vitiated for \( m = M = m(\Delta) \), then we stop at that time and accept \( H_0 \) or \( H_1 \) according as \( m\Delta[ A_m^{**} - (\theta_0 + \theta_1)/2 ] \) is \( < bs_m^2 \) or \( \geq as_m^2 \). Thus, \( M(\Delta) \) is the stopping variable. Note that in the current context, at the stopping number \( M \), we have the data relating to \( M \) completed cycles of total time length \( T_1^* + \ldots + T_M \).

Note that by virtue of \((3.9)\), \((4.14)\) and \((4.20)\), for every fixed \( \theta_0 \) and \( \Delta \),

\[ P\{ M(\Delta) > m \mid A_H \} \to 0 \text{ as } m \to \infty, \quad (4.48) \]

so that the proposed test terminates with probability \( 1 \); the proof of \((4.48)\) runs parallel to that in Section 6 of Scn(1977), and hence, is omitted.

To study the QC and ASN functions of the proposed test, as in Scn(1981, Ch.9), we consider an asymptotic setup wherein we allow \( \Delta \) to converge to \( 0 \) (comparable to \( d \to 0 \) in the earlier problem in Section 4.1), and, we set
\[ A_H = \theta_0 + \phi \Delta, \text{ where } \phi \in \phi = \{\phi : |\phi| \leq K < \infty\} \]  

\[ (4.49) \]

Also, we denote by \( L_H(\phi, \Delta) \) the OC (i.e., probability of accepting \( H_0 \) when actually \( A_H = \theta_0 + \phi \Delta \)) of the test based on (4.47). Note that as \( \Delta \to 0 \), \( L_H(\phi, \Delta) \), for every fixed \( \phi \in \phi \), converges to a limit \( P(\phi) \), and this is given by:

\[
\lim_{\Delta \to 0} L_H(\phi, \Delta) = P(\phi) = \begin{cases} 
\frac{(A^1-2\phi - 1)/(A^1-2\phi - B^1-2\phi)}{a/(a-b)}, & \phi \neq \frac{1}{2}, \\
\phi = \frac{1}{2}.
\end{cases} \]

\[ (4.50) \]

Note that \( P(0) = 1 - \alpha \) and \( P(1) = \beta \), so that the limiting strength of the test is \((\alpha, \beta)\). By virtue of (4.20), (4.25) and the Skorokhod-Strassen embedding of Wiener process for sums of i.i.d.r.v.'s, the proof of (4.50) follows precisely on the same line as in the case of Theorem 6.1 of Sen(1977), and hence, is omitted.

Under the additional regularity conditions needed to ensure the existence of the moments of \( s_m^2 \) in (4.41)-(4.43), we may again adapt the proof of Theorem 6.2 of Sen(1977) and arrive at the following:

Under the asymptotic setup in (4.49), for every (fixed) \( \phi \in \phi \),

\[
\lim_{\Delta \to 0} \{\Delta^2 E[M(\Delta)] \mid \theta_0 = \theta_0 + \phi \Delta \} = \psi(\phi, \sigma_A)
\]

\[ (4.51) \]

where

\[
\psi(\phi, \sigma_A) = \begin{cases} 
\{b P(\phi) + a [1-P(\phi)]\} \sigma_A^2/(\phi-\frac{1}{2}), & \phi \neq \frac{1}{2}, \\
-\sigma_A^2 a b, & \phi = \frac{1}{2}
\end{cases}
\]

\[ (4.52) \]

and \( \sigma_A^2 \) and \( P(\phi) \) are defined by (4.19) and (4.50), respectively.

The asymptotic results in (4.50)-(4.52) provide good approximations in practice when \( \Delta \) is small.

5. PROGRESSIVE CENSORING SCHEMES

We may note that for the statistical inference procedures described in Section 4, one needs the set of observations \((O_i, T_i)\), \( i \geq 1 \). In an operating scheme, each observation (vector) \((O_i, T_i)\) requires the running of the system for the total cycle time \( T_i \), and these are nonnegative random variables. Thus, given a predetermined duration of the study, one would have a random number of cycles completed, and based on this random number of observations, one would be required to draw inference on \( A_H \). Alternatively, one would allow the system to be operative
until we have a prespecified number (m) of cycles completed, and then to base
the statistical analysis on \((O_i, T_i), i = 1, \ldots, m\); in such a case, the duration
of the study is equal to \(T_1 + \ldots + T_m\) and is therefore stochastic in nature. In
the usual fashion, we may term the two schemes as \textit{Truncated (Type I Censored)}
and \textit{Censored (Type II Censored)} Schemes. In either way, when the joint d.f.
of the \((O_i, T_i)\) is not (at least roughly) known, we may have some drawback.
Either we may have very few observations to base a test or formulate an estimate
with reliable precisions, or we may have to wait too long to obtain the desired
number of observations to initiate the statistical inference procedures. It is
not unnatural to have interim analysis, so that if at any early stage, there
is any strong indication of sub-standard functionning of the system, one may
curtail the study and make corrective adjustments before putting it back to
operation. In this context, continuous monitoring of the system is often
advocated, and under such a monitoring scheme, \textit{progressively censored schemes}
can naturally be adopted with some advantage.

To be more general, we consider \(n (\geq 1)\) independent copies of the system,
and, for the \(i\)th copy, we denote the random variables by \((O_{ij}, T_{ij}), j \geq 1\), for
\(i = 1, \ldots, n\). Also, for every \(t > 0\), we define the non-negative integer-valued
random variables \(\{ m_i(t), i = 1, \ldots, n \}\) by letting \(T_{i0} = 0, i \geq 1\), and
\[
m_i(t) = \max \{ k \geq 0 : T_{i0} + \ldots + T_{ik} \leq t \}, \quad t > 0, \quad i \geq 1.
\] (5.1)

Then, if all the \(n\) copies of the system start operating at time 0, we have at
time \(t\), \(N(t) = m_1(t) + \ldots + m_n(t)\) observations (vectors) \((O_{ij}, T_{ij}), j = 1, \ldots,
m_i(t), i = 1, \ldots, n\), and we desire to draw inference on \(A_H\) (as in Sections 3 and 4)
in a repeated scheme, where we allow \(t\) to vary over an interval \([0, T]\), for some
positive and finite \(T\). This situation is quite comparable to the sequential
schemes in Section 4, excepting that the \(N(t)\) are non-negative integer valued
random variables, and hence, some additional adjustments are necessary to
make the theory applicable.
In Section 4.1, we have studied sequential confidence intervals for $A_H$, and, in this context, the stopping number $M(d)$, $d > 0$ plays the vital role. To adapt this sequential procedure in the setup of progressive censoring, we define for every $d > 0$,

$$\omega_d = \min \{ t > 0 : N(t) \leq \frac{s^2_{N(t)} d^2}{\tau^2_{\alpha/2}} \}.$$  \hspace{1cm} (5.2)

Thus, the stopping time is given by $\omega_d$, and, based on the number $N(\omega_d)$ of observations, we construct the confidence interval for $A_H$ as in (4.21), with $m = N(\omega_d)$. At this stage, we may make use of the elementary renewal theorem and conclude that

$$\frac{N(t)}{nt} \rightarrow \{ET_1\}^{-1} \text{ a.s.}, \text{ as } t \text{ increases,} \hspace{1cm} (5.3)$$

so that by (4.20), (4.30), (5.2) and (5.3), we obtain that as $d \downarrow 0$,

$$n \omega_d \sim m(d)E(T_1) \sim d^{-2} \frac{\tau^2_{\alpha/2} \sigma^2_A}{\{ET_1\} \text{ a.s.}}, \hspace{1cm} (5.4)$$

and,

$$\omega_d \sim n^{-1} d^{-2} \frac{\tau^2_{\alpha/2} \sigma^2_A}{(ET_1) \text{ a.s.}} \hspace{1cm} (5.5)$$

Note that in the above derivation, we have tacitly assumed that either $n$ is fixed or $n$ is allowed to be large, but $nd^2$ is small, so that (5.3) remains adaptable. The stopping time $\omega_d$ thus depends on $n$ as well. Having obtained these results, we are in a position to verify (4.31) and (4.34) without any difficulty. To verify (4.35), with $M(d)$ replaced by $N(\omega_d)$, we make use of the related elementary renewal theorem [viz., Ross (1970, pp. 40-41)], and conclude that

$$\lim_{t \to \infty} \{EN(t)/(nt) \} = \{E(T_1)\}^{-1}. \hspace{1cm} (5.6)$$

As such, the results in Section 4.1 can readily be adapted to show that (4.35) holds in the current context too. Further, given (5.3), (5.5) and the asymptotic normality result in (4.45), we have the parallel result for the stopping time $\omega_d$. The inversion theorem via (5.2) yields the desired result.

For the sequential testing problem in Section 4.2, we may replace in (4.47), $m$ by $N(t)$, $t \geq t_0$ for some positive $t_0$. The stopping number $M(A)$, defined after
(4.47), has to be replaced by the stopping time $\omega^*(\Delta)$, defined by

$$\omega^*(\Delta) = \min \{ t \geq t_0 : N(t)\Delta[A_{N(t)}^{**} - (\theta_0 + \theta_1)/2]/\sigma_{N(t)}^2 \leq (b,a) \},$$

(5.7)

where $a$ and $b$ are defined as in (4.47). With this minor change, we may prove (4.48) by replacing $M(\Delta)$ and $m$ by $\omega^*(\Delta)$ and $t$, respectively. Under (4.49), the limiting OC function in (4.50) remains valid, while, for the limiting ASN function in (4.51), if we replace $M(\Delta)$ by $\omega^*(\Delta)$, then in (4.52), we need to replace $\sigma_A^2$ by $\{ET_1\}^{-1} \sigma_A^2$; this will then be the average stopping time in the limiting case. Control chart type sampling inspection plans (for $A_H$) under progressive censoring may also be adopted under the usual setup.

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