Dependence Properties of Iterated Generalized
Bivariate Farlie-Gumbel-Morgenstern Distributions
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DEPARTMENT OF STATISTICS
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1. Introduction

The bivariate Farlie-Gumbel-Morgenstern (FGM) distribution has joint cumulative distribution function (cdf)

\[ F_{12}(x_1, x_2) = F_1(x_1)F_2(x_2)\{1 + \alpha S_1(x_1)S_2(x_2)\} \]  

(with \( |\alpha| \leq 1 \))

where \( F_j(x_j) \) is the cdf of \( X_j \) \((j=1,2)\) and \( S_j(x_j) = 1 - F_j(x_j) \) is the survival distribution function (sdf) of \( X_j \).

In Johnson and Kotz (1976) we discussed certain generalizations of (1). Motivated by some remarks of Farlie (1960), we suggested replacing the product \( S_1(x_1)S_2(x_2) \) (which would be the joint sdf of \( X_1 \) and \( X_2 \) if they were independent) in (1) by the joint sdf of another bivariate FGM distribution, namely

\[ S_{12}^{(1)}(x_1, x_2) = S_1(x_1)S_2(x_2)\{1 + \alpha_1 F_1(x_1)F_2(x_2)\} \]  

with \( |\alpha_1| \leq 1 \).

This procedure can be iterated in various ways. In particular, we can replace \( F_1(x_1)F_2(x_2) \) in (2) by the FGM type distribution

\[ F_{12}^{(2)}(x_1, x_2) = F_1(x_1)F_2(x_2)\{1 + \alpha_2 S_1(x_1)S_2(x_2)\}. \]  

(Of course \( F_{12}^{(2)} \) might itself be of the form of a first stage iterate FGM distribution, like (2), but here we will restrict ourselves to the relatively simple form (3).)

One purpose of introducing these iterated generalizations of bivariate FGM distributions was to check if one could remedy a defect of bivariate
FG1 distributions — namely, the relatively low correlation coefficients that can be attained with such distributions.

For FG1 distributions with normal marginals the correlation coefficient is \( \omega \), and, since \( |\omega| \leq 1 \), this means that the absolute magnitude of the correlation coefficient cannot exceed \( \pi^{-1} \) (= 0.32 approx.)

In the present note we find that iterations of the kind described above cannot increase the maximum possible dependence (as measured by an index suggested by Schweizer and Wolff (1976)), between the variables by more than about 0.12, no matter how many iterations are performed. We therefore investigate a few other modifications of bivariate FG1 distributions which might increase dependence more substantially, but we find that necessary restrictions on the values of the \( \alpha \)'s are such as to seriously limit the possibilities.

2. Schweizer & Wolff's Measure of Dependence

The measure of dependence introduced by Schweizer & Wolff (1976) is essentially (for continuous joint distributions), the expected value of

\[
12|F_{12}(x_1, x_2) - F_1(x_1)F_2(x_2)|
\]  

(4)

(or, equivalently, of

\[
12|S_{12}(x_1, x_2) - S_1(x_1)S_2(x_2)|
\]  

(4')

Denoting this by the symbol \( (\omega)_{X_1,X_2} \) it is clear that it is unchanged by any monotonic transformation is applied to either (or both) of the variables \( X_1, X_2 \). In particular we can suppose that such transformations have been
applied to produce variables $X_1, X_2$ which are each uniformly distributed between 0 and 1. Then

$$\langle \omega \rangle_{X_1, X_2} = 12 \int_0^1 \int_0^1 |F_{12}(x_1, x_2) - x_1 x_2| dx_1 dx_2. \quad (5)$$

If $X_1$ and $X_2$ are mutually independent, $\langle \omega \rangle_{X_1, X_2} = 0$. The maximum possible value of $\langle \omega \rangle_{X_1, X_2}$ is 1, and this is attained when, with probability 1, $X_1$ and $X_2$ are monotonic functions of each other.

(When $X_1$ and $X_2$ are each uniformly distributed over $(0, 1)$, if $X_1$ and $X_2$ are each monotonic functions of the other then we must have either $X_1 = X_2$ or $X_1 = 1 - X_2$.)

We will use this measure of dependence in our studies, rather than the better-known mean square contingency. This is quite closely related to the Schweizer-Wolff index, (see Section 5)—but the comparisons using it are not so clear-cut as using the latter measure.

3. **Dependence in Generalized FGM Distributions**

If the marginal distributions are each uniform over $(0, 1)$ then the FGM cdf (1) is

$$x_1 x_2 \{1 + \alpha (1-x_1)(1-x_2)\}$$

and

$$\langle \omega \rangle_{X_1 X_2} = 12 |\alpha| \int_0^1 \int_0^1 x_1 (1-x_1) x_2 (1-x_2) dx_1 dx_2$$

$$= 12 |\alpha| \left( \frac{1}{2} - \frac{1}{3} \right)^2 = \frac{1}{3} |\alpha|^2. \quad (6)$$

This is of course the value of $\langle \omega \rangle_{X_1 X_2}$ for any continuous FGM distribution with cdf (1).
For the once iterated FGGM distribution with uniform \((0,1)\) marginal distributions we have

\[
F_{12}(x_1,x_2) = x_1x_2[1 + \alpha(1-x_1)(1-x_2)(1 + \alpha_1x_1x_2)]
\]

whence

\[
(\sigma \omega)_{X_1 X_2} = 12 \int_0^1 \int_0^1 |\alpha \pi_1 (1-x_1)x_2(1-x_2) + \alpha_1x_1^2(1-x_1)x_2^2(1-x_1)| dx_1 dx_2.
\]

For given numerical values of \(\alpha, \alpha_1\) we will get the largest value for \((\sigma \omega)_{X_1 X_2}\) by having \(\alpha, \alpha_1\) of the same sign. Taking \(\alpha > 0\) and \(\alpha_1 > 0\) we obtain

\[
(\sigma \omega)_{X_1 X_2} = 12\{\alpha(\frac{1}{2} - \frac{1}{3})^2 + \alpha_1(\frac{1}{3} - \frac{1}{4})^2\} = \frac{1}{3}\alpha + \frac{1}{12}\alpha_1 = \frac{1}{3}(1 + \frac{1}{4}\alpha_1)\alpha. \quad (8)
\]

Similarly, we find for the twice iterated FGGM distribution, with \(\alpha > 0, \alpha_1 > 0, \alpha_2 > 0\)

\[
(\sigma \omega)_{X_1 X_2} = \frac{1}{3}\alpha + \frac{1}{12}\alpha_1 + \frac{1}{75}\alpha_1\alpha_2. \quad (9)
\]

Generally for the \(r\)-th iterated FGGM distribution with no negative \(\alpha\)'s

\[
(\sigma \omega)_{X_1 X_2} = 12 \sum_{j=1}^r a_j\alpha_j
\]

where \(\alpha_j = \alpha\)

\[
\alpha_j = \alpha \prod_{i=1}^{j-1} \alpha_i \quad (j \geq 2)
\]

and

\[
a_j = \begin{cases} 
(B(\frac{j+3}{2}, \frac{j+3}{2}))^2 & \text{for } j \text{ odd} \\
(B(\frac{j+4}{2}, \frac{j+2}{2}))^2 & \text{for } j \text{ even}
\end{cases}
\]

(11)

where \(B(i,j)\) is the Beta function.
Since $0 \leq \alpha_{r-1} \leq \alpha_r \leq \ldots \leq \alpha_1 = \alpha$, it follows that

$$(\sigma \omega)_{X_1X_2} \leq 12\alpha \sum_{j=1}^{\infty} a_j \leq 12\alpha \sum_{j=1}^{\infty} a_j$$

and so

$$(\sigma \omega)_{X_1X_2} \leq 12\alpha \sum_{h=2}^{\infty} \{B(h,h)\}^2 + \{B(h+1,h)\}^2
= 15\alpha \sum_{h=2}^{\infty} \{B(h,h)\}^2$$

(since $B(h+1,h) = \frac{1}{2}B(h,h)$).

Now $\sum_{h=2}^{\infty} \{B(h,h)\}^2 = 6^{-2} + 30^{-2} + 140^{-2} + 630^{-2} + \ldots$

$< 6^{-2} + 30^{-2} + 140^{-2} + 630^{-2}(1 + 4^{-2} + 4^{-4} + \ldots)$

$< 0.029$ (13)

and so

$$(\sigma \omega)_{X_1X_2} < 15 \times 0.029\alpha = 0.435\alpha$$ (14)

however many iterations are used.

This shows that iterations of the kind described in Section 1 cannot increase the measure of dependence $(\sigma \omega)_{X_1X_2}$ above 0.435.

The reason for the low limit on the measure of dependence is in the rapid decrease of $B(h,h)$ as $h$ increases. A natural way to avoid this would be to use distributions of form

$$F_{12}(x_1, x_2) = F_1(x_1)F_2(x_2)[1 + \alpha(S_1(x_1))^{\phi_1}S_2(x_2)^{\phi_2}]$$

with $0 < \phi_j < 1$ ($j = 1, 2$). Unfortunately this is not a proper distribution function for any absolutely continuous marginals $F_1(x_1), F_2(x_2)$. We have

$$\frac{\partial^2 F_{12}}{\partial x_1 \partial x_2} = f_1(x_1)f_2(x_2)[1 + \prod_{j=1}^{2} (1 - S_j(x_j))^{-(1+\phi_j)} f_j(x_j)]$$
By choosing \( x_1 \) so that \((1 + \phi_1)F(x_1) > 1\) and \( x_2 \) so that \( S_2(x_2) \) is sufficiently small we make the expression in square brackets negative.

(Since \( F_1(x_1) < 1 \) and \( S_2(x_2) > 0 \), we can always find \( x_1, x_2 \) so that \( f_1(x_1) > 0 \) and \( f_2(x_2) > 0 \).

Some other modifications of bivariate \( F_{2|1} \) distributions are discussed in the next section, with special reference to possible increases in dependence.

4. Some Other Generalized \( F_{2|1} \) Distributions

4.1 The following example shows directly how the restrictions on the values of the \( \alpha \)'s militates against realization of substantial increases in

\[(\sigma \omega)_{X_1, X_2} \].

We consider a family of distributions obtained by modifying (1) with replacement of

\[F_1(x_1)F_2(x_2) \text{ by } F_1(x_1)F_2(x_2)\{1 + \alpha_1S_1(x_1)S_2(x_2)\}\]

and

\[S_1(x_1)S_2(x_2) \text{ by } S_1(x_1)S_2(x_2)\{1 + \alpha_2F_1(x_1)F_2(x_2)\}.\]

Transforming so that each marginal distribution is uniform over \((0,1)\) we have

\[F_{12}(x_1, x_2) = x_1x_2\{1 + \alpha_1(1-x_1)(1-x_2)\}\{1 + \alpha(1-x_1)(1-x_2)(1 + \alpha_2x_1x_2)\}\]

\[0 < x_j < 1, j=1,2. \quad (15)\]

Hence

\[(\sigma \omega)_{X_1, X_2} = 12 \int_0^1 \int_0^1 |(\alpha + \alpha_1)x_1x_2(1-x_1)(1-x_2) + \alpha_1x_1x_2(1-x_1)^2(1-x_2)^2 + \alpha_2x_1^2x_2^2(1-x_1)(1-x_2) + \alpha_1\alpha_2x_1^2x_2^2(1-x_1)^2(1-x_2)^2| dx_1 dx_2. \]
If all $\alpha$'s are nonnegative then

$$ (\sigma \omega)_{x_1, x_2} = \frac{1}{3}(\alpha + \alpha_1) + \frac{1}{12}\alpha(\alpha_1 + \alpha_2) + \frac{1}{75}\alpha \alpha_1 \alpha_2. \quad (16) $$

If we only needed the conditions $|\alpha| \leq 1$, $|\alpha_1| \leq 1$, $|\alpha_2| \leq 1$ then it would appear that we could get values of $(\sigma \omega)_{x_1, x_2}$ as near to $\frac{2}{3} + \frac{1}{6} + \frac{1}{75} = 0.346$ as desired.

However, the $\alpha$'s must be such that

$$ \frac{\partial^2 F}{\partial x_1 \partial x_2} = 1 + (\alpha + \alpha_1)(1 - 2x_1)(1 - 2x_2) + \alpha \alpha_1 (1 - x_1)(1 - 3x_1)(1 - x_2)(1 - 3x_2) $$

$$ + \alpha \alpha_2 x_1(2 - 3x_1)(2 - 3x_2) + 4\alpha \alpha_1 \alpha_2 x_1(1 - x_1)(1 - 2x_1)x_2(1 - x_2)(1 - 2x_2) \geq 0 \quad (17) $$

for $x_1$, $x_2$ with $0 \leq x_1 \leq 1$, $0 \leq x_2 \leq 1$.

Putting $x_1 = 0$, $x_2 = 1$ gives

$$ \alpha + \alpha_1 \leq 1. $$

This implies that

$$ (\sigma \omega)_{x_1, x_2} \leq \frac{1}{3} + \frac{1}{12}\alpha(1 + \alpha_2 - \alpha) + \frac{1}{75}\alpha(1 - \alpha) \alpha_2 $$

$$ \leq \frac{1}{3} + \frac{1}{12}\alpha(2 - \alpha) + \frac{1}{75}\alpha(1 - \alpha) $$

$$ \leq 0.417 \quad \text{for} \ 0 \leq \alpha \leq 1. \quad (18) $$

4.2 The modified FGM family defined by

$$ F_{12}(x_1, x_2) = F_1(x_1)F_2(x_2)[1 + \alpha \min(S_1(x_1), S_2(x_2))] \quad (19) $$

or equivalently
\[ F_{12}(x_1, x_2) = F_1(x_1)F_2(x_2)[1 + \alpha(1 - \max(F_1(x_1), F_2(x_2)))] \] (19)

has a somewhat larger value of \((\omega)_{X_1X_2}\). We have

\[
(\omega)_{X_1X_2} = 12 \alpha \int_0^1 \int_0^1 x_1x_2(1 - \max(x_1, x_2))dx_1dx_2
\]

\[
= 12 \alpha \times 2 \int_0^1 \int_0^{x_2} x_1x_2(1-x_2)dx_1dx_2
\]

\[
= 12 \alpha \int_0^1 x_2^3(1-x_2)dx_2
\]

\[
= 0.6\alpha. \quad (20)
\]

This is considerably larger (if \(\alpha > 0\)) than the values so far obtained, but it still cannot exceed 0.6.

5. **Comparison with Mean Contingency**

The mean contingency (as defined by Rényi, (1970, p. 282)) for an absolutely continuous distribution with standard uniform marginal distribution is

\[
\phi_{X_1X_2}^2 = \int_0^1 \int_0^1 [f_{12}(x_1, x_2)]^2dx_1dx_2 - 1 \quad (21)
\]

where \(f_{12} = \frac{\partial^2 F_{12}}{\partial x_1 \partial x_2}\) is the joint probability density function of \(X_1\) and \(X_2\).

Table 1 summarizes values of \((\omega)_{X_1X_2}\) and the corresponding values of \(\phi_{X_1X_2}^2\), for the distributions discussed in this note.
Table 1: Comparison of \((c_0)_{X_1X_2}\) and \(\phi^2_{X_1X_2}\)

<table>
<thead>
<tr>
<th>Value of ((c_0)_{X_1X_2}) from eq.</th>
<th>Corresponding value of (\phi^2_{X_1X_2})</th>
</tr>
</thead>
<tbody>
<tr>
<td>((6)) (\frac{1}{3}\left</td>
<td>\alpha\right</td>
</tr>
<tr>
<td>((8)) (\frac{1}{3}(1+\frac{1}{4}a_1)\alpha)</td>
<td>(\frac{1}{9}(1+\frac{1}{2}a_1+\frac{4}{25}a_1^2)\alpha^2)</td>
</tr>
<tr>
<td>((9)) (\frac{1}{3}\alpha+\frac{1}{12}\alpha_1\alpha_2+\frac{1}{75}\alpha_1\alpha_2)</td>
<td>(\frac{1}{9}(1+\frac{1}{2}\alpha_1+\frac{4}{25}a_1^2)\alpha_2^2+\frac{2}{25}(\alpha_1+\alpha_2)\alpha_1\alpha_2+\frac{1}{75}\alpha_1\alpha_2(\alpha_1+\alpha_2+1)\frac{4}{1225}\alpha_1^2\alpha_2^2)</td>
</tr>
<tr>
<td>((16)) (\frac{1}{3}(\alpha+\alpha_1)+\frac{1}{12}\alpha(\alpha_1+\alpha_2)+\frac{1}{75}\alpha_1\alpha_2)</td>
<td>(\frac{1}{9}(1+\frac{1}{2}\alpha_1+\frac{4}{25}a_1^2)\alpha_1\alpha_2^2+\frac{2}{25}(\alpha+\alpha_1)(\alpha_1+\alpha_2)+\frac{4}{1225}\alpha_1^2\alpha_2^2+\frac{1}{75}\alpha_1\alpha_2(\alpha_1+\alpha_2+1))</td>
</tr>
</tbody>
</table>

We note that for \((6)\)

\[(c_0)_{X_1X_2}^2 = \phi_{X_1X_2}^2\]

and for \((8)\)

\[(c_0)_{X_1X_2}^2 = \phi_{X_1X_2}^2 - \frac{13}{1200}\alpha^2\alpha_1^2\]

and for \((9)\), approximately

\[(c_0)_{X_1X_2}^2 \approx \phi_{X_1X_2}^2 - \frac{13}{1200} \left[ (1 + \frac{4\alpha_2}{13})^2 + \left( \frac{16}{35} \alpha_2^2 \right)^2 \right] \alpha^2\alpha_1^2\]
REFERENCES


