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Abstract

In this note two characterizations of the exponential distribution are discussed, based on a generalization of the lack of memory property. These results were motivated by the notion of "relevation of distributions" introduced by Krakowski (1973).

Key words: exponential distribution; characterization; relevation; convolution; power series expansion; Laplace transform; non-negative random variables; replacement procedures.

1. Definitions

If the random variables Y_1, Y_2 with survival distribution functions (SDF's)

$$\Pr[Y_i > t] = S_i(t) \quad (i=1,2)$$

(with $S_i(0) = 1$) are mutually independent, then the SDF of $T = Y_1 + Y_2$ is

$$\Pr[T>t] = S_1(t) - \int_0^t S_2(t-x)dS_1(x) \ . \qquad (1)$$

This is the *convolution* of $S_1(t)$ and $S_2(t)$.

If, now, we think of $S_1(t)$ as the SDF of an item placed in service at time $t = 0$, and $S_2(t)$ as the SDF of an item placed in storage at time $t = 0$ and chosen independently from survivors in storage to replace the first item when it fails, then the SDF of the time of failure (T') in service of the second item is

$$\Pr(T'>t) = S_1(t) - \int_0^t [S_2(t)/S_2(x)]dS_1(x) \ . \qquad (2)$$

The first term on the right-hand side corresponds to survival in service of the first item; the second, to failure of the first, followed by survival of the replacement till time t. Formula (2) was given by Krakowski (1973) who termed it the *relevation* of $S_1(t)$ and $S_2(t)$. Note that, in general, the relevation of $S_1(t)$ and $S_2(t)$ differs from that of $S_2(t)$ and $S_1(t)$. For convolution, however, there is no difference.

We can write $T' = Y_1 + Y_2'$ where Y_1 is the failure time of the first item, and Y_2' is the time in service of the second item (from replacement to failure).

2. Relevation and Convolution

If Y_2' were independent of Y_1, then the relevation of $S_1(t)$ and $S_2(t)$ would be identical with their convolution. This will certainly be so, whatever $S_1(t)$ be, if $S_2(t) = \exp(-\lambda t)$ with $\lambda > 0$. It will also be so, if $S_1(t)$ corresponds to a discrete distribution taking only positive integer values, and $S_2(t)$ corresponds to a geometric distribution with

$$\Pr[Y_2=n] = (1-\theta)\ \theta^{n-1} \quad (n=1,2,\dots) \ .$$

This is because this distribution has a 'lack of memory' property

$$P(Y_2=t+n|Y_2>t) = (1-\theta)^{-1}\,\theta^{n-1} = P(Y_2=n) \quad (t,n=1,2,\ldots)$$

(since $P(Y_2>t) = (1-\theta)^{-1}\,\theta^{t}(1+\theta+.) = \theta^{t}$ and $P(Y_2=t+n) = (1-\theta)^{-1}\,\theta^{t+n-1}$) analogous to that of the exponential distribution.

We will show that if $S_2(t)$ is not only continuous but can be expressed in the form of a power series

$$S_2(t) = 1 + \sum_{j=1}^{\infty} a_j\, t^j \tag{3}$$

then relevation and convolution of $S_1(t)$ and $S_2(t)$ are identical only if $S_2(t) = \exp(-\lambda t)$. We do this by taking the special case $S_1(t) = S_2(t)$.

We first note that since $S_2(t)$ is monotonic and bounded, (3) implies that (i) the probability density function (PDF) is

$$f_2(t) = -dS_2(t)/dt = \sum_{j=0}^{\infty} (j+1)a_{j+1}\, t^j ,$$

and (ii) inversion of summation and integration over $[0,t]$ with $\underline{t}$ finite is justified.

Identity of relevation and convolution requires that the right hand sides of (1) and (2) be identical, that is

$$\int_0^t S_2(t-x)dS_1(x) = S_2(t) \int_0^t \{S_2(x)\}^{-1}\, dS_1(x). \tag{4}$$

In the special case $S_1(t) = S_2(t) = S(t)$, say, this implies

$$\int_0^t S(t-x)dS(x) = S(t)\log S(t). \tag{5}$$

If (3) is valid, then

$$-1 < \sum_{j=1}^{\infty} a_j\, t^j < 0 \text{ for } t > 0$$

and (with $a_0=1$)

$$S(t)\log S(t) = \{\sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} (\sum_{k=1}^{\infty} a_k t^k)^i\} \sum_{j=0}^{\infty} a_j t^j \tag{6}$$

$$= \sum_{g=1}^{\infty} t^g \sum_{\mu=1}^{g} a_{g-\mu} (a_\mu - \frac{1}{2} \sum_{\substack{m_1+m_2=\mu \\ m_i\geq 1}} a_{m_1} a_{m_2} + \frac{1}{3} \sum_{\substack{m_1+m_2+m_3=\mu \\ m_i\geq 1}} a_{m_1} a_{m_2} a_{m_3} - \ldots).$$

The left hand side of (5) equals

$$\int_0^t S(t-x)dS(x) = \int_0^t \{1 + \sum_{j=1}^{\infty} a_j (t-x)^j\} \sum_{h=0}^{\infty} (h+1)a_{h+1} x^h dx$$

$$= \sum_{h=0}^{\infty} \sum_{j=0}^{\infty} (h+1)a_{h+1} a_j B(h+1,j+1)t^{h+j+1} \tag{7}$$

$$= \sum_{g=1}^{\infty} t^g \sum_{h=0}^{g-1} (h+1)a_{h+1} a_{g-h-1} B(h+1,g-h) = \sum_{g=1}^{\infty} t^g \sum_{\mu=1}^{g} \mu a_\mu a_{g-\mu} B(\mu,g-\mu+1).$$

($B(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$ is the beta function.)

The coefficients of t^g in (6) and (7) must coincide. For $g=1$ we obtain only the trivial condition $a_1 = a_1$ and we are free to select this coefficient arbitrarily. By equating the coefficients of t^2, we obtain $a_2 + \frac{1}{2}a_1^2 = \frac{1}{2} a_1^2 + a_2$, which is identically satisfied for all choices of a_1 and a_2. For $g=3$ we obtain $a_2 = \frac{a_1^2}{2}$. In general, for $g\geq 3$, the equality of the coefficients of t^g in (6) and (7) yields

$$(a_g - \frac{1}{2} \sum_{\substack{m_1+m_2=g \\ m_i\geq 1}} a_{m_1} a_{m_2} + \ldots) + a_1(a_{g-1} - \frac{1}{2} \sum_{\substack{m_1+m_2=g-1 \\ m_i\geq 1}} a_{m_1} a_{m_2} + \ldots) + \ldots + a_{g-1}(a_1)$$

$$= g\, a_g B(g,1) + (g-1)a_{g-1} a_1 B(g-1,2) + \ldots + a_1 a_{g-1} B(1,g) . \tag{8}$$

Here the dots stand for terms that contain only coefficients with subscripts less than $g-1$. By using also the identities $gB(g,1) = gB(1,g) = g(g-1)B(g-1,2)=1$, the second member of (8) reduces to $a_g + (2a_1/g)a_{g-1} + h(a_1,a_2,\ldots,a_{g-2})$.

If we assume that, for a given choice of $a_1 \neq 0$, we have already determined $a_2,\ldots,a_{g-2}$ then (8) yields

$$a_g + a_{g-1} \cdot a_1 = a_g + (2a_1/g)a_{g-1} + h(a_1,\ldots,a_{g-2}),$$

where $h(a_1,\ldots,a_{g-2})$ is a certain polynomial in the known quantities $a_1,\ldots,a_{g-2}$. It follows that

$$a_1\, a_{g-1}\, (1-(2/g)) = h(a_1,\ldots,a_{g-2}) \tag{9}$$

and (9) (with g=3,4...) yields unique values for the coefficients a_g, $g \geq 2$. In fact, once we know that, given $a_1 \neq 0$, (9) leads to unique values for all coefficients of (3), we do not have to determine these coefficients recursively by (9). Indeed, with $a_n = \frac{a_1^n}{n!}$, S(t) becomes e^{at} and satisfies (5), as well as S(0) = 1. Finally, the conditions $-S'(t) = f(t) \geq 0$ and $\int_0^\infty f(t)dt = 1$ require that $a = -\lambda < 0$, so that $S(t) = e^{-\lambda t}$ is the unique solution with $a_1 = S'(0) = -\lambda$.

3. Remarks

Although we have established that the exponential distribution $S_2(t) = \exp(-\lambda t)$ is the only SDF which makes (1) and (2) identical for *all* $S_1(t)$, among SDF's which can be expressed in power series, there are at least two more general results which, one feels should be valid.

(i) The power series expansion condition could be weakened to require only that $S_2(t)$ is continuous (and perhaps with a continuous derivative).

(ii) It should be possible to show that under the conditions in (i), if $S_2(t)$ is not exponential, there is *no* $S_1(t)$ for which relevation and convolution are identical. We defer a discussion of these matters to a future paper.

4. Averaged lack of memory property

In a number of papers devoted to characterization of exponential distribution by spacings and order statistics, Ahsannulah (1976-8) investigated variants and extensions of the following equation:

$$\int_0^\infty f(x) \left[\frac{S(x+z)}{S(x)} - S(z)\right]dx = 0 \text{ for all } z \geq 0. \tag{10}$$

One interpretation of this equation is in terms of a weighted (or averaged) lack of memory property, the weight function being the underlying PDF $f(\cdot)$.

Another interpretation (in terms of relevations) is that the time in service of the second item (i.e. from replacement of the first item to failure of the second) has the same distribution as the lifetime of individual items chosen at random at time zero. It does not ensure that time in service is *independent* of time of starting in service (i.e. of failure of the first component). Of course, if the latter is true it is obvious that this ensures that distribution of time in service is the same as distribution of individual lifetimes (and indeed it ensures exponentiality).

Ahsannulah derives an exponential solution for these types of equations under the restriction that the underlying density possesses a monotone hazard rate. We shall prove the existence of a unique exponential solution for (10) without any assumptions except the existence of a density $f(\cdot)$ for the SDF $S(\cdot)$. In fact the following result is valid:

Theorem 4.1. If

$$\int_0^\infty f(x) \left\{\frac{S(x+z)}{S(x)} - S(z)\right\}dx = 0 \tag{11}$$

then $S(x) = e^{-\lambda x}$ for some $\lambda > 0$, and $f(x) = \lambda e^{-\lambda x}$.

Proof. a) If $S(x) = e^{-\lambda x}$, then the large bracket in (11) vanishes so that the functions $S(x) = e^{-\lambda x}$ are solutions.

b) To prove the converse, let $S(x)$ be any solution and let λ be an arbitrary positive number. Set $g(x) = e^{\lambda x} S(x)$, so that $S(x) = e^{-\lambda x} g(x)$. By substitution in (11), we obtain, after routine simplifications

$$\int_0^\infty g(x+z)\ \{\lambda - \frac{g'(x)}{g(x)}\}\ e^{-\lambda x}\, dx = g(z)\ . \tag{12}$$

The right hand member can be written as $\lambda \int_0^\infty g(z)\ e^{-\lambda x}\, dx$ and (12) is equivalent to

$$\int_0^\infty \{\lambda(g(x+z) - g(z)) - \frac{g(x+z)}{g(x)}\, g'(x)\}\ e^{-\lambda x}\, dx = 0\ . \tag{12$'$}$$

Set

$$\psi(x,z) = \int_0^x \frac{g(u+z)}{g(u)}\, g'(u)du\ . \tag{13}$$

We observe that: (i) $\psi(0,z) = 0$ for every z.

(ii) since $\lim_{x\to\infty} S(x) = 0$, $g(x) = o(e^{\lambda x})$.

(iii) from the monotonicity of $S(x)$,

$$\frac{g(u+z)}{g(x)} = e^{\lambda z}\ \frac{S(u+z)}{S(u)} < e^{\lambda z}\ .$$

From this it follows, in particular, that

$$\psi(x,z) = \int_0^x \frac{g(u+z)}{g(u)}\ g'(u)du < A(z) \int_0^x g'(u)du$$

$$= A(z)\cdot(g(x) - g(0)) < A(z)g(x)$$

with $A(z) = \limsup_{u\to\infty} \frac{g(u+z)}{g(z)} < e^{\lambda z}$

and

$$\lim_{x\to\infty} e^{-\lambda x}\ \psi(x,z) \le \lim_{x\to\infty} \{A(z)g(x)e^{-\lambda x}\} = A(z) \lim_{x\to\infty} S(x) = 0. \tag{14}$$

From (12)' we now obtain, by integration by parts and by use of (13) and (14),

$$\lambda \int_0^\infty \{g(x+z) - g(z)\} e^{-\lambda x} dx = \lambda \int_0^\infty \psi(x,z) e^{-\lambda x} dx.$$

For $\lambda \neq 0$, using the uniqueness of the inverse Laplace transform and the continuity of the functions $\psi(x,z)$ and $g(x)$,

$$g(x+z) - g(z) = \psi(x,z) .$$

Differentiating with respect to x, and using (13):

$$g'(x+z) = \frac{\partial\psi(x,z)}{\partial x} = \frac{g(x+z)}{g(x)} g'(x) ,$$

whence

$$\frac{g'(x+z)}{g(x+z)} = \frac{g'(x)}{g(x)}$$

for every z. It follows that $\frac{g'(x)}{g(x)}$ is constant, equal to c, say, where $\log g(x) = cx+d$, $g(x) = ae^{cx}$ and so the most general function S(x) is of the form $S(x) = e^{-\lambda x} g(x) = ae^{-(\lambda-c)x}$.

Since $S(0) = 1$, $a = 1$; and since $S(x) \to 0$ as $x \to \infty$, $\mu = \lambda - c > 0$. Hence $S(x) = e^{-\mu x}$, $\mu > 0$. □

5. Concluding Remarks

We conclude by noting that two methods of solving characterizing equations for an exponential c.d.f. were utilized in this paper. The first is a power series expansion and the second is a Laplace transform representation. To the best of our knowledge, neither one of these methods has been previously utilized in deriving characterizations of exponential distributions, of the kind considered in this paper. This fact may lead to some additional characterizations of this distribution as well as to alternative proofs of existing characterizations, possibly under milder assumptions.

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