EXPLICIT CHARACTERIZATION OF OPTIMAL STOPPING TIMES*

by

Anthony Mucci

Department of Biostatistics
University of North Carolina at Chapel Hill

Institute of Statistics Mimeo Series No. 1115

April 1977
ABSTRACT

A large class of continuous time optimal stopping problems is shown to have solutions explicitly determined by roots of equations \( xH(x) = 1 \) where \( H \) involves Laplace transforms. These results motivate the specification of discrete time optimal stopping problems whose solutions are approximated by solutions to corresponding continuous time problems, making rigorous a procedure sometimes employed in the literature. A fairly self-contained treatment of continuous time optimal stopping is also included, albeit for highly structured situations.
(I) This paper is centered around Section (2) where we determine explicit solutions for the problem of optimally stopping $e^{-\lambda t}X_t$, where $X$ is a Hunt process. More precisely, we determine $F$ and $T_\infty$ where

$$F(t,x) = \sup_T E\left\{ e^{-\lambda T}X_T | X_t = x \right\}$$

$$= E\left\{ e^{-\lambda T_\infty}X_{T_\infty} | X_t = x \right\}$$

for a representative class of Hunt processes. It is shown in the theorems of Section (2) that $T_\infty$ satisfies

$$T_\infty = \inf t \text{ such that } X_t \geq x_0$$

where $x_0$ is the largest $x$ for which

$$x \leq H(x) \leq 1$$

where, if $T_x$ is the hitting time for $x$, then

$$H(x) = \lim_{y \to x} \frac{d}{dy} \int e^{-\lambda T}X_{dy} , \quad y < x.$$ 

In particular, when $X$ has independent homogeneous increments, continuous paths:

$$x_0 = \frac{1}{K(\lambda)}$$

where

$$\int e^{-\lambda T}X_{dy} = e^{-K(\lambda)(x-y)}$$

The technique determining these results can be partially adapted to more general situations, for instance, it is easily established that the
optimal hitting time $T_\infty$ for the return function $\frac{X_t}{a+t}$ where 'X is standard Brownian motion has form

$$T_\infty = \inf t \text{ such that } X_t \geq K \sqrt{a+t}$$

although the constant $K$ must be determined by other methods.

(II) Section (1) represents an attempt to provide a simple treatment of continuous time optimal stopping by imposing strong conditions on the processes and payoffs under consideration. In many applications these restrictive conditions are shown to be more apparent than real—this is made precise in Remarks (1,8).

(III) Section (3) provides two representative situations in which the solution to a continuous time optimal stopping problem is a good approximation to the solution of a discrete time optimal stopping problem when the underlying processes are close in the sense of weak convergence.

The first two sections were motivated by a close reading of Taylor [13]. The last section developed from remarks made in Chernoff [3] and Shepp [12].

Section (1). Continuous Time Optimal Stopping

We restrict attention to a class of Markov processes $X = \{X_t, t \geq 0\}$ which are real valued and governed by an initial distribution $\mu$ and a transition $p$ in the sense that for all $t, s, x, y$: 

$$P\{X_{t+s} \leq y | X_t = x\} = P_x\{X_s \leq y\} = \int_0^y p(s, x, dz)$$
and

\[ P_u \{ X_t \leq y \} = \int_0^y p(t, x, dx) u(dx) . \]

The subclass of such processes to be considered is a restricted class of
Hunt processes which satisfies the following conditions

(1.1) \( X \) is strong Markov and \( P_x \{ X_0 = x \} = 1 \)

(1.2) \( X \) has right continuous paths with left limits

(1.3) **Feller property:** For all bounded continuous \( f, (p^t f)(x) = \int f(X_t) dP_x \)

is bounded continuous.

(1.4) \( X \) is extended quasi-left continuous: If \( T_n \) is an increasing

sequence of stopping times and if \( T_n \searrow T_\infty \leq \infty \), then \( X_{T_n} \) exists

as a finite value \( P_x \) a.e. and \( P_x \{ X_{T_n} \rightarrow X_{T_\infty} \} = 1 \).

Taylor [13] restricts \( X_{T_n} + X_{T_\infty} \) to the set where \( T_\infty < \infty \). Our more

restrictive condition, implying at least that \( X_\infty \) exist a.e. is convin-

cient for the development of the theory of this section and presents no real

obstacle for applications of the usual sort, as will soon become evident.

We remark also that the conclusion \( X_{T_n} + X_{T_\infty} \) or \( T_\infty < \infty \) obtains whenever

one has the following (see Dynkin [7], Volume 1, p. 104):

\[ \lim_{t \to 0} \sup_{x \in K} p(t, x, [x - \varepsilon, x + \varepsilon]^C) = 0 \]

all compact \( K \).

In all that follows our state space \( S \) will be the time-space struc-
ture \([0, \infty) \times \mathbb{R} \) with its product topology and Borel sets. The points in

this space will usually be denoted \( z = (t, x) \), and our process \( X \) will

be thought of as a process \( Z = \{(t, X_t), t \geq 0\} \) still governed by our
transition \( p \) in the sense that \( p(s, x, dy) \) becomes \( p(s, (t, x), (t+s, dy)) \).

We now define our optimal stopping problem. The elements are:

(1.5) \( f \), a non-negative and continuous function on \( S \),

(1.6) For all \((t, x) \in S, f \in L_1(t, x)\) where this means

\[
\int_{S} \sup_{s} f(t+s, X_{t+s}) dP(t, x) < \infty ,
\]

Let's call \( f \) the return function. The optimal stopping problem is the determination of the optimal payoff \( F \) and the optimal stopping time \( T_\infty \) where

(1.7) \[ F(t, x) = \sup_{T} \int f(T, X_{T}) dP(t, x) = \int f(T_\infty, X_{T_\infty}) dP(t, x) \]

where \( T \) runs through all stopping times.

(1.8) Remarks

The determination of

\[
\sup_{T \geq t} \int c(t+T)x^+ dP(t, x), \quad c(t) \geq 0
\]

where is the exclusive concern of the applications in later sections is the problem determined by the process \( X = \{X_t, t \geq 0\} \) and the return \( f(t, x) = c(t)x^+ \). It is seldom the case that \( X_\infty \) exists, i.e., that (1.4) holds.

Our development of the theory of continuous time optimal stopping uses (1.4) essentially. This discrepancy between theory and application is only apparent, not substantial. The accommodation of such applications under our theory merely involves a redefinition of processes, that is, we define \( Y = \{Y_t, t \geq 0\} \) where \( Y_t = c(t)X_t \) and consider
\[ \sup_{T \geq t} \int_{Y_T}^t dP(t, y) \]

where \( c(t) \) decreases rapidly enough so that \( c(t)X_t \to 0 \), i.e., so that \( Y_\infty = 0 \).

We now consider classes of processes \( X \) and well-behaved non-negative decreasing \( c(t) \) for which (1.1) through (1.3) hold and for which (1.4) and (1.6) hold in the form

\[
\begin{align*}
&\mathbb{P}(0,0) \left\{ \lim_{t \to \infty} c(t)X_t = 0 \right\} = 1 \\
&\left( \sup_{t} c(t) |X_t| dP(0,0) \right)^{< \infty}.
\end{align*}
\]

These are the versions of (1.4) and (1.6) proper to \( f(t,y) = y^+ \) where \( Y_t = c(t)X_t \). We consider only the starting point \( (t,x) = (0,0) \) for computational convenience.

**Class (1)**

(1.10) \[ X_t = X_0 + \int_0^t \sigma(X_s) dW_s + \int_0^t m(X_s) ds \]

where \( \sigma \) and \( m \) are bounded, \( X_0 \) is square integrable, \( W \) is standard Brownian motion and \( X_0 \) in independent of \( W \).

**Class (2)**

\( X_t \) is non-explosive pure jump on the integers with bounded intensities, i.e., there exists \( \{\lambda_x, x \text{ an integer}\} \) and \( \{Q_x, x \text{ an integer}\} \) where if \( T_x \) is the jump time from \( x \), then
\[ P_x \{ X_x > t \} = e^{-\lambda_x t} \]
\[ P_x \{ X_x = x+y \} = Q_x(y) \]
\[ \sup_x \lambda_x \leq \lambda_\infty < \infty \]
\[ \sup_x \int y^2 Q_x(dy) < \infty . \]

Both classes of processes are easily seen to satisfy the decomposition

\[ X_t = X_0 + Y_t + Z_t \quad \text{where} \]
\[ X_0 \text{ is square integrable} \]
\[ Y_t \text{ is a martingale and } \int (Y_{t+s} - Y_t)^2 ds \leq K s, \text{ small } s \]
\[ |Z_t| \leq M_t \]

Clearly, for decreasing \( c(t) \), \( \int \sup_t c(t) |X_t| dP(0,0) < \infty \) provided

\[ \int \sup_t c(t) |Y_t| dP(0,0) < \infty \]
\[ \text{and} \]
\[ \sup_t c(t) t < \infty . \]

Now, by right continuity and with obvious notation

\[ \int \sup_t c(t) |Y_t| dP(0,0) = \lim_{\Delta t \to 0} \int \sup_k c(k\Delta t) |Y_k\Delta t| dP(0,0) \]
\[ = \lim_{\Delta t \to 0} \int_{0}^{\infty} P(0,0) \left\{ \sup_k c(k\Delta t) |Y_k\Delta t| > a \right\} da \]
\[ \leq \lim_{\Delta t \to 0} \int_{0}^{\infty} \left\{ \frac{1}{a^2} \sum_k c^2(k\Delta t) \int (Y_{(k+1)\Delta t} - Y_{k\Delta t})^2 dP(0,0) \right\} da \]
by the Hajek-Renyi-Chow inequality, (see Chow, Robins, Siegmund [5]).

Using (1.12):

\[ \int \sup_t c(t) |Y_t| dP(0,0) \leq 1 + \int_1^\infty \left\{ \int_0^\infty c^2(t) dt \right\} \frac{da}{a^2} \]

from which

(1.14) \[ \int_0^\infty c^2(t) dt < \infty \Rightarrow \int \sup_t c(t) |Y_t| dP(0,0) < \infty \]

It is easily seen that \( \int_0^\infty c^2(t) dt < \infty \Rightarrow \sup_{t \geq 0} c(t) t < \infty \).

Further, it is shown in Chow [4] that the hypothesis in (1.14) also implies \( c(t) Y_t \to 0 \). Thus, in order that (1.9) hold, we need \( c(t) Z_t \to 0 \), which certainly holds if \( tc(t) \to 0 \). We collect these criteria for ready reference.

**Proposition (1.1)**

Let \( X_t = X_0 + Y_t + Z_t \) satisfy (1.12). Then

(A) If \( Z_t \equiv 0 \), and \( \int_0^\infty c^2(t) dt < \infty \), then

\[
\begin{align*}
\int \sup_s c(t+s) |X_{t+s}| dP(t,x) < \infty, \text{ all } (t,x) \\
\text{and} \\
P_{(t,x)} \left\{ \lim_{s \to \infty} c(t+s) X_{t+s} = 0 \right\} = 1, \text{ all } (t,x)
\end{align*}
\]

(1.15)

(B) The conclusions (1.15) obtain if the hypotheses in (A) are replaced by

\( tc(t) \to 0 \).

(1.16)
Remarks

The stationary Ornstein-Uhlenbeck process has representation

\[ X_t = e^{-at} \left( X_0 + \sqrt{2ab} \int_0^t e^{as} \, dW_s \right) \]

and appears in the applications. One can model the arguments above, via

the Hajek-Renyi-Chow inequality to determine that

\[ \int \sup_s e^{-\lambda(t+s)} |X_{t+s}| \, dP(t,x) < \infty \]

and

\[ \lim_{s \to \infty} e^{-\lambda(t+s)} X_{t+s} = 0 \]

All these results overlap those of Walker [14].

We now turn to the determination of \( F \) and \( T_\infty \) from (1.7). Our
development is a modification and generalization of that found in Taylor
[13] and Dynkin [8]. We always assume (1.1) through (1.6).

(1.17) Definition

(A) \( G : S \to R_+ \) is called excessive if \( G \) is universally measurable

and if

\[ G \geq P^t G \text{ all } t \geq 0 \]

and

\[ \lim_{t \to 0} P^t G = G \]

(B) An excessive \( G \) is called an excessive majorant for \( f \) if \( G \geq f \).

(C) An excessive majorant \( G \) for \( f \) is called the least excessive

majorant if \( H \geq G \) for all excessive majorants \( H \) of \( f \).
Grigelionis-Shiryae [9] show that \( f \) has a least excessive majorant, \( f_\infty \), determined by the recursion

\[
f_0 = f, \quad f_n = \sup_t p^t_{f_{n-1}}, \quad f_\infty = \eta f_n.
\]

Further, the continuity of \( f \) and the Feller property make \( f_\infty \) lower semi-continuous — see Taylor [13]. We define (allowing \( t = \infty \) as a value):

\[
(1.18) \quad \Gamma_\infty = \{(t,x); f(t,x) = f_\infty(t,x)\}.
\]

Again, the lower semi-continuity of \( f_\infty \) makes \( \Gamma_\infty \) a closed set. The hitting time

\[
(1.19) \quad T_\infty = \inf_{t} (t,X_t) \in \Gamma_\infty
\]

will be achieved with probability one from any starting point \((t,x)\) if the following holds

\[
(1.20) \quad P(t,x)\{f(\infty, X_\infty) = f_\infty(\infty, X_\infty)\} = 1
\]

We prove this by adapting Neveu [11] to our context:

For fixed \( t < t_0 < r \):

\[
f_\infty(r,X_r) \leq E(t,x)\left\{ \sup_{s \geq r} f(s,X_s) | B_r \right\} \\
\leq E(t,x)\left\{ \sup_{s \geq t_0} f(s,X_s) | B_r \right\},
\]

Let \( r \to \infty \), we have

\[
E(t,x)\left\{ \sup_{s \geq t_0} f(s,X_s) | B_r \right\} + \sup_{s \geq t_0} f(s,X_s) P(t,x) \text{ a.e.}
\]

so that, by lower-continuity of \( f_\infty \):
\[ f_{\infty}(\infty, X_\infty) \leq \lim_{r \to \infty} f_{\infty}(r, X_r) \leq \sup_{s \geq t_0} f(s, X_s) \quad P(t, x), \quad \text{a.e.} \]

Letting \( t_0 \to \infty \) and using the continuity of \( f \) we have

\[ f_{\infty}(\infty, X_\infty) \leq f(\infty, X_\infty) \]

and the other direction is obvious. We turn now to our principal result.

**Theorem (1.1)**

\[ f_{\infty}(t, x) = \int f_{\infty}(T_\infty, X_{T_\infty}) dP(t, x), \quad \text{all } (t, x) \in S. \]

**Proof:**

Dynkin [8] shows the following. If \( \epsilon > 0 \) and \( \Gamma_\epsilon = \{(t, x) : f_{\infty}(t, x) \leq f(t, x) + \epsilon\} \), then with \( T_\epsilon \) the hitting time for \( \Gamma_\epsilon \) and with \( f \) bounded:

\[ f_{\infty}(t, x) = \int f_{\infty}(T_\epsilon, X_{T_\epsilon}) dP(t, x), \]

It is clear that \( \Gamma_\epsilon \) is closed, that \( \Gamma_\epsilon + \Gamma_\infty \) and that \( T_\epsilon \uparrow T_\infty \) so that by our quasi-left continuity assumptions \( X_{T_\epsilon} \to X_{T_\infty} \). Further,

\[ f_{\infty}(t, x) \leq \int f(T_\epsilon, X_{T_\epsilon}) dP(t, x) + \epsilon. \]

Letting \( \epsilon \to 0 \) and using the quasi-left continuity of \( X \), and continuity and boundedness of \( f \), we have

\[ f_{\infty}(t, x) \leq \int f(T_\infty, X_{T_\infty}) dP(t, x), \]

Since \( f \leq f_{\infty} \) and since \( f_{\infty} \) is excessive and bounded, we've established our result in the bounded case. We now consider the unbounded case, subject as usual to (1.6). Set, for each \( a > 0 \):
\[ f_a = \min(f, a) \]
\[ \tilde{f}_a = \text{least excessive majorant of } f_a \]
\[ \Gamma_a = \{ f = \tilde{f}_a \} \]
\[ \tilde{\Gamma}_a = \{ f \geq \tilde{f}_a \} \]
\[ T_a = \text{hitting time for } \Gamma_a \]
\[ N_a = \text{hitting time for } \tilde{\Gamma}_a . \]

Note that \( \Gamma_a \subset \tilde{\Gamma}_a \), that \( N_a \leq T_a \) and that \( b \geq a \) implies \( \tilde{f}_b \geq \tilde{f}_a \) by properties of least excessive majorants. Let \( \tilde{f}_\infty = \sup_{a \to \infty} \tilde{f}_a \) as \( a \to \infty \).

Clearly \( \tilde{f}_\infty \) is excessive and \( \tilde{f}_\infty \geq f \), therefore \( \tilde{f}_\infty \geq f_\infty \). On the other hand, \( f_\infty \geq f_a \), hence \( f_\infty \geq \tilde{f}_a \), from which \( f_\infty = \tilde{f}_\infty \), so that \( N_a \uparrow T_\infty \), for if we set \( N_\infty = \sup_{a \to \infty} N_a \), then \( N_\infty \leq T_\infty \) while

\[
f(N_\infty, X_{N_\infty}) = \lim_{a \to \infty} f(N_a, X_{N_a}) \geq \lim_{a \to \infty} \tilde{f}_a(N_a, X_{N_a})
\]

\[
\geq \lim_{a \to \infty} \tilde{f}_b(N_a, X_{N_a}) \geq \tilde{f}_b(N_\infty, X_{N_\infty})
\]

by lower semi-continuity of \( \tilde{f}_b \). But then

\[
f(N_\infty, X_{N_\infty}) \geq \lim_{b \to \infty} \tilde{f}_b(N_\infty, X_{N_\infty}) = f_\infty(N_\infty, X_{N_\infty})
\]

which implies \( (N_\infty, X_{N_\infty}) \in \Gamma_\infty \), thus \( N_\infty \geq T_\infty \).

Next,\[
\int f(N_a, X_{N_a})dP(t, x) \geq \int \tilde{f}_a(N_a, X_{N_a})dP(t, x)
\]

\[
\geq \int \tilde{f}_a(T_a, X_{T_a})dP(t, x)
\]

since \( \tilde{f}_a \) is excessive and \( T_a \geq N_a \).
Since $\tilde{f}_a$ is bounded, we have by Dynkin's result,
\[
\int \tilde{f}_a(T_a, X_{T_a}) dP(t, x) = \tilde{f}_a(t, x),
\]
so that
\[
\int f(N_a, X_{N_a}) dP(t, x) \geq \tilde{f}_a(t, x),
\]
Now, (1.6) allows us to use Lebesgue dominated convergence:
\[
\int f(T_\infty, X_{T_\infty}) dP(t, x) = \lim_{a \to \infty} \int f(N_a, X_{N_a}) dP(t, x)
\]
\[
= \lim_{a \to \infty} \int f(N_a, X_{N_a}) dP(t, x) \geq \lim_{a \to \infty} \tilde{f}_a(t, x)
\]
\[
= f_\infty(t, x) = \tilde{f}_\infty(t, x).
\]
Since $f_\infty \geq f$ and $f_\infty$ is excessive, the result follows, Q.E.D.

**Theorem (1.2)**
\[
F(t, x) = \int f(T_\infty, X_{T_\infty}) dP(t, x), \text{ all } (t, x) \in S.
\]

**Proof:**
\[
F(t, x) = \sup_T \int f(T, X_T) dP(t, x) \geq \int f(T_\infty, X_{T_\infty}) dP(t, x)
\]
\[
= \int f_\infty(T_\infty, X_{T_\infty}) dP(t, x) = f_\infty(t, x)
\]
\[
\geq \sup_T \int f_\infty(T, X_T) dP(t, x)
\]
\[
\geq \sup_T \int f(T, X_T) dP(t, x)
\]
\[
= F(t, x), \text{ Q.E.D.}
\]
(1.21) Remarks on Discrete Time Optimal Stopping

Let \( X = \{X_{kr}, k = 0,1,2,\ldots\} \) where \( r > 0 \) be a Markov process which moves on the discrete time lattice \( \{kr\} \) and which is governed by the transition

\[
P(kr, x, dy) = P_x \{X_{kr} \in dy\}.
\]

In the case \( r = 1 \), we write

\[
p(r, x, dy) = p(x, dy).
\]

Suppose

(A) If \( f \) is bounded and continuous, then

\[
(P^\infty f)(x) = \int f(t + r, y)p(x, dy)
\]

is bounded and continuous.

(B) \( p(t, x)\{\lim_{k \to \infty} X_{kr} = X_\infty \text{ exists and is finite}\} = 1 \), all \((t, x)\).

(C) \( f \) is positive continuous and \( \int \sup_k f(t + kr, X_{t+kr}) dP(t, x) < \infty \).

Under (A), (B), and (C) the previous development for continuous time will hold for discrete time — it will in fact be simpler. Restricting stopping times, \( T \), to live on the lattice \( \{kr\} \), we have

\[
F(t, x) = \sup_T \int f(T, X_T) dP(t, x) = \int f(T_\infty, X_{T_\infty}) dP(t, x)
\]

where \( T_\infty \) is the hitting time for \( \Gamma_\infty = \{f = f_\infty\} \) where

\[
f_0 = f, \quad f_n = \max(f, Pf_{n-1}), \quad f_\infty = \sup_n.
\]

Section (2), Determination of Optimal Hitting Sets and Payoffs

Let \( X = \{X_t, t \geq 0\} \) be a Hunt process such that \( P(t, x) = P_x \) and

\[
\begin{align*}
\text{(a) } & \quad \sup_{s} e^{-\lambda(t+s)} |X_{t+s}| dP(t, x) < \infty \\
\text{(b) } & \quad P(t, x) \left\{ \lim_{s \to \infty} e^{-\lambda(t+s)} X_{t+s} = 0 \right\} = 1
\end{align*}
\]

We want to determine \( F, \Gamma_{\infty} \) and \( T_{\infty} \) for \( f(t, x) = e^{-\lambda t} x^+ \). Our theory applies since (1.1) through (1.6) obtain. It is easily seen that we can just as well consider \( f(t, x) = e^{-\lambda t} x \) since (2.1b) implies that it is unreasonable to examine stopping times \( T \) where \( X_T \) is negative, given that we can do better by continuing on to time infinity. Thus, we'll investigate the structure of \( F, \Gamma_{\infty}, T_{\infty} \) where

\[
F(t, x) = \sup_{T \geq t} e^{-\lambda T} X_T dP(t, x) = \int e^{-\lambda T_{\infty}} X_{T_{\infty}} dP(t, x).
\]

We assume throughout that \( P(t, x) = P_x \).

Theorem (2.1)

(A) Let \( X = \{X_t, t \geq 0\} \) be a Hunt process satisfying (2.1). Suppose that for all \( 0 \leq a < b < \infty \), there exists \( x \in (a, b) \) such that if \( T_{a, b} \) is the exist time from \( [a, b] \), then

\[
\begin{align*}
\left\{ \begin{array}{l}
P_x \{T_{a, b} < \infty\} = 1 \\
X \geq \int e^{-\lambda T_{a, b}} X_{T_{a, b}} dP_x
\end{array} \right.
\]

(2.2)

Then there exists \( x_0 \geq 0 \) such that \( \Gamma_{\infty} = \{(t, x) \mid x \geq x_0\} \).
(B) Let $X = \{X_t, t \geq 0\}$ be a Hunt process satisfying (2.1). Then the optimal hitting set for $f(t,x) = e^{-\lambda t} x$ again has form $\Gamma_\infty = \{(t,x): x \geq x_0\}$ if $X$ has homogenous independent increments, i.e., if

$$P_X\{X_t \leq a\} = P_0\{X_t \leq a-x\} = F_t(a-x)$$

where $F_t$ is a distribution for each $t \geq 0$.

**Proof:**

We write $P(t,x) = P_X$, $\tau = T-t$, $x_\tau = X_{T-t}$. Note then that if $(t,x) \not\in \Gamma_\infty$, then there exists $\tau$ with

$$e^{-\lambda t} x < \int e^{-\lambda (t+\tau)} (x + x_\tau) dP_x.$$

If $(t,x) \in \Gamma_\infty$, then for all $\theta$

$$e^{-\lambda t} x \geq \int e^{-\lambda (t+\tau)} (x + x_\tau) dP_x.$$

Consequently, by straightforward algebra

$$\text{(2.4)} \quad (t,x) \in \Gamma_\infty \iff x \geq \sup_{\tau > 0} \frac{\int e^{-\lambda \tau} x dP_x}{1 - e^{-\lambda \tau} dP_x} \equiv H(x)$$

where we can restrict $\tau$ to run through hitting times of closed sets in $S$.

If (2.3) holds, $H(x)$ is a constant and we're finished. So let's assume (2.2) holds. Now

$$\Gamma_\infty = \{(t,x); t \geq H(x)\}$$

and since $\Gamma_\infty$ is closed and contains no $(t,x)$ with $x$ negative, if $\Gamma_\infty \neq \{(t,x): x \geq x_0\}$, then there must exist $0 \leq a < b < \infty$ with $x < H(x)$, $x \in (a,b)$ while $a \geq H(a)$, $b \geq H(b)$. For $x$ as determined by (2.2), we'd have simultaneously
\[ x \geq \int e^{-\lambda T_{a,b} x} x^{T_{a,b}} dP_x \]

and

\[ e^{-\lambda t} x < \int e^{-\lambda (t+T_{a,b})} x^{T_{a,b}} dP_x , \]

a contradiction. Q.E.D.

An Explicit Characterization of \( x_0 \) for Diffusions

We restrict attention to \( X = \{ X_t, t \geq 0 \} \) satisfying the hypotheses of our theorem and having continuous paths. We impose a further requirement

(2.5) Let \( \tau_{x_0} \) be the time to reach \( x_0 \) and let \( x < x_0 \). Then for all \( \lambda > 0 \), the derivative \( \frac{d}{dx} \int e^{-\lambda T_{x_0}} dP_x \) exists as a continuous function with limit

\[ H_\lambda(x_0) = \lim_{x \uparrow x_0} \frac{d}{dx} \int e^{-\lambda T_{x_0}} dP_x . \]

Theorem (2.2)

Under the hypotheses in theorem (2.1) and the condition (2.5) we can characterize \( x_0 \) by

(2.6) \[ x_0 = \sup_x \text{ such that } xH_\lambda(x) \leq 1 . \]

Proof:

If \( x < x_0 \), then

\[ e^{-\lambda t} x < x_0 \int e^{-\lambda (t+T_{x_0})} dP_x \]

which we re-write as
\[ x < (x_0 - x) \int e^{-\lambda t x_0} dP_x \frac{-\lambda t x_0}{1 - \int e^{-\lambda t x_0} dP_x} \]

By the Mean Value Theorem of Calculus

\[ 1 - \int e^{-\lambda t x_0} dP_x = (x_0 - x) \frac{d}{dx} \int e^{-\lambda t x_0} dP_y \]

where \( x < y < x_0 \). Letting \( x \to x_0 \) we have our conclusion that

\[ x_0 \leq \frac{1}{H_\lambda(x_0)} \]

On the other hand, if \( x_0 \leq x < x_1 \), then

\[ e^{-\lambda t x} \geq x_1 \int e^{-\lambda(t+\tau x)} dP_x \]

and repeating our reasoning we have

\[ x_1 \geq \frac{1}{H_\lambda(x_1)} \]

Q.E.D.

**Applications**

(I) If \( X = \{X_t, t \geq 0\} \) has homogeneous independent increments, continuous paths, then there exists non-negative \( G(\lambda) \) with

\[ \int e^{-\lambda t x_0} dP_x = e^{-(x_0 - x)G(\lambda)} \]

Consequently

\[ H_\lambda(x_0) = G(\lambda) \]

so that

\[ x_0 = \frac{1}{G(\lambda)} \]
Thus, if $X$ is Brownian Motion with variance $\sigma^2$ and drift $m$, then

$$G(\lambda) = \frac{\sqrt{m^2 + 2\lambda \sigma^2} - m}{\sigma^2}$$

from which

$$X_0 = \frac{\sigma^2}{\sqrt{m^2 + 2\lambda \sigma^2} - m}$$

and

$$F(t, x) = \begin{cases} e^{-\lambda t} x & \text{if } x \geq x_0 \\ x_0 e^{-\lambda t} e^{-(x_0 - x)G(\lambda)} & \text{if } x < x_0 \end{cases}$$

This agrees with Taylor [13].

(II) The stationary Ornstein-Uhlenbeck process has representation

$$X_t = e^{-\alpha t} \left( X_0 + \sqrt{2\alpha \beta} \int_0^t e^{\alpha s} dW_s \right)$$

where $W$ is standard Brownian motion, $X_0$ is $\mathcal{N}(0, \beta)$, and $X_0$ and $W$ are independent. Conditions (2.1) and (2.2) are easily seen to hold for this process; for (2.2) one uses

$$P\{X_{Ta,b} = a\} = \frac{x}{\int_a^b e^{t^2/2\beta} dt} \int_a^b e^{t^2/2\beta} dt.$$

We consider the optimization

$$\sup_{T \geq t} \int e^{-T} X_t dP(t, x)$$
solved by Taylor [13]. The transform \( \int e^{-\lambda t} x_0 \, dp_x \equiv \phi(x, x_0) \) satisfies \( a b \phi'' - ax \phi' = \lambda \phi \), \( \phi(x_0) = 1 \)

which is solved in Darling-Siegert [6] and which, for the case \( \lambda = 1 \),
takes the relatively simple form

\[
(x, x_0) = \frac{\int_0^{\infty} e^{xt - \frac{t^2}{2}} \, dt}{\int_0^{\infty} e^{x_0 t - \frac{t^2}{2}} \, dt}.
\]

Straightforward calculations lead to

\[
H(1, x_0) = \lim_{x \to x_0} \frac{d}{dx} (x, x_0) = x_0 + \frac{\int e^{-\frac{1}{2} t^2} \, dt}{\int e^{-\frac{1}{2} \frac{x_0^2}{2}} \, dt}.
\]
Thus, from (2.6), we need

\[ x_0^2 + \frac{x_0 e^{\infty}}{\int_{-\frac{2}{2}}^{\frac{1}{2}} e^{-\frac{1}{2}t^2} dt} = 1 \]

which has solution \( x_0 \approx 0.839 \), agreeing with Taylor [ ]. The payoff \( F \) has form

\[
F(t,x) = \begin{cases} 
  e^{-\lambda t} & \text{if } x \geq x_0 \\
  x_0 \Phi(x,x_0) & \text{if } x < x_0 .
\end{cases}
\]

An Explicit Characterization of \( x_0 \) for Compound Poisson Processes

Let \( X = \{X_t, t \geq 0\} \) live on the integers as a pure jump process which has independent homogeneous increments. Thus, if \( T_x \) is the jump time from \( x \) to some other state, then

\[ P_x \{T_x > t\} = e^{-at}, \text{ some finite positive } a \]

\[ P_x \{X_{T_x} = x + Z\} = \Theta(Z), \text{ } \Theta \text{ a fixed probability } \]

Let's assume that \( \Theta(Z) = 0 \) if \( Z \leq 0 \) so that the process is increasing. We assume further that \( \sum Z^2 \Theta(Z) < \infty \) so that, using Section (1), Remarks (1.8), it is clear that \( X \) qualifies for the hypotheses of Theorem (2.1), in
fact, the existence of second moments is much more that is needed. Interpreting (2.4) in the present context, it is seen that

\[
\int e^{-\lambda T_{x_0}} x_{T_{x_0}}^{-1} dP_{x_{0}}^{-1} x_0^{-1} < \frac{\int e^{-\lambda T_{x_0}} x_{T_{x_0}}^{-1} dP_{x_{0}}^{-1}}{1 - \int e^{-\lambda T_{x_0}} dP_{x_0}}
\]

\[
x_0 \geq \frac{\int e^{-\lambda T_{x_0}} x_{T_{x_0}} dP_{x_0}}{1 - \int e^{-\lambda T_{x_0}} dP_{x_0}}
\]

where \( X_{T_x} \) is the size of the jump taken at jump time \( T_x \). Now the right side in the inequalities above is constant since the process has homogeneous independent increments. This constant is

\[
\int e^{-\lambda T_{0}} x_{T_{0}} dP_{0} = \frac{a}{\lambda} \sum Z\theta(Z)
\]

Thus \( x_0 \) is the smallest integer for which

\[
x_0 \geq \frac{a}{\lambda} \sum Z\theta(Z)
\]

The optimal payoff for the case \( \theta(1) = 1 \) is

\[
F(t, x) = \begin{cases} 
 e^{-\lambda t} x & \text{if } x \geq x_0 \\
 x_0 \left( \frac{a}{a + \lambda} \right)^{x_0 - x} e^{-\lambda t} & \text{if } x < x_0 
\end{cases}
\]

These results agree with Taylor [13].
Remarks

Let \( c(t) \) be smooth and decreasing to zero as \( t \to \infty \), let \( X \) be a Hunt process and consider

\[
F(t,x) = \sup_T \int C(T)X_T \ dP(t,x).
\]

Arguing as before, \( (t,x) \in \Gamma_\infty \) iff

\[
x \geq \sup_{\tau > 0} \frac{\int \frac{c(t)}{c(t+\tau)} X_\tau \ dP(t,x)}{1-\int \frac{c(t)}{c(t+\tau)} \ dP(t,x)}.
\]

If the process \( X \) has homogeneous independent increments, then \( P(t,x) = P_0 \) and the right side above is a function of \( t \) alone. The case most thoroughly treated in the literature has \( c(t) = \frac{1}{1+t} \) and \( X \) standard Brownian motion.

A generalization treated by Walker [15], slightly modified here, has \( X \) standard Brownian motion and

\[
c(t) = \frac{1}{(A+Bt)^r}, \quad r > \frac{1}{2}.
\]

The characterization of \( \Gamma_\infty \) for this case is \( (t,x) \in \Gamma_\infty \) iff

\[
x \geq \sup_{\tau} \frac{(A+Bt)^r}{1-(A+Bt)^r} \int \frac{X_\tau}{(A+B(t+\tau))^r}.
\]

Using the fact that \( X_t \overset{d}{=} \frac{a X_{t\frac{a^2}{2}}}{} \) for \( X \) standard Brownian motion, and making this substitution above with \( a = \sqrt{\frac{A+Bt}{B}} \), we calculate that...
\[(t,x) \in \Gamma_\infty \text{ iff} \]
\[
x \geq K_T \cdot \sqrt{\frac{A+bt}{B}}
\]
where
\[
K_T = \sup \int \frac{x^T}{(1+\tau)^T} \\
1 - \int \frac{1}{(1+\tau)^T}
\]

Thus, \(T_\infty\) is the hitting time for the square root boundary \(K_T \sqrt{\frac{A+bt}{B}}\), \(K_T\) some appropriate constant. The determination of \(K_T\) is achieved by Walker [15] and in special cases by Shepp [13], Taylor [12], and others; we won't pursue this problem here. We remark only that on the basis of this example and those considered previously, it is not too much to expect that the time \(T_\infty\) is the hitting time for some smooth curve \(G(t)\). This assumption is exploited in the next section.

Section (3). Weak Convergence in Optimal Stopping

We consider a class of discrete time optimal stopping problems whose solutions are approximated by solutions of related continuous time problems. To begin with, suppose \(\{X_n\}\) to be a sequence of processes in discrete time where \(X_n(t)\) has \(t\) restricted to the lattice \(\{m\phi(n), m = 0,1,2,\ldots\}\) where \(\phi(n) \to 0\). Let \(f\) be our return function having its usual domain \(S = [0,\infty) \times R\) and define

\[
F_n(t,x) = \sup \int f(T,X_n(T))dP(t,x)
\]

where \(T\) runs through stopping times on the lattice \(\{m\phi(n)\}\). Suppose we
interpolate $X_n$ so that it is a process in $C[0,\infty)$, i.e., we set

$$X_n(t) = \frac{((m+1)\phi(n)-t)}{\phi(n)} X_n((m+1)\phi(n))$$

$$+ \frac{(t-m\phi(n))}{\phi(n)} X_n((m+1)\phi(n))$$

for $t \in [m\phi(n), (m+1)\phi(n)]$.

Each discrete time process $X_n$ is Markovian and satisfies, for all $t, s$ in the lattice $\{m\phi(n)\}$

$$P\{X_n(t+s) \in dy | X_n(t) = x\} = p_n(s, x, dy)$$

for appropriate fixed transition probabilities $p_n$. For each fixed $(t, x)$, the continuous time process $X_n$ conditioned on $X_n(t) = x$ will be called $X_n$ under initial $(t, x)$ and the resulting probability on $C[0,\infty)$ will be denoted either $P_{(t,x)}$ or $P_x$, the latter denotation being adequate in view of (3.3). We assume in what follows that there exists a Markov process $X_\infty$ on $C[0,\infty)$ governed by a transition probability $p_\infty(t, x, dy)$ such that

$X_n \xrightarrow{P} X_\infty$

under all initial $(t, x)$ where the indicated convergence is weak convergence — see Billingsley [2] and Whitt [16].

Suppose there exists a closed set $\Gamma_\infty$ in $S$ whose hitting time $T_\infty$ satisfies

$$F_\infty(t, x) = \sup_T \int f(T, X_\infty(T)) dP(t, x) = \int f(T_\infty, X_\infty(T_\infty)) dP(t, x).$$
We define \( T_n \) as the approximate hitting time of \( \Gamma_\infty \) for the discrete time process \( X_n \):

\[
(3.4) \quad T_n = \text{minimal } m\phi(n) \text{ such that the continuous time process } X_n \\
\text{hits } \Gamma_\infty \text{ in } ((m-1)\phi(n), m\phi(n)) .
\]

It is reasonable to expect that \( T_n \) is approximately optimal for discrete time \( X_n \), i.e.,

\[
F_n(t,x) \sim \int F(T_n, X_n(T_n)) dP(t,x) .
\]

We now proceed to lay down assumptions which given this approximation a precise meaning. It is not assumed in this development that \( \lim_{t \to \infty} X_n(t) \) exists.

**Assumptions**

\[(3.5) \quad \text{For all } n \leq \infty \text{ and under initial } (t,x): \]

\[
P(t,x) \left\{ \lim_{s \to \infty} f(s, X_n(s)) = 0 \right\} = 1 .
\]

Thus, we interpret \( f(T_n, X_n(T)) = 0 \) on \( \{ T = \infty \} \).

\[(3.6) \quad f(t,x) \text{ is increasing in } x, \text{ decreasing in } t, \text{ uniformly continuous in } t .
\]

\[(3.7) \quad \text{There exists } G \in C[0,\infty), \text{ non-decreasing in } t \text{ with } \]

\[
\Gamma_\infty = \{(t,x); x \geq G(t)\} .
\]

We interpret \( T_\infty = \infty \) on paths where \( X_\infty(t) < G(t) \), all \( t \).
(3.8) For all \((t,x) \in S\)

\[ P_{(t,x)} \left( \bigcap_{\varepsilon > 0} \bigcup_{0 < s < \varepsilon} \{ X_\infty(T_\infty + s) \cap G(T + s) \} \right) = 1 \]

conditional on \( \{ T_\infty < \infty \} \).

(3.9) For all large \( n \), \( F_\infty \) is excessive for discrete time \( X_n \), i.e.,

\[ F_\infty(m\phi(n),x) \geq \int F_\infty((m+1)\phi(n),y)p_n(\phi(n),x,dy) \]

Theorem (3.1)

Let \( X_n \) be the discrete time Markov process on the time lattice \( \{ m\phi(n) \} \) whose continuous time interpolation converges weakly to \( X_\infty \) under all initial \((t,x)\). Under assumptions (3.5) through (3.8) we have it that for any \( \varepsilon > 0 \), there exists \( N(\varepsilon) \) such that \( n \geq N(\varepsilon) \) implies

\[ 0 \leq F_n(t,x) - \int f(T_n,X_n(T_n))dP_{(t,x)} \leq F_\infty(t,x) - \int f(T_n,X_n(T_n))dP_{(t,x)} \leq \varepsilon \]

Proof:

By Skorokhod's theorem — see Billingsley [2] — there exists a probability triple \( (\Omega, \mathcal{B}, P) \) supporting random elements \( Y_n : \Omega \to \mathcal{C}[0,\infty), n \leq \infty \) such that

\[ Y_n \overset{\mathcal{D}}{=} X_n, \quad n \leq \infty \]

\[ Y_n \to Y_\infty, \quad \text{all } \omega \in \Omega, \]

Since the metric on \( \mathcal{C}[0,\infty) \) is uniform convergence on all closed intervals
[0, t_0], it is easily seen from assumptions (3.7) and (3.8) that if \( V_n \) is the hitting time of \( G(t) \) for the continuous time process \( X_n \) and if \( S_n \) is the corresponding hitting time for \( Y_n \), then

\[
V_n \overset{P}{=} S_n, \text{ all } n \leq \infty
\]

and

\[
S_n \to S_\infty \text{ on } \{S_\infty < \infty\}.
\]

Clearly (3.8) implies (3.12) by making hitting times \( C[0, \infty) \) continuous where they occur. But then, by continuity of \( f \) and \( Y_n \), we have

\[
f(S_n, Y_n(S_n)) \to f(S_\infty, Y_\infty(S_\infty)), \text{ all } \omega \in \Omega.
\]

Here we use (3.5) on \( \{S_\infty = \infty\} \). Now (3.11) demands that

\[
\int f(V_n, X_n(V_n))dP(t, x) = \int f(S_n, Y_n(S_n))dP, \text{ all } n \leq \infty
\]

and then Fatou's lemma applied to the right and interpreted for the left yields (note \( V_\infty = T_\infty \)):

\[
\int f(T_\infty, X_\infty(T_\infty))dP(t, x) \leq \lim_{n \to \infty} \int f(V_n, X_n(V_n))dP(t, x),
\]

Now we want to replace \( V_n \) by \( T_n \) so that we can relate behaviour of discrete time \( X_n \) to continuous time \( X_\infty \). Since \( |T_n - V_n| \leq \phi(n) \), and \( \phi(n) \to 0 \), the uniform continuity of \( f \) in \( t \) demands that for large \( n \)

\[
f(V_n, X_n(V_n)) \leq f(T_n, X_n(V_n)) + \epsilon.
\]

Further, since \( G \) is non-decreasing and \( V_n \) is the hitting time for \( G \), and \( f \) is increasing in \( x \):

\[
f(T_n, X_n(V_n)) \leq f(T_n, X_n(T_n))
\]
Consequently:

\[ \int f(T_\infty, X_\infty(T_\infty)) dP(t, x) \leq \lim_{n \to \infty} \int f(T_n, X_n(T_n)) dP(t, x) . \]

Finally, assumption (3.9) and the properties of least excessive majorants gives \( F_\infty \geq F_n \) so that, using (3.18)

\[ F_\infty(t, x) = \int f(T_\infty, X_\infty(T_\infty)) dP(t, x) \leq \lim_{n \to \infty} \int f(T_n, X_n(T_n)) dP(t, x) \]

\[ \leq \lim_{n \to \infty} F_n(t, x) \leq F_\infty(t, x) . \]

Q.E.D.

A Related Result for \( D[0, \infty) \)

Let \( \{X_n\} \) be a discrete time sequence of integer-valued processes where \( X_n \) has time lattice \( \{m\phi(n)\} \), and where \( X_n \) has transitions governed by \( p_n \) where, for all integer \( x, y \):

\[ P(X_n((m+1)\phi(n)) = y | X_n(m\phi(n)) = x) = p_n(\phi(n), x, y) . \]

Let the continuous time version of \( X_n \) be defined by

\[ X_n(t) = X_n(m\phi(n)) \text{ if } t \in [m\phi(n), (m+1)\phi(n)) . \]

Each \( X_n \) is a pure-jump process with jumps occurring only at time values \( m\phi(n) \). Let \( X_\infty \) be a pure-jump continuous time process which lives on the integers, hence is a random element in \( D[0, \infty) \), and assume that \( X_\infty \) is non-explosive, i.e., \( X_\infty \) has at most finitely many jumps in finite time. That is,

\[ X_\infty(t) = \sum_{m} X_\infty(s_m) I_{[s_m, s_m+1)}(t) \]
where \( \{s_n\} \) is the sequence of jump times for \( X_\infty \) and this sequence satisfies

\[
P \left( \lim_{n \to \infty} s_n = \infty \right) = 1, \text{ all integer } x.
\]

Now suppose \( X_n \overset{D}{\to} X_\infty \) on \( D[0,\infty) \); this type of weak convergence is treated in Whitt [17] and Lindvall [10]. We want then to relate \( F_n \) to \( F_\infty \) where, as usual

\[
F_n(t, x) = \sup_{T \geq t} \int f(T, X_n(T)) dP(t, x), \quad n \leq \infty.
\]

**Assumptions**

(3.22) \( f(t, x) \) is increasing in \( x \), decreasing in \( t \), continuous in \( t \).

(3.23) \( P_{(t, x)} \left( \lim_{s \to \infty} f(s, x_n(s)) = 0 \right) = 1, \text{ all initial } (t, x) \).

(3.24) There exists non-decreasing \( G \in C[0,\infty) \), with

\[
\Gamma_\infty = \{(t, x) : x \geq G(t)\}
\]

and we again interpret \( T_\infty = \infty \) on paths where \( X_\infty(t) < G(t) \), all \( t \).

(3.25) For all large \( n \), \( F_\infty \) is excessive for discrete time \( X_n \).

**Theorem (3.2)**

If \( X_n, X_\infty \) above satisfy \( X_n \overset{D}{\to} X_\infty \) on \( D[0,\infty) \), and if (3.22) through (3.25) hold, then if \( T_n = \text{minimal } m_\phi(n) \) where \( X_n(m_\phi(n)) \geq G(t) \), we have:

\[
F_\infty(t, x) = \lim_{n \to \infty} F_n(t, x) = \lim_{n \to \infty} \int f(T_n, X_n(T_n)) dP(t, x),
\]
Proof:

Just as before, we replace $X_n$ be $Y_n$, $n \leq \infty$ where $X_n \overset{D}{=} Y_n$ and $Y_n \to Y_\infty$ in the $D[0,\infty)$ metric for each path. Since all paths are integer valued, and since $Y_\infty$ executes at most finitely many jumps in finite time, we see that for a particular path $\omega$, $Y_n(\omega) \to Y_\infty(\omega)$ if and only if $Y_n$ eventually makes exactly the same jumps as $Y_\infty$, i.e., $Y_n(s^m_n) = Y_\infty(s^\infty_m)$ where $s^m_n$ is the time of the $m$-th jump and further $s^m_n \to s^\infty_m$. Since $G$ is continuous and since $Y_n$ looks like $Y_\infty$ except for a slight distortion of the time axis, we have $S_n + S_\infty$ on $\{S_\infty < \infty\}$ where $S_n, S_\infty$ are defined as in the last theorem. Also, since $Y_n(S_n) = Y_\infty(S_\infty)$ for large $n$ on $\{S_\infty < \infty\}$, and since $f$ is continuous in $t$, we have, given (3.23), that $f(S_n, Y_n(S_n)) \to f(S_\infty, Y_\infty(S_\infty))$. Continuing the logic and notation of the previous theorem, noting that $T_n = V_n$, we see that

$$\int f(T_\infty, X_\infty(T_\infty)) dP(t,x) \leq \frac{\lim}{n} \int f(T_n, X_n(T_n)) dP(t,x)$$

and that

$$F_\infty \geq F_n$$

so that all conclusions follow.

Q.E.D.
REFERENCES


