A Note on the Campbell Sampling Theorem

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ABSTRACT. Campbell's 1968 sampling theorem is examined and a more explicit formula for the truncation error is given. The result is shown to apply to random processes bandlimited in a general sense.

§1. INTRODUCTION. Many versions of the Shannon sampling series have been proposed; see e.g. [1] for a recent review. In 1968 Campbell [2] proposed the sampling series

$$f(t) = \sum_{n=-\infty}^{\infty} f(nh) \frac{\sin \pi h^{-1}(t-nh)}{\pi h^{-1}(t-nh)} \hat{\psi}(\beta(t-nh)) \quad -\infty < t < \infty$$

valid for $h^{-1} > 2w$, $\beta < h^{-1}/2 - w$ where $\hat{\psi}$ is the Fourier transform of a $C^\infty$ function $\psi$ supported by $[-1,1]$ with $\int \psi dx = 1$ and $f$ is any function whose Fourier transform (in the distributional sense) is supported by $[-w,w]$.

The series (1) is also valid for stationary random processes. Campbell also gives an expression for the truncation error committed in using $2N + 1$ terms of (1) to evaluate $f$. In this note we give a more explicit form of the truncation error and show that the sampling theorem is valid for a general type of bandlimited random process, giving an expression for the truncation error in this case also.

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§2. BANDLIMITED FUNCTIONS. Call a function $f$ bandlimited to $w$ if
\[
\|f\|_{k}^{2} = \int |f(t)|^2 (1+t^2)^{-k} dt < \infty \quad \text{for some } k \geq 0 \quad \text{and if the Fourier transform of } f \text{ is supported by } [-w,w].
\]
The following form of the Paley-Wiener theorem will be useful.

Theorem. For all real $t$, a function $f$ bandlimited to $w$ satisfying
\[
\|f\|_{k} < \infty \quad \text{satisfies}
\]
\[
|f(t)| \leq C_{k}(w)(1+|t|)^{k}\|f\|_{k}
\]
where $C_{k}(w) \leq 4(W+1)^{K_{k}}$ for constants $K_{k}$ defined below.

Proof. Let $\phi$ be any $C^{\infty}$ function with compact support, i.e. a testfunction. Then if $F$ is the Fourier transform of $f$,
\[
|F(\phi)| = \left| \int f(u)\hat{\phi}(u)du \right| \leq \left\{ \int |f(u)|^2 (1+u^2)^{-k}du \cdot \int |\hat{\phi}(u)|^2 (1+u^2)^{k}du \right\}^{\frac{1}{2}}
\]
\[
= \|f\|_{k} \left\{ \int |\hat{\phi}(u)|^2 (1+u^2)^{k}du \right\}^{\frac{1}{2}}
\]
Now the distribution $F$ can be extended to a continuous linear functional on the space $E$ of all $C^{\infty}$ functions topologized by the usual family of semi norms (see e.g. [3] p 88) and if $\chi$ is a testfunction equal to 1 on $[-w,w]$ then for all $\xi \in E \ F(\xi) = F(\xi \chi)$, so in particular, setting
\[
\xi(x) = e^{2\pi i x t}
\]
we obtain
\[
|f(t)| = |F(e^{2\pi i x t})| = |F(e^{2\pi i x t} \chi(x))| \leq \|f\|_{k} \left\{ \int |\hat{\chi}(u-t)|^2 (1+u^2)^{k}du \right\}^{\frac{1}{2}}
\]
Now let $\gamma(x) = \begin{cases} K \exp\left(1/(x^2-1)\right) & |x| \leq 1 \\ 0 & |x| > 1 \end{cases}$
where $K^{-1} = \int \exp\left(1/(x^2-1)\right)dx = 2.2523$; then $\int \gamma(x)dx = 1$
and $\gamma(x)$ is a testfunction supported by $[-1,1]$. Also let for $\delta > 0$
\[ I(x) = \begin{cases} 1 & |x| \leq w + \delta, \\ 0 & |x| > w + \delta. \end{cases} \]

Then choose for \( \chi \) the function \( \chi(x) = \frac{1}{\delta} \int_{-\infty}^{w+\delta} I(v) \gamma(\frac{x-v}{\delta}) dv \)

then \( \chi(x) \) is 1 on \([-w,w]\) and is supported by \((-w-\delta, w+\delta)\), and is \( C^\infty \). Then denoting the Fourier transform of a function \( \lambda \) by

\[ \hat{\lambda}(u) = \int_{-\infty}^{\infty} e^{-2\pi iux} \lambda(x) dx \]

we have

\[ \chi(u) = \gamma(\delta) \frac{\sin 2\pi (w+\delta) u}{\pi u} \]

and so

\[ |f(t)| \leq \|f\|_k \int_{-\infty}^{\infty} \left| \hat{\gamma}(u) \right|^2 \left( \frac{\sin 2\pi (u) (u-t)}{\pi (u-t)} \right)^2 (1+u^2)^k du \]

\[ = \|f\|_k \int_{-\infty}^{\infty} \left| \hat{\gamma}(u) \right|^2 \left( \frac{\sin 2\pi (u) u}{\pi u} \right)^2 (1+(u+t)^2)^k du \]

\[ \leq \|f\|_k (1+|t|)^k \left\{ \int \left| \hat{\gamma}(u) \right|^2 \frac{\sin^2(2\pi (u) u)}{(\pi u)^2} (1+|u|^2)^k du \right\}^{1/2} \]

using the inequality \( (1+(u-t)^2)^k \leq (1+|t|)^2 (1+|u|^2)^k \).

Then (2) is true with \( C_k^2(w) = \inf_{\delta>0} \int \left| \hat{\gamma}(u) \right|^2 \frac{\sin^2(2\pi (w+\delta) u)}{(\pi u)^2} (1+|u|^2)^k du \).

For an upper bound on \( C_k^2(w) \), consider setting \( \delta = 1 \), then since

\[ \left| \frac{\sin x}{x} \right| \leq \frac{2}{1+|x|} \]

\[ C_k^2(w) \leq 16(w+1)^2 \int \left| \hat{\gamma}(u) \right|^2 (1+2\pi (w+1) |u|)^{-2} (1+|u|^2)^k du \]

\[ \leq 16(w+1)^2 \int \left| \hat{\gamma}(u) \right|^2 (1+u^2)^{k-1} du \]

\[ = 16(w+1)^2 \kappa_k^2 \]

say, and hence

(3) \[ |f(t)| \leq 4K_k (w+1) (1+|t|)^k \|f\|_k. \]
The constant $K_k$ can be calculated from

$$K_k^2 = \int |\hat{\gamma}(u)|^2 (1+u^2)^{k-1} du = \sum_{j=0}^{k-1} \binom{k-1}{j} \int |\hat{\gamma}(u)|^2 u^{2j} du$$

$$= \sum_{j=0}^{k-1} \binom{k-1}{j} \int \left| \frac{\gamma^{(j)}(x)}{(2\pi)^{3/2}} \right|^2 dx.$$  

The derivatives of $\gamma$ can be generated by a simple recursive scheme described in [2], a numerical integration then allows the calculation of the $K_k$, given below in Table 1 for $k = 1, 2, 3, 4, 5$.

Table 1. Values of $K_k$ to 4 significant figures.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$K_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$8.217 \times 10^{-1}$</td>
</tr>
<tr>
<td>2</td>
<td>$1.003 \times 10^0$</td>
</tr>
<tr>
<td>3</td>
<td>$1.649 \times 10^0$</td>
</tr>
<tr>
<td>4</td>
<td>$8.446 \times 10^0$</td>
</tr>
<tr>
<td>5</td>
<td>$1.252 \times 10^2$</td>
</tr>
</tbody>
</table>

We note in passing that for $k = 0$, the inequality takes the simple form

$$|f(t)| = \left\{ \int_{-W}^W e^{2\pi i t x} \hat{f}(x) dx \right\} \leq \left\{ \int_{-W}^W dx \int_{-\infty}^{\infty} |\hat{f}(x)|^2 dx \right\}^{1/2}$$

$$= (2W)^{1/2} \| f \|_0$$

so (2) is valid for $k = 0$ with $C_0(w) = (2w)^{1/2}$.

For convenience, and following [2] we propose to take for $\psi$ in the series (1) the function $\gamma$ defined above. A version of (2) appropriate for $\hat{\gamma}$ is obtained simply by integrating the f.t. of $\gamma^{(\nu)} \left\{ \begin{array}{c} d^\nu \gamma \\ \partial x^\nu \end{array} \right\}$ by parts, obtaining
\[(2\pi i u)^v \gamma(u) = \int e^{-2\pi i u x} \gamma^{(v)}(x) dx \leq \int |\gamma^{(v)}(x)| dx\]

and so for \(u \neq 0\) we obtain

\[(4) \quad |\hat{\gamma}(u)| \leq \frac{1}{(2\pi)^v} \int |\gamma^{(v)}(x)| dx |u|^{-v} \]

\[= c_v |u|^{-v}, \text{ say, for any integer } v \geq 0.\]

Again the constants \(c_v\) may be calculated simply by numerical quadrature. An upper bound for the \(c_v\) may be obtained by the methods of [2]. Table 2 below gives the first few values of \(c_v\):

<table>
<thead>
<tr>
<th>(v)</th>
<th>(c_v)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.000</td>
</tr>
<tr>
<td>1</td>
<td>0.2637</td>
</tr>
<tr>
<td>2</td>
<td>0.1822</td>
</tr>
<tr>
<td>3</td>
<td>0.3237</td>
</tr>
<tr>
<td>4</td>
<td>1.5550</td>
</tr>
<tr>
<td>5</td>
<td>13.6874</td>
</tr>
</tbody>
</table>

By means of (3) and (4) we can obtain a precise inequality on the truncation error of (1) in terms of the norm \(\|f\|_k\) of \(f\), using the method of [2]:

**Theorem.** Let \(f\) be a function bandlimited to \(w\) and satisfying \(\|f\|_k < \infty\). Then if \(\gamma\) is the function defined in the proof of theorem 1 and if \(\rho_N(t)\) denotes the truncation error using the function \(\gamma\) in (1) i.e. if

\[\rho_N(t) = f(t) - \sum_{|n| \leq N} f(nh) \frac{\sin \pi h^{-1}(t-nh)}{\pi h^{-1}(t-nh)} \gamma^{(b_t-nh)}(t-nh)\]
then for \( v > k \) \[
\rho_N(t) \leq \frac{2C_k(w)C_v \|f\|_k \sin \pi h^{-1}t (1+Nh)^k}{\beta^v \pi |Nh - |t||^v}
\]

\[
\leq \frac{8(w+1)K_kC_v \|f\|_k \sin \pi h^{-1}t |(1+nh)^k}{\beta^v \pi |Nh - |t||^v}
\]

for \( |t| < Nh \), \( \beta < h^{-1/2} - w \) and \( h^{-1/2} > w \).

Proof. Similar to §4 of [2].

§3. BANDLIMITED PROCESSES. Consider a zero mean second order process \( x(t) \), and let \( R(t,s) = E(x(t)x(s)) \) be the covariance function of \( x(t) \). Such a process will be termed band-limited to \( w \), if for some non negative integer \( k \), \( \int R(t,t)(1+t^2)^{-k}dt < \infty \) and the Fourier transform of \( R(t,s) \) is a distribution in the plane supported by \( [-w,w] \times [-w,w] \).

For the properties of such processes see [4].

A key property is the following. Consider the Hilbert space \( H_k \) consisting of all functions \( f \) satisfying \( \int |f|^2(1+t^2)^{-k}dt \). Then the operator \( R \) defined by

\[
Rf(s) = \int R(t,s)f(t)(1+t^2)^{-k}dt
\]

is a trace-class operator from \( H_k \) to \( H_k \). Let \( \lambda_j, f_j, j = 1,2,3, \cdots \) be the eigenvalues and eigenvectors of this operator. Then the following results are true (details may be found in [4] and [5]):

1. There exist random variables \( e_j \) satisfying \( E(e_j e_j^*) = \delta_{jj} \lambda_j \) such that

\[
x(t) = \sum_{j=1}^{\infty} f_j(t) e_j
\]

the convergence of (5) being in mean square;
2. Each function $f_j$ satisfies $\|f_j\|_k = 1$ and is bandlimited to $w$;

3. For each $t, s$, $R(t,s) = \sum_{j=1}^{\infty} \lambda_j f_j(t) \overline{f_j(s)}$, the series converging absolutely.

Using these we may prove the following

**Theorem 3.** Let $x(t)$ be a zero mean process whose covariance function $R$ satisfies $\int R(t,t)(1+t^2)^{-k}dt < \infty$ and that is bandlimited to $w$.

Then $x(t)$ satisfies the sampling expansion (1) and the root mean square truncation error

$$\left\{ E \left| x(t) - \sum_{|n| \leq N} x(nh) \frac{\sin \pi h^{-1}(t-nh)}{\pi h^{-1}(t-nh)} \hat{\gamma}(p(t-nh)) \right| \right\}^{1/2}$$

is bound by $\left( \int R(t,t)(1+t^2)^{-k}dt \right)^{1/2} \frac{8(w+1)K_v C_v}{\beta^v \pi (Nh - |t|)^v} |\sin \pi h^{-1} t | (1+Nh)^k$ for $k < v, \beta < h^{-1/2} - w, h^{-1/2} > w$, and $|t| < Nh$.

**Proof.** Using (5) above,

$$E \left| x(t) - \sum_{|n| \leq N} x(nh) \frac{\sin \pi h^{-1}(t-nh)}{\pi h^{-1}(t-nh)} \hat{\gamma}(\beta(t-nh)) \right|^2$$

$$\leq E \left| \sum_{j=1}^{\infty} f_j(t) - \sum_{|n| \leq N} f(nh) \frac{\sin \pi h^{-1}(t-nh)}{\pi h^{-1}(t-nh)} \hat{\gamma}(\beta(t-nh)) e_j \right|^2$$

$$= \sum_{j=1}^{\infty} \lambda_j \left| f_j(t) - \sum_{|n| \leq n} f_j(nh) \frac{\sin \pi h^{-1}(t-nh)}{\pi h^{-1}(t-nh)} \hat{\gamma}(\beta(t-nh)) \right|^2$$

$$\leq \sum_{j=1}^{\infty} \frac{8(w+1)K_v C_v}{\beta^v \pi (Nh - |t|)^v} \left| \frac{\sin \pi h^{-1} t | (1+Nh)^k}{\beta^v \pi (Nh - |t|)^v} \right|^2$$
Now consider \( R(t,t) = \sum_{j=1}^{\infty} \lambda_j |f_j(t)|^2 \). Integrating term by term with respect to \((1+t^2)^{-k} \) \( dt \) we obtain

\[
\int R(t,t)(1+t')^{-k} dt = \sum_{j=1}^{\infty} \lambda_j \int |f_j(t)|^2 (1+t^2)^{-k} dt \\
= \sum_{j=1}^{\infty} \lambda_j \|f_j\|^2_k \\
= \sum_{j} \lambda_j j^2 \] 

For \( k < \nu \), we see that the truncation error converges to zero as \( N \to \infty \), thus proving the theorem.

The series (1) converges in mean square for a bandlimited process \( x(t) \); since the sample paths of \( x(t) \) are bandlimited to \( W \) with probability 1, (see [4]); it follows that \( x(t) \) satisfies the series (1) with probability 1.

**REFERENCES**


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