Parameter Estimation in Groundwater Flow Models with Distributed and Pointwise Observations*

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Abstract. We present in this paper some results concerning the least sqaures estimation of parameters in a groundwater flow model. As is typically the case in field studies, the form of the data is pointwise observation of hydraulic head and hydraulic conductivity at a discrete collection of observation well sites. We prove continuous dependence results for the solution of the groundwater flow equation, with respect to conductivity and boundary values, under certain types of numerical approximation.

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1 Introduction

Understanding the flow of groundwater is an important scientific and engineering problem, from many points of view. From resource conservation to municipal planning to contaminant tracking and clean up, models of groundwater flow provide essential decision making tools. One of the more difficult issues in using models in field situations is the calibration of model parameters to the values at a particular site. In this paper, we address analytical and numerical problems associated with estimating parameters in a groundwater model.

The basic model of steady state flow of groundwater in saturated soil is the elliptic equation

$$\nabla \cdot (k \nabla h) = 0 \qquad x \in \Omega, \tag{1}$$

where k is the hydraulic conductivity, h is the hydraulic head, and Ω is a subsurface region (in $\mathbb{R}^2$ or $\mathbb{R}^3$) of interest. The equation (1) is based on Darcy's empirical law, which states that groundwater velocity is proportional to the hydraulic gradient

$$v = -k \nabla h, \tag{2}$$

which, when combined with incompressibility and conservation of mass arguments, leads to (1). The references [FC, Chapter 2] and [Be, Chapter 7] provide a detailed discussion of the Darcy model.

In order to determine the groundwater flow velocity, one must solve (1), for which we need to know the function k and the boundary values of h. The inverse problem of interest here is the determination of the function k and $g = h|_{\partial\Omega}$ from pointwise or distributed observations. In general one expects k to be discontinuous (due, e.g., to layering in the subsurface), and it is not known a priori where these discontinuities lie. Thus, in the inverse problem we must search for k's from a very general class of functions, and in the forward problem we must consider weak solutions.

We assume that we are given data points $\{\hat{h}_i\}_{i=1}^{n_1}$ and $\{\hat{k}_j\}_{j=1}^{n_2}$, which correspond to observations of h at the points x_i and of $(1/|\omega_j|)\int_{\omega_j} k(x)\,dx$ in which ω_j denotes a "small" set in Ω and $|A|$ denotes the Lebesgue measure of A (the averaging nature of this observation operator is based on the borehole flowmeter measurement device: see [RBG])

To determine parameters based on these data, we define the least squares cost functional

$$J(k,g) = \sum_{i=1}^{n_1} |h(x_i;k,g) - \hat{h}_i|^2 + \sum_{j=1}^{n_2} \Big| \frac{1}{|\omega_j|} \int_{\omega_j} k(x)\,dx - \hat{k}_j \Big|^2 \tag{3}$$

which is to be minimized over some appropriate collection $\mathcal{K} \times \mathcal{G}$ of functions k and g. In general, there are two main obstacles to implementation of this approach: the function h mush be computed numerically and the set $\mathcal{K} \times \mathcal{G}$ must be approximated by a finite dimensional set. Thus, for computational purposes, we minimize

$$J^N(k,g) = \sum_{i=1}^{n_1} |h^N(x_i;k,g) - \hat{h}_i|^2 + \sum_{j=1}^{n_2} \Big| \frac{1}{|\omega_j|} \int_{\omega_j} k(x)\,dx - \hat{k}_j \Big|^2 \tag{4}$$

over a set $\mathcal{K}^M \times \mathcal{G}^M$. The abstract least squares theory of Banks (see, e.g., [BK, pp. 143ff.]) lays out general conditions that must be satisfied in order to guarantee convergence of minimizers of of J^N to minimizers of J. The continuous dependence and convergence results we present here allow the application of these abstract least squares methods.

We note here that the pointwise nature of the observation operator introduces some difficulties due to the fact that $H^1(\Omega)$ is not continuously imbedded in $C(\bar{\Omega})$ in dimensions higher than 1. We shall use a form of Harnack's inequality to obtain the appropriate continuous dependence. Typically, one either assumes a distributed measurement $\hat{h}(\cdot)$ in which one observes the hydraulic head (see, e.g. [G]), or one averages the hydraulic head over a small region, which changes the observation operator to $\frac{1}{|\omega_j|}\int_{\omega_j} h(x; g, k)\, dx$. Either case requires only L^2 continuity of the solution h with respect to the parameters. Another possibility is to use parameters k having much more regularity than L^∞ (see [KW] or [BK, Chapter VI] and the references therein). Here, we derive pointwise continuity of h with no more than the usual coercivity and boundedness requirements for wellposedness. For approximations, we use additional regularity on k, and we relax the regularity as the mesh becomes finer.

The paper is organized as follows. In Section 2, we briefly review wellposedness issues and a prioi estimates for solutions of elliptic boundary value problems. Section 3 contains the main continuous dependence and convergence of approximation results, and we conclude in Section 4 with some remarks on implementation and future studies.

2 Properties of the Elliptic Boundary Value Problem

In this section, we recall some important results from analysis and approximation in elliptic partial differential equations such as (1). We denote by $H^k = H^k(\Omega)$

and $H_0^k = H_0^k(\Omega)$ the usual Sobolev spaces of weakly differentiable functions. Throughout, we assume that Ω is a bounded open subset of $\mathbb{R}^m$. The notation $f|_{\partial\Omega}$ is used in general to denote the trace of an H^1 function. We also use the notation $H^s(\partial\Omega)$ to denote the fractional order (i.e., $s > 0$ real) trace spaces of Lions (see, e.g., [W, Chapter I.3]).

To solve the differential equation (1) subject to the boundary condition $h|_{\partial\Omega} = g$, we seek a function $h \in H^1$ of the form $h = u + G$, where $G \in H^1$ satisfies $G|_{\partial\Omega} = g$ and where $u \in H_0^1$ satisfies

$$\nabla \cdot \Big(k\nabla u - k\nabla G \Big) = 0 \tag{5}$$

in Ω. It is well known (see, e.g., [W, p. 129]) that there is a bounded linear extension operator $Z\colon H^{1/2}(\partial\Omega) \to H^1$ such that the trace of Zf is f, for each $f \in H^{1/2}(\partial\Omega)$. The determination of h is then based on solving (5) for u. Mulitplying in (5) by a test function ϕ and integrating by parts, we have

$$\sigma_k(u, \phi) = F_{g,k}(\phi), \tag{6}$$

in which the bilinear form $\sigma_k\colon H_0^1 \times H_0^1 \to \mathbb{R}$ is given by

$$\sigma_k(\phi, \psi) = \int_\Omega k(x)\nabla\phi(x) \cdot \nabla\psi(x)\, dx, \tag{7}$$

and the bounded linear functional $F_{g,k}\colon H_0^1 \to \mathbb{R}$ is given by $F_{g,k}(\phi) = \int_\Omega k(x)\nabla G(x) \cdot \nabla\phi(x)\, dx$, in which $G = Z(g) \in H^1$. We summarize the application of the Lax-Milgram theorem and the weak maximum principle in the following theorem (see [T,p. 267 and W, p. 271] for details of the proof).

Theorem 2.1 *If the domain Ω is a bounded open set with Lipschitz continuous boundary $\partial\Omega$, if $g \in H^{1/2}(\partial\Omega)$ and $k \in L^\infty(\Omega)$, and if there exist positive numbers α, β such that $\alpha \le k(x) \le \beta$ a.e., then the equation (6) has a unique solution $u \in H_0^1$, and the equation (1) has a unique solution $h \in H^1$ satisfying $h|_{\partial\Omega} = g$.*

We remark that in the proof of this theorem, which is essentially contained in the references [T, W], the coercivity and boundedness of σ are crucial: σ satisfies

$$\alpha\|\phi\|^2_{H_0^1} \le \sigma_k(\phi, \phi),$$

$$|\sigma_k(\phi, \psi)| \le \beta\|\phi\|_{H_0^1}\,\|\psi\|_{H_0^1}$$

in which $\|\phi\|^2_{H_0^1} = \int_\Omega |\nabla\phi(x)|^2\,dx$ is the norm on $H_0^1(\Omega)$.

Two other crucial estimates involve the sup norm and higher Sobolev norms of the solution. Below we state these results, which may be found in their full generality in [GT, Chapter 8]. First, we need some notation. If Ω_0 and Ω_1 are open sets, we write $\Omega_0 \subset\subset \Omega_1$ if the closure $\bar{\Omega}_0$ is a compact subset of Ω_1.

Theorem 2.2 *(Harnack's inequality) If Ω is a bounded open set and if $\omega \subset\subset \Omega$, then there exist constants $C, \lambda > 0$ such that*

$$\|u\|_{C^\lambda(\bar{\omega})} \le C\|u\|_{L^2(\Omega)} \tag{8}$$

for each pair $u \in H^1, k \in L^\infty(\Omega)$ such that $0 < \alpha \le k(x) \le \beta$, such that $\nabla\cdot(k\nabla u) = 0$, where C^λ denotes as usual the space of Hölder continuous functions of exponent λ.

As noted in [GT], the particular choice of the constants C and λ depends only on β/α and $\mathrm{dist}(\partial\Omega, \omega)$, and not on the particular coefficient function $k \in L^\infty$

or solution $u \in H^1$. In order to establish the desired convergence with respect to approximations, we shall make use of additional regularity for smooth coefficients.

Theorem 2.3 *Suppose that* Ω *is a bounded open set,* $k \in C^{\ell,1}(\bar{\Omega})$*, such that* $0 < \alpha \leq k(x) \leq \beta$*, and* $h \in H^1(\Omega)$ *satisfies (1). Then* $h \in H^{2+\ell}(\Omega_1)$*, for each open set* $\Omega_1 \subset\subset \Omega$*. Furthermore, there exists a constant* M_ℓ *such that*

$$\|h\|_{H^{2+\ell}(\Omega_1)} \leq M_\ell \|h\|_{H^1(\Omega)},$$

in which the constant M_ℓ *depends only on* $\alpha, \beta, \|k\|_{C^{\ell,1}}, \Omega_1$ *and* Ω*, and not on* h*.*

This result can be found, for example, in [GT, Chapter 8]. Note the additional regularity required for k. Since both of the above results are "interior" results, the constants involved do not depend on the boundary values g.

Our goal in this paper is to demonstrate continuous dependence and convergence of approximation results relevant to the identification of k and g from data. In the next section we analyze solutions of (1) as functions of the coefficient k and the boundary data g, and we also analyze the effect of approximation.

3 Parameter Identification and Approximation

To begin the parameter estimation problem, we define sets of parameters over which we will minimize J and J^N. We let

$$\mathcal{K} = \{\phi \in L^\infty(\Omega) : 0 < \alpha \leq \phi(x) \leq \beta, a.e. x \in \Omega, TV(\phi) < \rho\}$$

where the total variation $TV(\phi)$ is defined as

$$TV(\phi) = \sup\Big\{\int_\Omega \phi(x)(\nabla \cdot F)(x)\, dx \colon F \in C^1(\bar{\Omega}; \mathbb{R}^m), \sup_{x\in\Omega} |F(x)| \leq 1\Big\},$$

where $\Omega \subset \mathbb{R}^m$ with $m \leq 3$. In [G] it is proved that $\mathcal{K}$ is a compact subset of $L^1(\Omega)$, when Ω is a bounded open set in $\mathbb{R}^m$. For the boundary conditions, we fix a positive constant Δ, and we set

$$\mathcal{G} = \{\phi \in H^{\frac{1}{2}}(\partial\Omega) : \phi \in H^1(\partial\Omega), \|\phi\|^2_{H^1(\partial\Omega)} \leq \Delta\},$$

which is compact in $H^{\frac{1}{2}}(\partial\Omega)$ if the boundary $\partial\Omega$ is Lipschitz continuous (see, e.g., [W, p. 120]).

The wellposedness theorem of the previous section guarantees that for each $k \in \mathcal{K}$ and $g \in \mathcal{G}$ there exists a unique weak solution to the partial differential equation. The following will show that the weak solution h depends continuously on k and g.

Lemma 3.1 *Let $(g_n) \in \mathcal{G}$ be a sequence of functions such that $g_n \to g$ in $\mathcal{G}$. Let $(k_n) \in \mathcal{K}$ be a sequence of functions such that $k_n \to k \in \mathcal{K}$ in the L^1-norm. For each n, define $u_n \in H^1_0(\Omega)$ to be the Lax-Milgram solution of (6) using k_n and G_n, where $G_n = Z(g_n) \in H^1(\Omega)$ is the extension of g_n. Then, $u_n \to u$ in $H^1(\Omega)$ where u is the Lax-Milgram solution using k and G, and $G = Z(g) \in H^1(\Omega)$ is the extension of g.*

Proof. Let (g_n), (k_n), (u_n), and (G_n) be the sequences described above. Using the coercivity and boundedness of σ (the constants for which are independent of $k \in \mathcal{K}$, we have that

$$\begin{aligned} \alpha\|u_n - u\|^2_{H^1_0} &\leq \sigma_{k_n}(u_n - u, u_n - u) \\ &= \sigma_{k_n}(u_n, u_n - u) - \sigma_{k_n}(u, u_n - u) \end{aligned}$$

$$
\begin{aligned}
&= \sigma_{k_n}(u_n, u_n - u) - \sigma_k(u, u_n - u) + \sigma_k(u, u_n - u) - \sigma_{k_n}(u, u_n - u) \\
&= F_{k_n,g_n}(u_n - u) - F_{k,g}(u_n - u) + \sigma_k(u, u_n - u) - \sigma_{k_n}(u, u_n - u) \\
&= \int_\Omega k\nabla G \cdot \nabla(u_n - u) - \int_\Omega k_n \nabla G_n \cdot \nabla(u_n - u) \\
&\quad + \int_\Omega k\nabla u \cdot \nabla(u_n - u) - \int_\Omega k_n \nabla u \cdot \nabla(u_n - u) \\
&= \int_\Omega (k - k_n)\nabla G \cdot \nabla(u_n - u) + \int_\Omega k_n \nabla(G - G_n) \cdot \nabla(u_n - u) \\
&\quad + \int_\Omega (k - k_n)\nabla u \cdot \nabla(u_n - u) \\
&\leq \Big((\int_\Omega \mid k - k_n \mid^2 \mid \nabla G \mid^2 \ dx)^{\frac{1}{2}} + \beta \|G - G_n\|_{H^1(\Omega)} \\
&\quad + (\int_\Omega \mid k - k_n \mid^2 \mid \nabla u \mid^2 \ dx)^{\frac{1}{2}} \Big) \|u_n - u\|_{H_0^1(\Omega)}
\end{aligned}
$$

Thus, in order to argue convergence of u_n, then, we must show that the three terms above, namely,

- $\int_\Omega \mid k - k_n \mid^2 \mid \nabla G \mid^2$
- $\int_\Omega \mid k - k_n \mid^2 \mid \nabla u \mid^2$
- $\|G - G_n\|_{H^1(\Omega)}$

tend to 0 as $n \to \infty$. That the last item tends to 0 is a direct result of the continuity of the extension operator $Z\colon H^{\frac{1}{2}} \to H^1$. Since the sequence of functions $|k_n - k|$ is uniformly bounded by 2β and goes to 0 in measure, we may apply the dominated convergence theorem to obtain that the first two items tend to 0 (a similar argument is made in [G]).

Based on this result, it is then a simple task to obtain convergence of the functions $h_n = u_n + G_n$ to $h = u + G$, since the H^1 and H_0^1 norms are equivalent on H_0^1.

Theorem 3.1 *Under the same assumptions as the above theorem, the sequence of weak solutions* (h_n) *(where* $h_n = u_n + G_n$ *for each* n*) converges to* h *(where* $h = u + G$*) in the* H^1*-norm.*

Our next task is to convert this convergence to pointwise convergence through the use of Harnack's inequality. The following theorem provides the pointwise convergence.

Theorem 3.2 *Under the same assumptions as the previous theorem, we have that for each open set* ω_0 *with* $\omega_0 \subset\subset \Omega$*,* $h_n(\,\cdot\,; g_n, k_n) \to h(\,\cdot\,; g, k)$ *uniformly on* ω_0*.*

Proof. Let ω_0 be an open set with $\omega_0 \subset\subset \Omega$. Choose ω open so that $\omega_0 \subset\subset \omega \subset\subset \Omega$. From Harnack's inequality (8), as stated above, there exist constants $C, \lambda > 0$ such that

$$\|u\|_{C^\lambda(\bar{\omega})} \leq C\|u\|_{L^2(\Omega)} \tag{9}$$

for each pair $u \in H^1, k \in \mathcal{K}$ such that $\nabla \cdot (k\nabla u) = 0$. We again remark that the particular choice of the constants C and λ depends only on β/α and $\mathrm{dist}(\partial\Omega, \omega)$.

Let B be a bound for the $L^2(\Omega)$ norms of the functions h_n, and let $\epsilon > 0$. Given $x_0 \in \bar{\omega}_0$, we choose an open set $\omega(x_0)$ containing x_0 such that $\bar{\omega}(x_0) \subset \Omega$ and such that $|x - x_0|^\lambda CB < \frac{\epsilon}{3}$ for every $x \in \omega(x_0)$. Since $\bar{\omega}_0$ is compact, we may choose a finite subcollection, say $\omega(x_1), \ldots, \omega(x_\ell)$ which covers $\bar{\omega}(x_0)$. Next, from the L^2 convergence, we choose N such that $n \geq N$ implies that

$$\frac{1}{|\omega(x_i)|}\int_{\omega(x_i)} |h_n(x) - h(x)|dx < \frac{\epsilon}{3},$$

for each i, where again $|A|$ denotes the measure of A. We further note that

$$\frac{|h(x)-h(x_0)|}{|x-x_0|^\lambda} \leq \|h\|_{C^\lambda(\overline{\omega})} \leq C\|h\|_{L^2(\Omega)}$$

and

$$\frac{|h_n(x)-h_n(x_0)|}{|x-x_0|^\lambda} \leq \|h_n\|_{C^\lambda(\overline{\omega})} \leq C\|h_n\|_{L^2(\Omega)}$$

for every $x \in \omega$ and $n \geq 1$, in which we are using the fact that C and λ depend on α, β, ω, and Ω, and not on the solutions and parameter values. Finally, we have that, if $x_0 \in \omega$, and if $n \geq N$, then there is a set $\omega_i = \omega(x_i)$ such that $x_0 \in \omega_i$, and we obtain

$$\begin{aligned}
|h_n(x_0)-h(x_0)| &\leq |h(x_0) - \frac{1}{|\omega_i|}\int_\omega h(x)dx| + |\frac{1}{\mu(\omega)}\int_{\omega_i} h(x)dx - \frac{1}{|\omega_i|}\int_{\omega_i} h_n(x)dx| \\
&\quad + |\frac{1}{|\omega_i|)}\int_{\omega_i} h_n(x)dx - h_n(x_0)| \\
&\leq \frac{1}{|\omega_i}\int_{\omega_i} |h(x_0)-h(x)|dx + \frac{1}{|\omega_i|}\int_{\omega_i} |h(x)-h_n(x)|dx \\
&\quad + \frac{1}{|\omega_i|}\int_{\omega_i} |h_n(x_0)-h_n(x)|dx \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon
\end{aligned}$$

which gives us the uniform convergence as desired.

The results above combine to guarantee that J attains a minimum over the set $\mathcal{K} \times \mathcal{G}$. In order to compute minimizers, we need a sequence $\mathcal{K}^M \times \mathcal{G}^M$ of finite dimensional approximating parameter sets and a sequence of finite dimensional approximations to the differential equation. For the differential equation, we use the standard Galerkin approximation technique. Let $V^N \subset H_0^1$ be a sequence of approximating spaces such that the following holds:

(Appr1) For each $\phi \in H_0^1$, there exist $\phi^N \in V^N$ such that $\|\phi - \phi^N\|_{H_0^1} \to 0$ as $N \to \infty$.

Many of the usual finite element schemes satisfy this assumption (see, e.g., [BK, J]). One then seeks $u^N \in V^H$ such that (6) holds for all $\phi \in V^N$. The following theorem gives us the necessary convergence for parameter estimation.

Lemma 3.2 *Suppose that $k_N \to k$ in $\mathcal{K}$, $g_N \to g$ in $\mathcal{G}$, and that u^N satisfies $\sigma_{k_N}(u^N, \phi) = F_{k_N,g_N}(\phi)$, for all $\phi \in V^N$, where V satisfies the property* **(Appr1)** *above. Then $u^N \to u$ in H_0^1, where u satisfies the weak equation (6) with parameters k and g.*

Proof. The proof is very similar to the convergence proof above. We begin by choosing a sequence $\hat{u}^N \in V^N$ satisfying $\|\hat{u}^N - u\|_{H_0^1} \to 0$. It is then sufficient to show that $\|\hat{u}^N - u^N\|_{H_0^1} \to 0$. Toward that end, we note that

$$
\begin{aligned}
\alpha\|u^N - \hat{u}^N\|_{H_0^1}^2 &\leq \sigma_{k_N}(u^N - \hat{u}^N, u^N - \hat{u}^N) \\
&= F_{k_N,g_N}(u^N - \hat{u}^N) - \sigma_{k_N}(\hat{u}^N, u^N - \hat{u}^N) \\
&= F_{k_N,g_N}(u^N - \hat{u}^N) - \sigma_k(u, u^N - \hat{u}^N) \\
&\quad + \sigma_k(u, u^N - \hat{u}^N) - \sigma_{k_n}(u, u^N - \hat{u}^N) + \sigma_{k_n}(u - \hat{u}^N, u^N - \hat{u}^N) \\
&\leq \int_\Omega |k_N \nabla G_N - k\nabla G| \cdot |\nabla(u^N - \hat{u}^N)|\, dx \\
&\quad + \int_\Omega |k_N - k| \cdot |\nabla u| \cdot |\nabla(u^N - \hat{u}^N)|\, dx \\
&\quad + \beta\|u\|_{H_0^1} \|u^N - \hat{u}^N\|_{H_0^1},
\end{aligned}
$$

which as in the previous proof gives us the desired convergence.

We remark here that this theorem provides a convergence results for cost functionals involving measurements of average h values. That is, if we set

$$\tilde{J}(k,g) = \sum_{i=1}^{n}\Big|\frac{1}{|\tilde{\omega}_i|}\int_{\tilde{\omega}_i} h(x;k,g)\,dx - \hat{h}_i\Big|^2 + \sum_{j=1}^{m}\Big|\frac{1}{|\omega_j|}\int_{\omega_j} k(x)\,dx - \hat{k}_j\Big|^2$$

and we define $\tilde{J}^N$ is an analogous manner, we have that $\tilde{J}^N(k_N,g_N) \to \tilde{J}(k,g)$ as $N \to \infty$, whenever $k_N \to k$ in $\mathcal{K}$ and $g_N \to g$ in $\mathcal{G}$. If $\mathcal{K}^N \subset \mathcal{K}$ and $\mathcal{G}^N \subset \mathcal{G}$ are compact and are chosen in such a way that for each pair $(k,g) \in \mathcal{K} \times \mathcal{G}$ there is a sequence $(k_N,g_N) \in \mathcal{K}^N \times \mathcal{G}^N$ which converges to (k,g), then minimizers of $\tilde{J}^N$ over $\mathcal{K}^M \times \mathcal{G}^M$ converge subsequentially to minimizers of $\tilde{J}$ over $\mathcal{K} \times \mathcal{G}$ (see [BK, pp. 143ff]). We summarize these ideas in the following theorem.

Theorem 3.3 *Suppose that the hypotheses of the previous theorem are satisfied, and that $\mathcal{K}^M \subset \mathcal{K}$ and $\mathcal{G}^M \subset \mathcal{G}$ are compact. Further, we assume that for each pair $(k,g) \in \mathcal{K} \times \mathcal{G}$, there exists a sequence of pairs $(k^M,g^M) \in \mathcal{K}^M \times \mathcal{G}^M$ such that $(k^M,g^M) \to (k,g)$. Then any sequence of minimizers (k_N^M,g_N^M) of $\tilde{J}^N$ over $\mathcal{K}^M \times \mathcal{G}^M$ has a convergent subsequence whose limit is a minimizer of $\tilde{J}$ over $\mathcal{K} \times \mathcal{G}$.*

Proof. The proof of this result is a direct application of the abstract least squares theory of Banks (see, e.g., [BK, pp. 142ff]), based on the convergence of $\tilde{J}^N(k^N,g^N)$ to $\tilde{J}(k,g)$. We remark here that the approximating parameter spaces are easily constructed. For example, one may take $\mathcal{G}^M$ to be continuous piecewise linear finite elements on a grid of M nodes (as long as $\mathcal{G}^M \subset H^1(\partial\Omega)$) which is H^1 bounded by Δ (which is the bound for $\mathcal{G}$). The spaces $\mathcal{K}^M$ can be piecewise constant functions on a grid of M nodes that satisfy the compactness constraints of $\mathcal{K}$.

In order to extend these results to the pointwise evaluation cost functionals, we need some uniform convergence of the the approximate solutions, the uniformity being with respect to the spatial variable as well as the parameters. We shall need the following estimate from finite element analysis, whose general form and proof may be found in [BNS].

Theorem 3.4 *Suppose that Ω is the m-fold Cartesian product of bounded open intervals (a_i, b_i), and that $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega$. We define V^N to be the m-fold tensor product of linear B-splines with nodes $x_k^i = a_i + k(b_i - a_i)/N$, $1 \le i \le m, 0 \le k \le N$. If $h \in H^3(\Omega_1)$ and $h^N \in V^N$ such that $\sigma_k(h - h^N, \phi) = 0$, for each $\phi \in V^N$, then there exist N_0 and M^* depending on Ω, Ω_0 and $\|k\|_{C^{1,1}}$ such that*

$$\sup_{\Omega_0} |h(x) - h^N(x)| \le \frac{M^*}{N} \|h\|_{H^3(\Omega_1)},$$

where $N \ge N_0$.

We remark here that a careful examination of the constants M_ℓ and M^* reveals that the only dependence on k is through the Hölder norms mentioned in the statements above. Moreover, since these estimates are interior estimates, there is no dependence on $g \in H^{\frac{1}{2}}$.

We have stated the estimate of [BNS] in a very limited form, but which is sufficient for our purposes here. The following theorem gives us the convergence result of interest for pointwise observations.

Theorem 3.5 *Suppose that Ω, Ω_0, and V^N are as in the previous theorem. Suppose that $\mathcal{K}^N \subset \mathcal{K}$ is compact and that there exist constants $B_N \to \infty$ such that*

$\|k\|_{C^{1,1}(\bar{\Omega})} \leq B_N$. Moreover, we assume that B_N is chosen so that the product $M^*(B_N)M_1(B_N)/N \to 0$ as $N \to \infty$. Let $k^N \in \mathcal{K}^N, g^N \in \mathcal{G}$ such that $k^N \to k \in \mathcal{K}$ and $g^N \to g \in \mathcal{G}$. that u^N satisfies $\sigma_{k_N}(u^N, \phi) = F_{k_N, g_N}(\phi)$, for all $\phi \in V^N$. Set $h^N = u^N + G^N$ and $h = u + G$, where u solves (6) with parameters k and g. Then $h^N \to h$ uniformly on Ω_0.

Proof. Let $\tilde{u}^N$ be the solution of (6) with parameters k^N and g^N, and let $\omega \subset\subset \Omega$. Then by Theorem 3.2 we have that $\tilde{h}^N \to h$ uniformly on ω. Our next step is to relate $\tilde{h}^N$ and h^N. Toward that end, we note that $\tilde{h}^N - h^N$ satisfies the equation

$$\sigma_{k^N}(\tilde{h}^N - h^N, \phi) = 0, \qquad \forall \phi \in V^N.$$

Thus, we may appeal to the previous theorem to obtain

$$sup_\omega |\tilde{h}^N(x) - h^N(x)| \leq M^* \delta_N \|\tilde{h}^N\|_{H^3(\Omega)} \leq M^* M_1 \delta_N \|\tilde{h}^N\|_{L^2(\Omega)}.$$

By our previous results, we have that $\|\tilde{h}^N\|_{L^2(\Omega)}$ is bounded uniformly in N, so by our choice of bounding constants B_N we have the desired uniform convergence.

4 Conclusion

We have developed here a convergence theory for parameter estimation in groundwater flow problems. The problem has been posed in sufficient generality to allow the identification of discontinuous hydraulic conductivities. Moreover, by using approximating parameter sets which contain sufficiently smooth conductivities, we obtain uniform convergence of the approximations, so that we may implement pointwise observation operators.

We are in the process of developing computational algorithms for implementation of these ideas. Preliminary computations are very encouraging, and we are pursuing the use of field data such as that obtained at the MADE (MAcroDispersion Experiment) site at Columbus Air Force Base, Mississippi [BBLMMAS]. An extensive computational study is in preparation, and the results will be reported elsewhere.

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