SINGULARITIES IN THE DISTRIBUTION OF THE INCREMENTS OF A SMOOTH FUNCTION

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INSTITUTE OF STATISTICS MIMEO SERIES #1109
March 1977

DEPARTMENT OF STATISTICS
Chapel Hill, North Carolina
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§1. By the "distribution of the increments" of a Borel function \( F : [0,1] \to \mathbb{R} \), I mean the measure

\[
\lambda(B) = \int_0^1 \int_0^1 1_B(F(s)-F(t))dsdt ,
\]

where \( B \) is a Borel set in \( \mathbb{R} \). \( \lambda \) is the convolution of the "occupation measure" \( \mu(B) = m(F^{-1}(B)) \) with \( \mu(-B) \); here \( m \) is the Lebesgue measure. When \( \mu \ll m \), write \( \alpha(x) \) for the Radon-Nikodym derivative \( \frac{d\mu}{dm}(x) \) (the "local time" of \( F \) at \( x \)). Of course \( \mu \ll m \) implies \( \lambda \ll m \) and

\[
\Lambda(x) \equiv \frac{d\lambda}{dm}(x) = \int_{-\infty}^{\infty} \alpha(y)\alpha(x+y)dy .
\]

Although this paper treats only smooth \( F \)'s (at least \( C^1 \)), the relevant background consists of two general results from [3]. Throughout, \( \psi \) will denote a nonnegative,

This work was partially supported by National Science Foundation grant MCS 76-06599 and by the Office of Naval Research Contract N00014-75-C-0809.
Borel measurable function. Define

$$I(\psi;F) = \int \int_0^1 \psi(F(s) - F(t)) ds dt = \int_0^1 \psi d\lambda \leq \infty.$$ 

Then (a) if \( \psi \) is even, decreasing on \((0,\infty)\), and nonintegrable on \((0,1)\), then \( I(\psi;F) = \infty \) for any \( F \); (b) \( \mu \ll m \) with \( \alpha \in L^2 \) if and only if \( I(\psi;F) < \infty \) \( \forall \psi \in L^1 \).

Now for \( F \) differentiable a.e., \( \mu \ll m \) if and only if \( D_0 = \{ t : F'(t) = 0 \} \) has Lebesgue measure 0. Suppose \( F \in C^1 \) (i.e. has a continuous derivative, with the usual conventions about the endpoints) and \( D_0 \neq \emptyset \), \( m(D_0) = 0 \). Then, as the Theorem states, \( \lim_{x \to 0} \Lambda(x) = \infty \). (The additional assumptions made on \( F \) below are not needed for this.) Hence \( I(\psi;F) = \infty \) for some \( \psi \in L^1 \) (and so \( \alpha \in L^2 \)) because \( \lambda \ll m \) implies

(2) \[ I(\psi;F) = \int \Lambda(x) \psi(x) dx. \]

So, the question arises: for which \( \psi \)'s - in particular, which monotone ones - is \( I(\psi;F) < \infty \)? This depends on the nature of the singular points of \( \Lambda \).

Assume now \( D_0 \neq \emptyset \), \( F \in C^2 \) and that \( F''(t) \neq 0 \) for all \( t \in D_0 \). Then \( D_0 \) is finite, say \( D_0 = \{ a_i \}_{i=1}^N \), \( 0 < a_1 < a_2 < \ldots < a_N \leq 1 \). Let \( A_i = F(a_i) \) and let \( \{ B_i \}_{i=1}^L \) denote the (distinct) elements of \( \{ A_i - A_j \} \) for which there exist \( t_1, t_2 \in D_0 \) with \( F''(t_1) F''(t_2) > 0 \) and \( F(t_1) - F(t_2) = A_i - A_j \). \( \{ B_i \} \) is symmetric about 0 and contains 0. For the version of \( \Lambda(x) \) given by (1):

**THEOREM.** \( \Lambda(x) \) is continuous on \( \mathbb{R} \setminus \{ B_i \} \) and

(3a) \[ 0 < \lim_{x \to B_i} \frac{\Lambda(x)}{-\log|x - B_i|} \leq \lim_{x \to B_i} \frac{\Lambda(x)}{-\log|x - B_i|} < \infty \quad 1 \leq i \leq L. \]

Consequently, for \( \psi \in L^1 \)

(3b) \[ I(\psi;F) < \infty \iff \psi \in L^1 \left\{ \sum_{i=1}^L |\log|x - B_i|| dx \right\}. \]

In particular, if \( \psi \) is even and decreasing on \((0,\infty)\), then
\[ I(\psi;F) < \infty \iff \int_0^1 \psi(x) \log \frac{1}{x} \, dx < \infty. \]

§2. The fact that the singularities of \( \Lambda \) occur among the points \( \{A_1 - A_j\} \) is fairly obvious. Indeed for \( F \) as above ([4])

\[ \alpha(x) = \sum_{\sigma \in F^{-1}(\{x\})} |F'(s)|^{-1}. \]

(Since \( F(D_0) \) has measure zero, it doesn't matter how \( \alpha \) is defined there.) Clearly \( \alpha \) is well-behaved off \( \{A_i\} \), and, in turn, \( \Lambda \) off \( \{A_1 - A_j\} \). (Actually, (4) is valid for any \( F \) such that \( F' \) exists a.e., although "\( \sigma \in F^{-1}(\{x\}) \)" must be replaced by "\( \sigma \in F^{-1}(\{x\}) \cap \{F' \text{ exists, finite}\} \)" and neither \( F(\{ |F'| = \infty \}) \) nor \( F(\{F' \text{ doesn't exist}\}) \) need have measure 0.)

The Co-Area Theorem [2], applied to the Lipschitz function \( s,t \mapsto F(s) - F(t) \), leads to this expression for \( \Lambda \):

\[ \Lambda(x) = \int \int_{U_x} \left[ (F'(s))^2 + (F'(t))^2 \right]^{-1/2} H(dsdt); \]

here \( U_x = \{(s,t) : F(s) - F(t) = x\} \) and \( H \) is one-dimensional Hausdorff measure in \( \mathbb{R}^2 \). This shows clearly where \( \Lambda \) might explode. Nonetheless, I will not refer again to (5), but instead work with the version of \( \Lambda \) given by (1) with \( \alpha \) as in (4).

That the singularities of \( \Lambda \) are logarithmic is perhaps not as evident, and emerged in a curious way. To get an idea of when \( I(\psi;F) \) is finite, \( \psi \in L^1 \), choose a convenient random function \( X(t,\omega) \), \( 0 \leq t \leq 1 \), \( \omega \in \Omega \), with smooth trajectories and compute the expected value \( E[I(\psi;X(\cdot,\omega)) \] of the random variable \( \omega \mapsto I(\psi;X(\cdot,\omega)) \). For instance, let \( X(t,\omega) \) be Gaussian, mean 0, \( \sigma^2(s,t) = E[(X(s) - X(t))^2] \). Then
\[ E[I(\psi;X(\cdot,\omega))] = \int_0^1 \int_0^\infty \psi(x) [2\pi \sigma(s,t)]^{-1} x^2 \exp\left(-\frac{x^2}{2\sigma^2(s,t)}\right) \, dx \, ds \, dt. \]

For simplicity, and to insure the differentiability of the sample functions, suppose there are constants \(0 < C_1 < C_2 < \infty\), \(C_1 |s-t| < \sigma(s,t) < C_2 |s-t| \) \(\forall s,t\). (For example, \(X(t,\omega)\) is stationary, \(r(t) \equiv \mathbb{E}_t X_0 \neq r(0), t \neq 0\), and \(-r''(0) < \infty\).) A straightforward computation yields (for \(\psi\) even):

\[ E[I(\psi;X(\cdot,\omega))] < \infty \iff \int_0^\infty -y^2 \frac{1}{y} \int_0^y \psi(x) \, dx \, dy < \infty; \]

equivalently, \(M_{\psi}(x) \equiv \frac{1}{x} \int_0^x \psi(u) \, du\) is integrable around the origin, say over [0,1]. If \(\psi\) is decreasing on (0,\(\infty\)), then \(M_{\psi}(x)\) is the usual maximal function:

\[ M_{\psi}(x) = \sup_{0<u<x<v<1} \frac{1}{v-u} \int_u^v \psi(y) \, dy, \quad 0 < x < 1; \]

hence \(M_{\psi} \in L^1[0,1]\) if and only if \(\psi \in L^{1,\log} \), i.e.

\[ \int_0^1 \psi(x) \log \psi(x) \, dx < \infty. \]

Whether or not \(\psi\) is monotone, Fubini's theorem shows

\[ \int_0^1 M_{\psi}(x) \, dx = \int_0^1 \psi(x) \log \frac{1}{x} \, dx. \]

Consequently, \(I(\psi;X(\cdot,\omega)) < \infty\) a.s. for any \(0 \leq \psi \in L^1\) with \(\psi(x) \log \frac{1}{x} \in L^1[0,1]\), and likewise for any stochastic process which satisfies several mild conditions concerning the distribution of its derivative \(X'(s,\omega)\). This is a "stochastic version" of the real-variable theorem above: only the "fixed" singularity of \(\Lambda\) at 0 is picked up; the others - at \(\{B_1\} \setminus 0\) - depend on the specific function and will generally occur at any fixed point \(x_0\) with probability 0.

Rounding out the picture, it follows from a theorem of Bulinskaya [1] that the hypotheses of the theorem are valid for almost every sample function of a stochastic process \(X(t,\omega)\) for which: (i) \(X(\cdot,\omega)\) is \(C^2\) a.s.,
(ii) for each $0 \leq t \leq 1$, $X'(t, \omega)$ has a density $p_t(x)$ which is bounded in $t$ and $x$.

Condition (ii) guarantees that $\{t: X'(t, \omega) = X''(t, \omega) = 0\}$ is empty a.s. Our earlier statement "$I(\psi; X(\cdot, \omega)) < \infty$ a.s. for any $\psi$ in $L\log L$" can then be strengthened to "$I(\psi; X(\cdot, \omega)) < \infty$ for all $\psi$ in $L\log L$, a.s.," i.e. the exceptional $\omega$-set no longer depends on the particular $\psi$.

§3. Here is the proof of the theorem, which uses little else than ordinary calculus. Recall that $\Lambda$ is the version of $d\lambda/dm$ given by

$$\Lambda(x) = \int_{-\infty}^{\infty} \sum_{s \in F(x)} \frac{1}{|F'(s)|} \sum_{s \in F^{-1}(x)} |F'(s)|^{-1} dy, \quad -\infty < x < \infty.$$ 

(i) $\Lambda$ is continuous off $\{B_i\}_i^L$. I will show that $\Lambda$ is continuous on $\{A_i - A_j\}_{i,j=1}^N \setminus \{B_i\}_i^L$, the proof of continuity at $x \in \mathbb{R} \setminus \{A_i - A_j\}$ goes about the same, except is easier.

Let $A_0 = \inf_s F(s)$, $A_{N+1} = \sup_s F(s)$, and $\nu(x) = \text{Card}\{s: F(s) = x\} \leq 1 + \text{Card}\{D_0\} < \infty$. First, notice that $\alpha$ is continuous off $\{A_i\}_i^N$ because $F'$ and $F^{-1}$ (defined piecewise) are continuous, and because $\nu(x+\varepsilon) = \nu(x)$ for all small $\varepsilon$ if $x \notin \{A_i\}_i^N$.

Now fix $A_i - A_j \notin \{B_i\}_i^L$, $1 \leq i,j \leq N$, and let $(k_\ell, r_\ell)$, $\ell = 1, \ldots, q$, be those pairs of integers among $\{1, 2, \ldots, N\}$ for which $A_{k_\ell} - A_{r_\ell} = A_i - A_j$. Assume $F''(a_i) < 0 < F''(a_j)$; then $F''(a_k) < 0$ (resp. $F''(a_k) > 0$) for each $1 \leq k \leq N$ with $A_k = A_i$ (resp. $A_k = A_j$). It follows that $\alpha(A_j^-) = \lim_{\varepsilon \to 0} \alpha(A_j - \varepsilon)$ and $\alpha(A_i^+) = \lim_{\varepsilon \to 0} \alpha(A_i + \varepsilon)$ exist, finite. The same argument applies to each $A_{k_\ell}, A_{r_\ell}$ and yields:

$$\alpha(A_{k_\ell}^+) < \infty, \quad \alpha(A_{r_\ell}^-) < \infty, \quad 1 \leq \ell \leq q.$$ 

Since $\Lambda(x)$ is an even function and $\{A_i - A_j\}_{i,j=1}^N \setminus \{B_i\}_i^L$ is symmetric about
0, it will be enough to check that $\Lambda$ is right-continuous at $A_1-A_j$. Set $K(x,y) = \alpha(y)\alpha(y+x+A_1-A_j)$ and let

$$T_\delta = \bigcup_{k=1}^{N} (A_k-\delta, A_k+\delta), \quad W_\delta = \bigcup_{l=0}^{q} (A_{r_l}-\delta, A_{r_l}+\delta);$$

also, let $\eta > 0$ be the distance from $A_1-A_j$ to $\{A_n-A_m\}$, $(n,m) \neq (k_l,r_l)$.

Then

$$\Lambda(x+A_1-A_j) = \int_{W_\delta \cap T_\delta} K(x,y)dy + \int_{W_\delta \cap T_\delta^c} K(x,y)dy + \sum_{\ell \in \Gamma} \int_{A_{r_\ell}-\delta} K(x,y)dy$$

$$= P_1(x) + P_2(x) + \sum_{\ell \in \Gamma} P_{3,\ell}(x)$$

where the $A_{r_\ell}, \ell \in \Gamma \subseteq \{1, \ldots, q\}$, are distinct and $\delta$ is small enough that the intervals $(A_{r_\ell}-\delta, A_{r_\ell}+\delta), \ell \in \Gamma$, are disjoint.

If $0 < x < \delta/2$ and $\delta < \eta/2$, $y \in W_\delta \cap T_\delta \Rightarrow y+A_1-A_j \in T_\delta^c \Rightarrow y+x+A_1-A_j \in T_\delta/2$. In particular, $sup \alpha(y+x+A_1-A_j) < \infty$ for such $x$'s. Consequently, recalling that $\alpha \in L^1$ and $\alpha$ is continuous a.e., $P_1(x) + P_1(0) < \infty$ as $x \downarrow 0$ (dominated convergence theorem). Similarly, $P_2(x) < \infty \forall x \geq 0$ and

$$|P_2(x) - P_2(0)| \leq \sup_{y \in T_\delta^c} \int_{y}^{\infty} |\alpha(x+y) - \alpha(y)|dy + 0 \text{ as } x \downarrow 0.$$ 

Finally,

$$P_{3,\ell}(x) = \int_{A_{r_\ell}-\delta} K(x,y)dy + \int_{A_{r_\ell} \cap T_\delta} \alpha(y-A_1-A_j)\alpha(y+x)dy$$

which converges to $P_{3,\ell}(0) < \infty$ as $x \downarrow 0$ by using (*) and arguing as above with $P_1$ and $P_2$. 


Next, $F^\circ F^{-1}$ satisfies upper and lower H"{o}lder conditions of order $1/2$ at each $A_i$, $1 \leq i \leq N$. For convenience, assume $0 < a_0 < a_N < 1$; the other cases only need some additional notation. For each $a_i \in D_0$ and $s \in [0,1]$ there are numbers $\xi_s, \overline{\xi}_s$ between $s$ and $a_i$ with $F'(s) = F''(\xi_s)(s-a_i)$ and $F(s) - A_i = \frac{1}{2}F''(\overline{\xi}_s)(s-a_i)^2$.

It follows that there are constants $0 < C_1, C_2, C_3, C_4 < \infty$ and a $\delta_0 > 0$ such that for each $1 \leq i \leq N$ and $\delta \leq \delta_0$,

\begin{align*}
(6a) & \quad C_2 |s-a_i| \leq |F'(s)| \leq C_1 |s-a_i|, \quad s \in (a_i-\delta, a_i+\delta) \\
(6b) & \quad C_4 |s-a_i|^2 \leq |F(s) - A_i| \leq C_3 |s-a_i|^2, \quad s \in (a_i-\delta, a_i+\delta).
\end{align*}

Let $\hat{F}_i$ denote the inverse of $F$ on $J_i = [a_i, a_{i+1}]$, $1 \leq i \leq N-1$. From (6b) and the continuity of the $\hat{F}_i$'s, there is a $\delta_0 > 0$ such that, for each $1 \leq i \leq N-1$, $\delta \leq \delta_0$,

\begin{align*}
(7) & \quad \frac{1}{C_3} |y-A_i|^{1/2} \leq |\hat{F}_i(y) - a_i| \leq \frac{1}{C_4} |y-A_i|^{1/2}, \quad y \in (A_i-\delta, A_i+\delta) \cap F(J_i) \\
& \quad \frac{1}{C_3} |y-A_{i+1}|^{1/2} \leq |\hat{F}_i(y) - a_{i+1}| \leq \frac{1}{C_4} |y-A_{i+1}|^{1/2}, \quad y \in (A_{i+1}-\delta, A_{i+1}+\delta) \cap F(J_i).
\end{align*}

Let $D(i, \delta) = (A_i, A_{i+\delta})$ if $F''(a_i) > 0$, and $A_i - \delta, A_i)$ if $F''(a_i) < 0$, $1 \leq i \leq N$.

Combining (6a) and (7), and reducing $\delta_0$ if necessary, there are constants $0 < C_5, C_6 < \infty$ such that for each $1 \leq i \leq N-1$, $\delta \leq \delta_0$,

\begin{align*}
(8) & \quad C_5 |y-A_i|^{1/2} \leq |F'(\hat{F}_i(y))| \leq C_6 |y-A_i|^{1/2}, \quad y \in D(i, \delta)
\end{align*}

and likewise (in case $a_N = 1$) with $A_i, D(i, \delta)$ replaced by $A_i+1$, $D(i+1, \delta)$.

We can assume that for each $i, j$ and each small $\delta$, either $D(i, \delta) = D(j, \delta)$ or $D(i, \delta) \cap D(j, \delta) = \emptyset$. Defining $J_0 = [0, a_1], J_N = [a_N, 1]$ and the corresponding inverses $\hat{F}_0, \hat{F}_N$, it is clear that (8) extends to $F^\circ \hat{F}_0$ and $F^\circ \hat{F}_N$ at the appropriate places. (By the way, both inequalities in (8) depend on $F'' \neq 0$ on $D_0$.)
(ii) \( \lim_{x \to B_1} \Lambda(x)/-\log|y-B_1| > 0, \ 1 \leq i \leq L. \) Suppose \( B_1 = A_k - A_k, \ 1 \leq \ell, k \leq N, \)

and \( F''(a_k) < 0, F''(a_\ell) < 0; \) the other case, namely \( F''(a_k), F''(a_\ell) > 0 \) is the same.

\[
\Lambda(x+B_1) = \int_{-\infty}^{\infty} \alpha(y+x+A_\ell)\alpha(y+A_k)dy \\
\geq \int_{-\varepsilon}^{\varepsilon} \alpha(y+x+A_\ell)\alpha(y+A_k)dy,
\]

\(|x| < \varepsilon .\)

Now for \( \varepsilon \) small, the conditions \(|x| < \varepsilon \) and \(-\varepsilon < y < -|x|\) together imply that \( y+x+A_\ell \in \mathcal{D}(\ell, \delta_0) \) and \( y+A_k \in \mathcal{D}(k, \delta_0) \). Consequently,

\[
\Lambda(x+B_1) \geq C_5^2 \int_{-\varepsilon}^{\varepsilon} |y+x|^{-1/2}|y|^{-1/2}dy
\]

\[
= C_5^2 \log \left| \frac{2\varepsilon^2 - 2\varepsilon x + 2\varepsilon - x}{2\sqrt{\varepsilon^2 - |x| x + 2|x|-x}} \right|
\]

\[
\geq C \log \frac{1}{|x|},
\]

for all small \( x \), for some \( C > 0. \)

(iii) \( \Lambda(x) \leq \text{const.} \times [1 + \sum_{i=1}^{L} |\log|x-B_1||]\forall x. \) (This is equivalent to the "\( \tilde{\text{Im}} \)"

part of (3a).) Evidently,

\[
\alpha(y) = \sum_{i=0}^{N+1} 1_F(J_i)(y) |F' \circ \hat{F}_i(y)|^{-1}.
\]

Off \( T_\delta, \) \( \alpha \) is bounded. Let \( y \in T_\delta \), say \( A_1 - \delta < y < A_1 + \delta, \ y \in F[0,1]. \) Keeping (8) in mind and that non-identical \( D(j, \delta) \)'s are disjoint:

\[
\alpha(y) = \sum_{j:A_1 = A_j} 1_F(J_j)(y) |F' \circ \hat{F}_j(y)|^{-1} + \sum_{j:A_1 \neq A_j} 1_F(J_j)(y) |F' \circ \hat{F}_j(y)|^{-1}
\]

\[
\leq \nu(y)C_5^2 |y-A_1|^{-1/2} + \nu(y) \sup_{s \in \mathcal{H}_\delta} |F'(s)|^{-1}, \ H_\delta = F^{-1} \{ \cap_{i=1}^{N} (A_i - \delta, A_i + \delta) \}
\]

\[
\leq \text{const.} \times [1 + \sum_{i=1}^{N} |y-A_1|^{-1/2}].
\]
Let $V = F[0,1]$ and $U = \bigcup_{i=1}^{N} V - A_i$, which is bounded.

$$\Lambda(x) = \int_{V} \alpha(y)\alpha(x+y)dy$$

$$\leq \text{const.} \times [1 + 2 \sum_{i=1}^{N} \int_{U} |y-A_i|^{-1/2}dy + \sum_{i,j=1}^{N} \int_{U} |y-A_i|^{-1/2}|y+x-A_j|^{-1/2}dy]$$

$$\leq \text{const.} \times [1 + \sum_{i,j=1}^{N} \int_{U} |y+A_j-A_i|^{-1/2}|y+x|^{-1/2}dy]$$

$$\leq \text{const.} \times [1 + \sum_{i,j=1}^{N} |\log|x-(A_i-A_j)|||]$$

since $\int_{U} |y+\varepsilon|^{-1/2}|y|^{-1/2}dy = O(\log \frac{1}{|\varepsilon|})$ as $\varepsilon \to 0$.

As for (3b), let $H(x) = 1 + \sum_{i=1}^{L} |\log|x-B_i|||$. Then $I(\psi;F) < \infty$ $\forall \psi \in L^1(\text{Hdm})$ if and only if

$$\int_{-\infty}^{\infty} \frac{\psi(x)}{H(x)} \Lambda(x)dx < \infty \quad \forall \psi \in L^1(dx),$$

if and only if $\text{ess sup}_{x} \frac{\Lambda(x)}{H(x)} < \infty$. Since $\Lambda, H$ are continuous from $\mathbb{R}$ to $\mathbb{R} \cup \{\infty\}$, this is the same as $\sup_{x} \frac{\Lambda(x)}{H(x)} < \infty$. In other words, the "lim" part of (3a) is equivalent to "$I(\psi;F) < \infty$ $\forall \psi \in L^1(\text{Hdm})"$. Now if $I(\psi;F) < \infty$ and $\psi \in L^1(dx)$, then it is easy to see, using the "lim" part of (3a) that $\psi H$ is integrable. The last statement of the theorem follows from (3b) and the aforementioned fact that $I(\psi;F) < \infty$ and $\psi^+$ imply $\psi \in L^1[0,1]$.

§4. Let $F(t) = t^2$. Then $D_0 = \{B_1\} = \{0\}$ and

$$\Lambda(x) = \frac{1}{2} \log \left[ \frac{1 + \sqrt{1-|x|}}{\sqrt{|x|}} \right], \quad |x| \leq 1.$$

For $F(t) = \sin 2\pi t$, $\Lambda(x)$ is an elliptic integral (of the first kind). I would give more examples, especially in "closed form" and with $L > 1$, if I could; the computations (even for $F$ a third degree polynomial) are formidable.
References

1. E.V. Bulinskaya, On the mean number of crossings of a level by a stationary Gaussian process, Teoriya Veroyatnostei i ee Primeneniya, 6 (1961), pp. 474-477.


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<th>13. NUMBER OF PAGES</th>
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<tbody>
<tr>
<td>10</td>
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<table>
<thead>
<tr>
<th>14. MONITORING AGENCY NAME &amp; ADDRESS (IF different from Controlling Office)</th>
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<table>
<thead>
<tr>
<th>15. SECURITY CLASS. (OF THIS REPORT)</th>
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<tbody>
<tr>
<td>UNCLASSIFIED</td>
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<table>
<thead>
<tr>
<th>15a. DECLASSIFICATION/DOWNGRADING SCHEDULE</th>
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<thead>
<tr>
<th>16. DISTRIBUTION STATEMENT (OF THIS REPORT)</th>
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<tbody>
<tr>
<td>Approved for public release: distribution unlimited.</td>
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<thead>
<tr>
<th>17. DISTRIBUTION STATEMENT (OF THE ABSTRACT ENTERED IN BLOCK 20, IF DIFFERENT FROM REPORT)</th>
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<tr>
<th>18. SUPPLEMENTARY NOTES</th>
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<tr>
<th>19. KEYWORDS (CONTINUE ON REVERSE SIDE IF NECESSARY AND IDENTIFY BY BLOCK NUMBER)</th>
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<tbody>
<tr>
<td>C² function, local time, distribution of the increments</td>
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<tr>
<th>20. ABSTRACT (CONTINUE ON REVERSE SIDE IF NECESSARY AND IDENTIFY BY BLOCK NUMBER)</th>
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<tbody>
<tr>
<td>Let $F(t)$, $0 \leq t \leq 1$, be a real function with two continuous derivatives such that ${F' = F'' = 0}$ is empty. Then $B \rightarrow \text{meas.}{(s,t): F(s) - F(t) \in B}$ in absolutely continuous; its density is continuous on $\mathbb{R}\setminus{B_i}$, ${B_i} = {y: y = F(t_1) - F(t_2), F'(t_1) = F'(t_2) = 0, F''(t_1)F''(t_2) &gt; 0}$, and has a logarithmic singularity at each $B_i$.</td>
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</tbody>
</table>