INEQUALITIES FOR POWER ONE TESTS FOR SUMS OF DEPENDENT VARIABLES

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ABSTRACT

Let S_n be the sum of n bounded dependent variables, L_n the sum of squares of these variables, and ϕ a positive increasing concave function. We determine explicit and tight upper bounds on the probability that $S_n \geq \phi(L_n)$ under the hypothesis that S_n is a martingale. We further determine explicit upper bounds for the mean intrinsic time for S_n to pass $\phi(L_n)$ under the hypothesis that S_n has positive drift. These inequalities can be used in power one tests for dependent variables in the manner proposed by Darling, Robbins, and Siegmund for independent variables

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(I) Introduction

Let ϕ: $[0,\infty)\to[0,\infty)$ satisfy

(1.1) $$\sqrt{t} < \phi(t) < t \quad \text{all} \quad t \ge t_* > 0$$

(1.2) $$\frac{\phi(t)}{\sqrt{t}} \uparrow \infty \quad \text{and} \quad \frac{\phi(t)}{t} \downarrow 0 \quad \text{as} \quad t \uparrow \infty$$

(1.3) ϕ has a continuous derivative ϕ' such that

$$p(t) \equiv \frac{t\phi'(t)}{\phi(t)} \downarrow p \in [\tfrac{1}{2}, 1) \quad \text{as} \quad t \uparrow \infty$$

(1.4) $$\phi'(t) < \frac{\phi(t)}{t}, \quad \text{all} \quad t \ge t_* > 0$$

Let (Ω, B, P) be a fixed probability space supporting the adapted sequence of random variables $\{f_n, B_n\}$. We assume throughout that

(1.5) $$\begin{cases} \text{(a)} \quad |f_n| \le 1, \quad \text{all} \quad n \ge 1 \\ \quad\quad \sum_1^\infty \sigma^2\{f_n | B_{n-1}\} = \infty \end{cases}$$

where $\sigma^2\{f_n|B_{n-1}\} = E\{f_n^2|B_{n-1}\} - E^2\{f_n|B_{n-1}\}$.

We fix some notation:

(1.6) $$\begin{cases} S_n = \sum_1^n f_k \\ T_n = \sum_1^n E\{f_k^2|B_{k-1}\} \\ L_n = \sum_1^n f_k^2 \\ R_n = \sum_1^n E\{f_k|B_{k-1}\} \end{cases}$$

We fix two hypotheses:

$$(1.7)\quad \begin{cases} H_0: & E\{f_n|B_{n-1}\} = 0, \quad \text{all} \quad n \geq 1 \\ H_1: & E\{f_n|B_{n-1}\} \geq \theta\sigma^2\{f_n|B_{n-1}\}, \ \text{all} \quad n \geq 1, \quad \text{some} \quad \theta > 0. \end{cases}$$

We define, for each $\theta > 0$ and small enough, a unique positive value t_θ where

$$(1.8)\qquad \frac{\phi(t_\theta)+1}{t_\theta} = \frac{\theta}{1+\theta}$$

We will prove the following two theorems:

Theorem (1)

Under H_0:

(A) For all $t_0 \geq t_*$:

$$P\{S_n \geq \phi(T_n), \quad \text{some} \quad T_n \geq t_0\} \leq 3\int_{t_0}^{\infty} \frac{\phi(t)}{t^{3/2}} e^{-\left\{1 - \frac{\sqrt{t}}{\phi(t)} - \frac{\phi(t)}{t}\right\} \frac{\phi^2(t)}{2t}} dt$$

(B) For all $t_0 \geq t_{**}$ where $\dfrac{\phi(t_{**})}{t_{**}} < \dfrac{1}{\sqrt{2}}$:

$$P\{S_n \geq \phi(L_n), \quad \text{some} \quad L_n \geq t_0\}$$

$$\leq 3\left\{\frac{1+2\,\dfrac{\sqrt{t_0}}{\phi(t_0)}}{1-\sqrt{2}\,\dfrac{\phi(t_0)}{t_0}}\right\} \int_{t_0}^{\infty} \frac{\phi(t)}{t^{3/2}} e^{-\left\{1 - \frac{\sqrt{t}}{\phi(t)} - \frac{\phi(t)}{t}\right\} \frac{\phi^2(t)}{2t}} dt$$

Theorem (2)

Assume H_1

Define

$$\tau_\theta = \inf n \ni S_n \ge \phi(T_n)$$

$$\tau_{\theta'} = \inf n \ni S_n \ge \phi(L_n)$$

Then τ_θ and $\tau_{\theta'}$ are proper and

$$\text{(A)} \quad \frac{1}{t_\theta}\int T_{\tau_\theta} \le 1 + \frac{1}{(1-p(t_\theta))^2}\left\{\sqrt{3\pi} + \frac{12\sqrt{t_\theta}}{\phi(t_\theta)}\right\}\frac{\sqrt{t_\theta}}{\phi(t_\theta)}$$

$$\text{(B)} \quad \frac{1}{t_\theta}\int L_{\tau_{\theta'}} \le 1 + 16\left(\frac{\theta}{1+\theta}\right)^2 + \frac{2}{(1-p(t_\theta))^2}\left\{3.2 + \frac{12\sqrt{t_\theta}}{\phi(t_\theta)}\right\}\frac{\sqrt{t_\theta}}{\phi(t_\theta)}$$

(1.9) Remarks

These theorems are analogues of results in the literature for sequences of i.i.d. randomvariables. For instance, when $\{f_n\}$ is i.i.d., with $\int f_1 = 0$, $\int f_1^2 = 1$ and $\int e^{\lambda f_1} < \infty$, some positive λ, then modulo some further regularity conditions, we have, assuming $\int_1^\infty \frac{\phi(t)}{t^{3/2}} e^{-\frac{\phi^2(t)}{2t}} dt < \infty$:

Darling-Robbins [3]

$$P\{S_n \ge \phi(n), \quad \text{some} \quad n \ge t_0\} \le \frac{\sqrt{e}}{2}\int_{t_0}^\infty \frac{\phi(t)}{t^{3/2}} e^{-\frac{\phi^2(t)}{2t}} dt$$

Strassen [10] Theorem (1,4) For p as in (1.3):

$$\lim_{t_0 \uparrow \infty} \frac{P\left\{S_n \geq \phi(n), \quad \text{some} \quad n \geq t_0\right\}}{\int_{t_0}^{\infty} \frac{\phi(t)}{t^{3/2}} e^{-\frac{\phi^2(t)}{2t}} dt} = \frac{p}{\sqrt{2\pi}}$$

Strassen's result establishes that $\int_{t_0}^{\infty} \frac{\phi(t)}{t^{3/2}} e^{-\frac{\phi^2(t)}{2t}} dt$ is the rate at which the probabilities under consideration tend to zero. Darling-Robbins bound all such probabilities by a fixed constant times such rates. As our Theorem (1) reveals, we bound the probabilities of the conditional versions of the crossings $\{S_n \geq \phi(n), \text{ some } n \geq t_0\}$ by a small perturbation of such rates.

Our second theorem establishes that

$$\overline{\lim_{\theta \downarrow 0}} \frac{1}{t_\theta} \int T_{\tau_\theta} \leq 1$$

$$\lim_{\theta \downarrow 0} \frac{1}{t_\theta} \int L_{\tau_{\theta'}} \leq 1$$

and gives rates for these lim sups.

If $\{f_n\}$ is i.i.d., with $\int f_1 = 0$, $\int f_1 = 1$ and with other regularity conditions, very general but too numerous to mention here, for t_θ defined through $\frac{\phi(t_\theta)}{t_\theta} = \theta$ and for T_θ defined through $T_\theta = \inf n \ni S_n + n\theta \geq \phi(n)$, we have

Lai [7] Corollary (1)

$$(*) \qquad \lim_{\theta \downarrow 0} \frac{1}{t_\theta} \int T_\theta = P\{S_n < \phi(n), \text{ all } n \geq t_0\} \leq 1$$

Our second theorem is the closest we've been able to come to a conditional analogue of Lai's result. We mention here that Gut [6] has produced limit theorems related to (*) for dependent sums.

(1.10) Consider again the following hypotheses:

$$H_0: \quad E\{f_n | B_{n-1}\} = 0, \quad \text{all} \quad n$$

$$H_1: \quad E\{f_n | B_{n-1}\} \geq \theta\sigma^2\{f_n | B_{n-1}\}, \quad \text{all} \quad n$$

One statistical test of these hypotheses consists in deciding for H_1 if $S_n \geq \phi(L_n)$ for some $L_n \geq t_0$. Using our theorems and assuming t_0 large, θ small, we have

$$P\{S_n \geq \phi(L_n), \quad \text{some} \quad L_n \geq t_0 | H_0\} \leq \varepsilon$$

and

$$P\{S_n \geq \phi(L_n), \quad \text{some} \quad L_n \geq t_0 | H_1\} = 1$$

and

$$E\{L_{\tau_{\theta'}} | H_1\} \leq (1+\varepsilon)t_{\theta'} \quad .$$

Darling-Robbins [2], [3] have considered such "tests of power one" for sums of i.i.d.'s. There has been subsequent work in this area, [8], [9], with the focus on i.i.d.'s also. Our theorems are an attempt at extending such tests to dependent sums. This extension does not seem statistically unrealistic. Consider the following contexts:

(A) A drug level, d_n, is administered to a subject whose condition, C_{n-1}, at time n, depends on past conditions and dosage levels received. The objective of the treatment is the stabilization of the subject's response where this response, f_n, is a random function of C_{n-1} and d_n, ad where stabilization is interpreted as $E\{f_n | B_{n-1}\} = 0$.

(B) We wish to test, through discrete sampling, if a diffusion Z_t, has positive drift. Under reasonable assumptions on the variance coefficient, $\sigma^2(Z_t)$, and with $f_n = Z_n - Z_{n-1}$ suitably truncated, H_0 vs H_1 as defined above will constitute a test of power one.

(C) To decide, as quickly as possible, that a game is not fair, i.e., that the win or loss increment, f_n, obtained at time n by a gambling house, favors the house in the sense that $E\{f_n | B_{n-1}\} \geq \theta\sigma^2\{f_n | B_{n-1}\}$.

(1.11) Under the mild hypothesis $\int f_n^2 \geq a > 0$, we see that

$$\int \tau_{\theta'} \leq \frac{1}{a} \int L_{\tau_{\theta'}}$$

so that upper bounds become available on the expected time $\tau_{\theta'}$.

(1.12) The proofs of theorems (1) and (2) are almost entirely dependent on the use of three supermartingales devised by Freedman [4], [5]. We introduce these here.

(a) Let $\{g_n, B_n\}$ be adapted, with $0 \leq g_n \leq 1$.

Let

$$C_n = \sum_1^n g_k$$

$$M_n = \sum_1^n E\{f_k | B_{k-1}\}$$

$$G(\lambda) = e^{\lambda} - 1$$

$$F(\lambda) = 1 - e^{-\lambda}$$

Then

$$e^{\lambda C_n - G(\lambda) M_n} \quad \text{and} \quad e^{F(\lambda) M_n - \lambda C_n}$$

are supermartingales.

(b) Let $K(\lambda) = e^{\lambda} - 1 - \lambda$. Then, with (1.5a) and the notation (1.6),

$$e^{\lambda S_n - K(\lambda) T_n}$$

is a supermartingale.

Proposition (1.1)

For all $\theta \geq 0$, all $t \geq 0$, and provided $C_\infty = \sum^{\infty} g_k = \infty$ a.e:

(1) $$P\{M_n \geq (1+\theta) C_n, \text{ some } C_n \geq t\} \leq e^{-\frac{\theta^2 t}{2(1+\theta)}}$$

(2) $$P\{M_n \leq \frac{1}{1+\theta} C_n, \text{ some } C_n \geq t\} \leq e^{-\frac{\theta^2 t}{2(1+\theta)}}$$

(3) $$P\{M_n \notin [\frac{1}{1+\theta} C_n, (1+\theta) C_n], \text{ some } C_n \geq t\} \leq 2e^{-\frac{\theta^2 t}{2(1+\theta)}}$$

Proof:

Let $0 \leq t < S$ and let

$$\tau_1 = \inf k \ni C_k \geq t$$

$$\tau_2 = \inf k \ni C_k \geq S$$

$$\tau = \begin{cases} \text{first } k \varepsilon [\tau_1, \tau_2) \text{ with } M_k \geq (1+\ )C_k \\ \tau_2 \text{ if no such } k \text{ occurs} \end{cases}$$

Then

$$1 \geq \int_{\{M_\tau \geq (1+\theta)C_\tau\}} e^{F(\lambda)M_\tau - \lambda C_\tau}$$

$$\geq \int_{\{M_\tau \geq (1+\theta)C_\tau\}} e^{F(\lambda)((1+\theta))C_\tau - \lambda C_\tau}$$

$$\geq e^{(F(\lambda)(1+\theta)-\lambda)t} \quad P\{M_\tau \geq (1+\theta)C_\tau\}$$

provided $\frac{F(\lambda)}{\lambda} \geq \frac{1}{1+\theta}$.

Under these circumstances, letting $S \uparrow \infty$:

$$P\{M_n \geq (1+\theta)C_n, \quad \text{some} \quad C_n \geq t\} \leq \inf_\lambda e^{(\lambda - F(\lambda)(1+\theta))t}$$

Now the exponent minimizes for $\lambda = \ln(1+\theta)$; we can use this value since

$$\frac{F(\lambda)}{\lambda} = \frac{\theta}{(1+\theta)\ln(1+\theta)} \geq \frac{1}{1+\theta} \quad , \qquad \theta \geq 0$$

Using $\ln(1+\theta) - \theta \leq -\frac{\theta^2}{2(1+\theta)}$, we have our first result.

For the second part, let $0 < t < S$ as before, let τ_1, τ_2 be as before, and define

$$\tau = \begin{cases} \text{first} \quad k \in [\tau_1, \tau_2) \quad M_k \leq \frac{1}{1+\theta} C_k \\ \tau_2 \quad \text{if no such } k \text{ occurs} \end{cases}$$

Then

$$1 \geq \int_{\{M_\tau \leq \frac{1}{1+\theta} C_\tau\}} e^{\lambda C_\tau - G(\lambda) M_\tau}$$

$$\geq \int_{\{M_\tau \leq \frac{1}{1+\theta} C_\tau\}} e^{(\lambda - \frac{1}{1+\theta} G(\lambda)) C_\tau}$$

$$\geq e^{(\lambda - \frac{1}{1+\theta} G(\lambda))t} \quad P\{M_\tau \leq \frac{1}{1+\theta} C_\tau\}$$

provided $\lambda \geq \frac{1}{1+\theta} G(\lambda)$.

Letting $S \to \infty$ we have

$$P\{M_n \leq \frac{1}{1+\theta} C_n, \quad \text{some} \quad C_n \geq t\} \leq \inf_\lambda e^{(\frac{1}{1+\theta} G(\lambda) - \lambda)t}$$

where λ is subject to $\lambda \geq \frac{1}{1+\theta} G(\lambda)$.

The best choice is $\lambda = \ell n(1+\theta)$ which satisfies this constraint.

Using $\frac{\theta}{1+\theta} - \ell n(1+\theta) \geq -\frac{\theta^2}{2(1+\theta)}$, we have the second case. From these two cases the third case follows straightforwardly.

Q.E.D.

Proposition (1.2)

Let $\{f_n, B_n\}$ satisfy

$$\sum_1^\infty f_n^2 = \infty \qquad \text{a.e.}$$

Then

$$P\{T_n \notin [\frac{1}{1+\theta} L_n, (1+\theta)L_n], \text{ some } L_n \geq t\} \leq 2e^{-\frac{\theta^2 t}{2(1+\theta)^2}}.$$

Proof:

Let $f_n^2 = g_n$ and use the previous proposition. We've replaced $\frac{\theta^2}{(1+\theta)}$ by $\frac{\theta^2}{(1+\theta)^2}$ for later technical reasons. Q.E.D.

(II) Proof of Theorem (1)

Fix $0 < t_0 < t_1$. Define

$$\tau_1 = \text{first } n \ni T_n \geq t_0$$

$$\tau_2 = \text{last } n \ni T_n \leq t_1$$

$$\tau = \begin{cases} \text{first } n \in [\tau_1, \tau_2] & S_n \geq \phi(T_n) \\ \tau_2 \text{ if no such } n \text{ occurs} \end{cases}$$

Then, from Optimal Stopping:

$$1 \geq \int_{\{S_\tau \geq \phi(T_\tau)\}} e^{\lambda S_\tau - K(\lambda) T_\tau}$$

$$\geq e^{\lambda\phi(t_0) - K(\lambda)t_1} \quad P\{S_\tau \geq \phi(T_\tau)\}$$

Letting

$$P_0 = P\{S_n \geq \phi(T_n), \text{ some } T_n \in [t_0, t_1]\},$$

we have

$$P_0 \le \inf_{\lambda\ge 0} e^{K(\lambda)t_1-\lambda\phi(t_0)}$$

The infimum occurs for $\lambda = \ell n\left\{1 + \frac{\phi(t_0)}{t_1}\right\}$.

Using $(1+x)\ell n(1+x) \ge x + \frac{x^2}{2} - \frac{x^3}{6}$, valid for $0 \le x \le 1$, we have

$$P_0 \le e^{\frac{\phi^3(t_0)}{6t_0^2} - \frac{\phi^2(t_0)}{2t_1}}$$

Define

$$t_0 = t_1 - \frac{1}{3}\frac{t_1^{3/2}}{\phi(t_1)}$$

Then

$$\frac{\phi^2(t_0)}{t_1} = \frac{\phi^2(t_1)}{t_1} - \left(\frac{\phi^2(t_1)-\phi^2(t_0)}{t_1}\right)$$

From the Mean Value Theorem and $\phi'(t) \le \frac{\phi(t)}{t}$, we have

$$\frac{\phi^2(t_1)-\phi^2(t_0)}{t_1} \le 2\left(\frac{t_1}{t_0} - 1\right)\frac{\phi^2(t_1)}{t_1}$$

$$\le \frac{2\sqrt{t_1}}{3\phi(t_1)-\sqrt{t_1}}\;\frac{\phi^2(t_1)}{t_1}$$

Thus

$$\frac{\phi^2(t_0)}{2t_1} \ge \left(1 - \frac{\sqrt{t_1}}{\phi(t_1)}\right)\frac{\phi^2(t_1)}{2t_1}$$

Next

$$\frac{\phi^3(t_0)}{t_0^2} \le \frac{\phi^3(t_1)}{t_1^2}\left(\frac{t_1}{t_0}\right)^2 \le \frac{9}{4}\,\frac{\phi^3(t_1)}{t_1^2}$$

Consequently

$$P_0 \le e^{-\left(1 - \frac{\sqrt{t_1}}{\phi(t_1)} - \frac{\phi(t_1)}{t_1}\right)\frac{\phi^2(t_1)}{2t_1}}$$

Suppose we now repeat this argument over intervals $[t_{n-1}, t_n]$ where

$$t_{n-1} = t_n - \frac{1}{3}\frac{t_n^{3/2}}{\phi(t_n)} \quad .$$

We'd have

$$P_{n-1} \le e^{-\left(1 - \frac{\sqrt{t_n}}{\phi(t_n)} - \frac{\phi(t_n)}{t_n}\right)\frac{\phi^2(t_n)}{2t_n}}$$

Then

$$P\{S_n \ge \phi(T_n), \quad \text{some} \quad T_n \ge t_0 \le \sum_{n=1}^{\infty} e^{-\left(1 - \frac{\sqrt{t_n}}{\phi(t_n)} - \frac{\phi(t_n)}{t_n}\right.\frac{\phi^2(t_n)}{2t_n}}$$

Note next that $t_n \uparrow \infty$ and

$$\frac{1}{t_n - t_{n-1}} = \frac{3\phi(t_n)}{t_n^{3/2}}$$

Thus, dominating the sum above by an integral:

$$P\{S_n \ge \phi(T_n), \quad \text{some} \quad T_n \ge t_0\} \le 3\int_{t_0}^{\infty} \frac{\phi(t)}{t^{3/2}}\, e^{-\left(1 - \frac{\sqrt{t}}{\phi(t)} - \frac{\phi(t)}{t}\right)\frac{\phi^2(t)}{2t}}\, dt$$

Next, for θ momentarily unspecified, and using Proposition (1.2):

$$P\{S_n \ge \phi(L_n),\ \text{some } L_n \ge t_0\} \le P\{T_n \notin [\tfrac{1}{1+\theta} L_n, (1+\theta)L_n],\ \text{some } L_n \ge t_0\}$$

$$+ P\{S_n \ge \phi\left(\frac{T_n}{1+\theta}\right),\ \text{some } T_n \ge \frac{t_0}{1+\theta}\} \le$$

$$2e^{-\frac{\theta^2 t_0}{2(1+\theta)^2}} + P\{S_n \ge \frac{1}{1+\theta}\phi(T_n),\ \text{some } T_n \ge \frac{t_0}{1+\theta}\}$$

$$\le 2e^{-\frac{\theta^2 t_0}{2(1+\theta)^2}} + 3\int_{\frac{t_0}{1+\theta}}^{\infty} \frac{\phi_0(t)}{t^{3/2}} e^{-\left(1-\frac{\sqrt{t}}{\phi_0(t)}-\frac{\phi_0(t)}{t}\right)\frac{\phi_0^2(t)}{2t}} dt$$

where $\phi_0(t) = \frac{1}{1+\theta}\phi(t)$.

Substituting $(1+\theta)t$ for t in the integral and letting $\int_{t_0}^{\infty}$ denote this integral, and noting that the integral decreases in t, θ

$$P\{S_n \ge \phi(L_n),\ \text{some } L_n \ge t_0\} \le 2e^{-\frac{\theta^2 t_0}{2(1+\theta)^2}} + 3(1+\theta)\int_{t_0}^{\infty}$$

Our choice for θ will be made so that

$$\lim_{t_0 \uparrow \infty} e^{-\frac{\theta^2 t_0}{2(1+\theta)^2}} \Big/ \int_{t_0}^{\infty} = 0 ,$$

so that the right side in the inequality above is dominated by $\int_{t_0}^{\infty}$ multiplied by some constant.

We reason as follows;

Letting $\frac{\phi(t_0)}{\sqrt{t}} = x$, we have

$$\int_{t_0}^{\infty} \geq \int_{\frac{\phi(t_0)}{\sqrt{t_0}}}^{\infty} e^{-\frac{x^2}{2}} dx;$$

this inequality follows from $\phi'(t) \leq \frac{\phi(t)}{t}$.

By a classical inequality:

$$\int_{x}^{\infty} e^{-\frac{x^2}{2}} dz \geq \left(\frac{1}{x} - \frac{1}{x^3}\right) e^{-\frac{x^2}{2}},$$

so that, subject to $\frac{\phi(t_0)}{t_0} \leq \frac{1}{\sqrt{2}}$, we have

$$\int_{t_0}^{\infty} \geq \frac{\sqrt{t_0}}{\phi(t_0)} \left\{1 - \frac{\sqrt{2}}{\phi(t_0)}\right\} e^{-\frac{\phi^2(t_0)}{2t_0}}$$

But then, letting $\frac{\theta}{1+\theta} = \sqrt{2}\,\frac{\phi(t_0)}{t_0}$, we have

$$e^{-\frac{\theta^2 t_0}{2(1+\theta)^2}} \Big/ \int_{t_0}^{\infty} \leq e^{-\frac{1}{2}\frac{\phi^2(t_0)}{t_0}} \Big/ \left(1 - \frac{\sqrt{2}}{\phi(t_0)}\right) \quad \frac{\phi(t_0)}{\sqrt{t_0}} \to 0 \text{ as to } \to \infty.$$

Staying with this value of θ, we have

$$P\{S_n \geq \phi(L_n), \text{ some } L_n \geq t_0\} \leq 2e^{-\frac{\phi^2(t_0)}{t_0}} + \frac{3}{1-\sqrt{2}\,\frac{\phi(t_0)}{t_0}} \int_{t_0}^{\infty}$$

from which Theorem (1B) follows.

Q.E.D.

(III) Proof of Theorem (2)

Lemma

$$P\{S_n \geq \phi(T_n) \quad \text{eventually}\} = 1$$

$$P\{S_n \geq \phi(L_n) \quad \text{eventually}\} = 1$$

Proof of Lemma

A special case of Brown [1], Theorem (1), shows that if

(a) $$E\{f_n | B_{n-1}\} \leq C\, E\{f_n | B_{n-1}\}\ , \quad \text{some } C > 0, \quad \text{and}$$

$$\sum_1^\infty E\{f_n | B_{n-1}\} = \infty \quad \text{a.e.},$$

then

$$\frac{S_N}{R_N} = \frac{\sum_1^N f_n}{\sum_1^N E\{f_n | B_{n-1}\}} \to 1 \quad \text{a.e.}$$

This result clearly holds in the present context. Given this result and the hypotheses on the function ϕ, we have for all $k > 0$ and eventually on a set of measure one:

$$S_n \geq \frac{1}{2} R_n \geq \frac{\theta}{2(1+\theta)} T_n \geq \frac{k\theta}{2(1+\theta)} \phi(T_n) \ ; \quad \text{now let}$$

$$k = \frac{2(1+\theta)}{\theta} .$$

Next, since, from Freedman [5]

$$\frac{L_n}{T_n} \to 1 \quad \text{a.e.},$$

and since $\frac{\phi(s)}{\phi(t)} \to 1$ as $t \to \infty$, $\frac{s}{t} \to 1$, we have

$$S_n \geq \phi(L_n) \quad \text{eventually} .$$

Q.E.D.

Now define

$$g_n = \frac{E\{f_n | B_{n-1}\} - f_n}{2}$$

$$\tilde{S}_n = \sum_1^n g_k = \tfrac{1}{2}(R_n - S_n)$$

$$\tilde{T}_n = \sum_1^n E(g_k^2 | B_{k-1}) = \tfrac{1}{4} \sum_1^n \sigma^2(f_k | B_{k-1})$$

Then

$$e^{\lambda \tilde{S}_n - K(\lambda)\tilde{T}_n} \quad \text{is a}$$

positive supermartingale.

Define

$$\tau = \inf n \geq 0 \quad S_n \geq \phi(T_n)$$

Our lemma tells us that τ is proper, so that

$$1 \geq \int_{\{T_\tau \geq t\}} e^{\lambda \tilde{S}_\tau - K(\lambda)\tilde{T}_\tau}$$

$$\geq \int_{\{T_\tau \geq t\}} e^{-\frac{\lambda}{2}(S_\tau - R_\tau) - \frac{K(\lambda)}{4} T_\tau}$$

$$\geq \int_{\{T_\tau \geq t\}} e^{-\frac{\lambda}{2}((\phi(T_\tau)+1) - \frac{\theta}{1+\theta} T_\tau) - \frac{K(\lambda)}{4} T_\tau} = \int_{\{T_\tau \geq t\}} e^{-\frac{\lambda}{2} T_\tau \left\{\frac{\theta}{1+\theta} - \frac{K(\lambda)}{2\lambda} - \frac{(\phi(T_\tau)+1)}{T_\tau}\right\}}$$

Suppose

$$(*) \qquad \frac{K(\lambda)}{2\lambda} \le \frac{\theta}{1+\theta} - \frac{(\phi(t)+1)}{t}$$

Then the integrand increases in T_τ, so that

$$(**) \qquad P\{T_\tau \ge t\} \le e^{\frac{K(\phi)t}{4} + \frac{(\phi(t)+1)\lambda}{2} - \frac{\theta t\lambda}{2(1+\theta)}}$$

It is easily seen that the right side minimizes for

$$\lambda_0 = \ell n\{1 + \frac{2\theta}{1+\theta} - 2\,\frac{(\phi(t)+1)}{t}\}$$

We have this λ available to us if it satisfies (*). This follows immediately from

$$\lambda(e^\lambda - 1) \ge k(\lambda), \quad \text{valid for all} \quad \lambda \ge 0 \ .$$

Let's use λ_0 in (**) and also

$$(1 + x)\ell n(1 + x) \ge x + \frac{x^2}{2(1+x)}, \quad \text{all} \quad x \ge 0$$

to arrive at

$$P\{T_\tau \ge t\} \le \exp\left(- \frac{t}{1+2\left(\frac{\theta}{1+\theta} - \frac{(\phi(t)+1)}{t}\right)} \left(\frac{\theta}{1+\theta} - \frac{(\phi(t)+1)}{t}\right)^2\right)$$

Defining t_θ through

$$\frac{\phi(t_\theta)+1}{t_\theta} = \frac{\theta}{1+\theta}, \quad \text{we have}$$

$$P\{T_\tau \ge t\} \le e^{-\frac{t}{6}\left(\frac{\phi(t_\theta)}{t_\theta} - \frac{\phi(t)}{t}\right)^2}, \quad \text{all} \quad t \ge t_\theta$$

But then

$$\int T_\tau \le t_\theta + \int_{t_\theta}^{\infty} e^{-\frac{t}{6}\left(\frac{\phi(t_\theta)}{t_\theta} - \frac{\phi(t)}{t}\right)^2} dt$$

Now

$$(***)\qquad \int_{t_\theta}^{\infty} e^{-\frac{t}{6}\left(\frac{\phi(t_\theta)}{t_\theta} - \frac{\phi(t)}{t}\right)} dt \le \int_{t_\theta}^{2t_\theta} e^{-\frac{t}{6}\left(\frac{\phi(t_\theta)}{t_\theta} - \frac{\phi(t)}{t}\right)} dt$$

$$(**)\qquad + \int_{2t_\theta}^{\infty} e^{-\frac{t_\theta}{6}\left(\frac{\phi(t_\theta)}{t_\theta} - \frac{\phi(2t_\theta)}{2t_\theta}\right)^2} dt$$

Use the Mean Value Theorem on the exponent in the first integral on the right. One has

$$\int_{t_\theta}^{2t_\theta} e^{-\frac{t_\theta}{6}\left(\frac{\phi(t_\theta)}{t_\theta} - \frac{\phi(t)}{t}\right)^2} dt \le \int_{t_\theta}^{2t_\theta} e^{-\frac{(1-p(t_\theta))^2}{48}\,\frac{\phi^2(t_\theta)}{t_\theta^3}(t-t_\theta)^2}$$

Letting $t = (1+r)t_\theta$ in the integral on the right, and then letting

$$z = \frac{(1-p(t_\theta))}{\sqrt{24}}\,\frac{\phi(t_\theta)}{\sqrt{t_\theta}}\,s, \quad \text{we have}$$

$$\int_{t_\theta}^{2t_\theta} e^{-\frac{t_\theta}{6}\left(\frac{\phi(t_\theta)}{t_\theta} - \frac{\phi(t)}{t}\right)^2} dt \le \frac{\sqrt{24}}{(1-p(t\theta))}\,\frac{t_\theta^{3/2}}{\phi(t\ )} \int_0^{\frac{(1-p(t_\theta))}{\sqrt{24}}\frac{\phi(t_\theta)}{\sqrt{t_\theta}}} e^{-\frac{z^2}{2}} dz$$

$$\le \frac{\sqrt{12\pi}}{(1-p(t_\theta))}\,\frac{t_\theta^{3/2}}{\phi(t_\theta)}$$

As for the second integral on the right in (***), we have

$$\int_{2t_\theta}^{\infty} e^{-\frac{t}{6}\left(\frac{\phi(t_\theta)}{t_\theta}-\frac{\phi(2t_\theta)}{2t_\theta}\right)^2} dt \le \int_{2t_\theta}^{\infty} e^{-\frac{t}{12}(1-p(t_\theta))^2\frac{\phi^2(t_\theta)}{t_\theta^2}} dt$$

$$= \frac{12t_\theta^2}{(1-p(t_\theta))^2\phi^2(t_\theta)} \; e^{-\left(\frac{(1-p(t_\theta)}{6}\right)^2 \frac{\phi^2(t_\theta)}{t_\theta}}$$

$$\frac{1}{t_\theta}\int_{t_\theta}^{\infty} P\{T_\tau \ge t\}dt \le \frac{1}{(1-p(t_\theta))^2}\{\sqrt{3\pi} + 12\frac{\sqrt{t_\theta}}{\phi(t_\theta)}\}\frac{\sqrt{t_\theta}}{\phi(t_\theta)}$$

Thus

$$\frac{1}{t_\theta}\int T_\tau \le 1 + \frac{1}{(1-p(t\ ))^2}\{\sqrt{3\pi} + 12\frac{\sqrt{t_\theta}}{\phi(t_\theta)}\}\frac{\sqrt{t_\theta}}{\phi(t_\theta)}$$

We turn next to the consideration of upper bounds for

$$\frac{1}{t_\theta}\int L_\tau$$

where

$$\tau = \inf n \to S_n \ge \phi(L_n).$$

Arguring as before;

$$1 \ge \int_{\{L_\tau \ge t\}\ \{T_\tau \varepsilon[(1-\varepsilon)L_\tau,(1+\varepsilon)L_\tau]\}} e^{-\frac{\lambda}{2}(S_\tau - R_\tau) - \frac{K(\lambda)}{4}T_\tau}$$

For notational convenience, let the set in the integral above be denoted $A(t, \varepsilon)$. We have

$$1 \geq \int_{A(t,\varepsilon)} e^{-\frac{\lambda}{2} L_\tau \left\{(1-\varepsilon)\frac{\theta}{1+\theta} - \frac{K(\lambda)}{2\lambda}(1+\varepsilon) - \frac{(\phi(L_\tau)+1)}{L_\tau}\right\}}$$

Continuing to mimic the previous argument, we choose

$$\lambda = \ell n\left\{1 + \frac{2}{1+\varepsilon}\left(\frac{\theta}{1+\theta}(1-\varepsilon) - \left(\frac{\phi(t)+1}{t}\right)\right)\right\}$$

and we have

$$P\{A(t,\varepsilon)\} \leq \exp\left\{\frac{t}{2(1+\varepsilon)} \frac{\left(\frac{\theta(1-\varepsilon)}{1+\theta} - \left(\frac{\phi(t)+1)^2}{t}\right)\right)^2}{1 + \frac{2}{1+\varepsilon}\left(\frac{\theta(1-\varepsilon)}{1+\theta} - \left(\frac{\phi(t)+1}{t}\right)\right)}\right\}$$

We define t_ε through

$$\frac{\phi(t_\varepsilon)+1}{t_\varepsilon} = \frac{\theta(1-\varepsilon)}{1+\theta}, \quad \text{and we then}$$

have for $t \geq t_\varepsilon$:

$$P\{A(t,\varepsilon)\} \leq e^{-\frac{t}{6}\left\{\frac{\phi(t_\varepsilon)}{t_\varepsilon} - \frac{\phi(t)}{t}\right\}^2} .$$

But then

$$\frac{1}{t_\varepsilon}\int_{t_\varepsilon}^{\infty} P\{A(t,\varepsilon)\}dt \leq \frac{1}{(1-p(t_\varepsilon))^2}\left\{\sqrt{3\pi} + 12\frac{\sqrt{t_\varepsilon}}{\phi(t_\varepsilon)}\right\}\frac{\sqrt{t_\varepsilon}}{\phi(t_\varepsilon)}$$

Next, for $\varepsilon \le \frac{1}{2}$, and using Proposition (1.2);

$$\int_{t_\varepsilon}^{\infty} P\{T_\tau \notin [(1-\varepsilon)L_\tau,\ (1+\varepsilon)L_\tau],\ \ L_\tau \ge t\}dt \le$$

$$2\int_{t_\varepsilon}^{\infty} e^{-\frac{-2\varepsilon^2 t}{(1+2\varepsilon)^2}} dt \le \frac{4}{\varepsilon^2} e^{-\frac{\varepsilon^2}{2} t_\varepsilon}$$

$$\le \frac{4}{\varepsilon^2} e^{-\frac{\varepsilon^2}{2} t_\theta} \quad \text{since} \quad t_\theta \le t_\varepsilon .$$

We then have, again using $t_\theta \le t_\varepsilon$;

$$\frac{1}{t_\varepsilon}\int_{t_\varepsilon}^{\infty} P\{L_\tau \ge t\}dt \le \frac{4}{\varepsilon^2 t_\theta} e^{-\frac{\varepsilon^2}{2} t_\theta}$$

$$+ \frac{1}{(1-p(t_\theta))^2}\left\{\sqrt{3\pi} + 12\frac{\sqrt{t_\theta}}{\phi(t_\theta)}\right\}\frac{\sqrt{t_\theta}}{\phi(t_\theta)}$$

Now

$$\frac{t_\varepsilon}{t_\theta} \le \frac{1}{1-\varepsilon}\frac{\phi(t_\varepsilon)}{\phi(t_\theta)} \le \frac{1}{1-\varepsilon}\left\{1 + (t_\varepsilon - t_\theta)\ \frac{\phi'(t_\theta)}{t_\theta}\right\}$$

$$= \frac{1}{1-\varepsilon}\left\{1 + p(t_\theta)\left(\frac{t_\varepsilon}{t_\theta} - 1\right)\right\},$$

from which

$$\frac{t_\varepsilon}{t_\theta} \le 1 + \frac{\varepsilon}{1-p(t_\theta)-\varepsilon}$$

Replace ε with $(1-p(t_\theta))\varepsilon$, and we have

$$\frac{t_\varepsilon}{t_\theta} \le 1 + \frac{\varepsilon}{1-\varepsilon}$$

But then

$$\frac{1}{t_\theta}\int_{t_\theta}^{\infty} P\{L_\tau \ge t\}dt \le \left(\frac{t_\varepsilon}{t_\theta} - 1\right) + \frac{t_\varepsilon}{t_\theta} \cdot \frac{1}{t_\varepsilon}\int_{t_\varepsilon}^{\infty} P\{L_\tau \ge t\}dt$$

$$\le \frac{\varepsilon}{1-\varepsilon} + \frac{1}{1-\varepsilon}\,\frac{1}{t_\varepsilon}\int_{t_\varepsilon}^{\infty} P\{L_\tau \ge t\}dt$$

Setting $\varepsilon = \dfrac{\sqrt{t_\theta}}{\phi(t_\theta)}$, we have

$$\frac{1}{t_\theta}\int_{t_\theta}^{\infty} P\{L_\tau \ge t\}dt \le 16\,\frac{\phi^2(t_\theta)}{t_\theta^2} + \frac{1}{4(1-p(t_\theta))^2}\left\{1 + 8\left[\sqrt{3\pi} + 12\,\frac{\sqrt{t_\theta}}{\phi(t_\theta)}\right]\right\}\frac{\sqrt{t_\theta}}{\phi(t_\theta)}$$

$$\le \frac{4}{(1-p(t_\theta))^2}\;\frac{\phi^2(t_\theta)}{t_\theta^2} + \frac{1}{4(1-p(t_\theta))^2}\left\{1 + 8\left[\sqrt{3\pi} + 12\,\frac{\sqrt{t_\theta}}{\phi(t_\theta)}\right]\right\}\frac{\sqrt{t_\theta}}{\phi(t_\theta)}$$

$$\le 16\,\frac{\phi^2(t_\theta)}{t_\theta^2} + \frac{2}{(1-p(t_\theta)^2}\left\{3.2 + 12\,\frac{\sqrt{t_\theta}}{\phi(t_\theta)}\right\}\frac{\sqrt{t_\theta}}{\phi(t_\theta)}$$

Q.E.D.

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