ON TIME-SEQUENTIAL POINT ESTIMATION OF THE MEAN OF AN EXponential DISTRIBUTION

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ABSTRACT

In the context of life testing, an asymptotically risk-efficient time-sequential procedure for estimating the mean of an exponential distribution is considered and its various properties are studied.

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Key Words & Phrases: Asymptotic normality, asymptotic risk-efficiency, loss function, risk function, stopping number, stopping time, time-sequential procedure.

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1. INTRODUCTION

Let \( \{X_i, i \geq 1\} \) be a sequence of independent and identically distributed (i.i.d.) non-negative random variables (r.v.) with the distribution function (d.f.) \( F_\theta(x) = 1 - e^{-x/\theta} \), \( x \in [0, \infty) \), where \( \theta(>0) \) is an unknown parameter. For \( n(\geq 1) \), items under life testing, the failures \( X_{n,1}, \ldots, X_{n,n} \) are the order statistics corresponding to \( X_1, \ldots, X_n \) and, from cost and time considerations, one may curtail experimentation at the \( k^{th} \) failure \( X_{n,k} \) and estimate \( \theta \) by

\[
\hat{\theta}_{nk} = k^{-1} V_{nk}, \quad \text{where} \quad V_{nk} = \sum_{i=1}^{k} X_{n,i} + (n-k)X_{n,k} \quad \text{for} \quad 1 \leq k \leq n.
\]

Note that \( V_{nk} \) is the total life under test up to the \( k^{th} \) failure, \( EV_{nk} = k\theta \) and \( E(V_{nk} - k\theta)^2 = k\theta^2 \), for \( k = 1, \ldots, n \). Thus, if \( a_1(>0) \) and \( a_2(>0) \) be respectively the cost of recruitment (per individual) and of follow-up (per unit of test-life), then one may conceive of the loss incurred in estimating \( \theta \) by \( \hat{\theta}_{nk} \) as

\[
L_{nk} = a_0(\hat{\theta}_{nk} - \theta)^2 + a_1 n + a_2 V_{nk} \quad (1 \leq k \leq n),
\]

where the weights \( a_0(>0), a_1 \) and \( a_2 \) are all known. Thus, the risk in estimating \( \theta \) by \( \hat{\theta}_{nk} \) is

\[
R_{nk}(\alpha, \theta) = EL_{nk} = k^{-1} a_0^2 + a_1 n + a_2 k\theta \quad (\alpha = (a_0, a_1, a_2)'),
\]

and, naturally, one would seek to minimize (1.3) by a proper choice of \( k \). However, as \( \theta \) is unknown, no single value of \( k \) minimizes \( R_{nk}(\alpha, \theta) \) for all \( \theta(>0) \), and hence, a time-sequential procedure for choosing such a value of \( k \) is desirable.

Motivated by the works of Robbins (1959), Starr and Woodroofe (1972) and Ghosh and Mukhopadhyay (1979) [all dealing with the classical
sequential point estimation case], in Section 2, we formulate a time-sequential procedure for our problem and under asymptotic setup (similar to their cases) study its various properties. The derivation of the main results are postponed to concluding section.

2. **TIME-SEQUENTIAL POINT ESTIMATION OF $$\theta$$**

Note that by (1.3),

$$R_{n k}^{(a)}(a, \theta) - R_{n k+1}^{(a)}(a, \theta) \geq 0$$ according as $$k(k+1) \leq \frac{\theta a_0}{a_2}$$.

Thus, if $$n(n-1) < \frac{\theta a_0}{a_2}$$, then $$R_{n k}(a, \theta)$$ is \(\downarrow\) in $$k(1 \leq k \leq n)$$, and hence, $$k = n$$ is an optimal choice. On the other hand, if $$n(n-1) \geq \frac{\theta a_0}{a_2}$$, then there exists an optimal $$k_n = k_n(a, \theta)$$ for which $$k_n < n$$ and $$R_{n k}(a, \theta)$$ is minimized for $$k = k_n$$. Since $$\hat{\theta}_{n k} = k^{-1}V_{n k}$$ is an unbiased estimator of $$\theta$$, motivated by the above, we consider the following stopping number

$$N_n = N_n(a) = \begin{cases} \text{smallest } k(1 \leq k \leq n-1) \text{ for which } V_{n k} \leq k^2(k+1)a_2/a_0, \\ n \text{ if } V_{n k} > k^2(k+1)a_2/a_0, \text{ for every } 1 \leq k \leq n - 1. \end{cases}$$

The corresponding stopping time is $$X_{n N_n}$$ and the point estimator of $$\theta$$ is $$\hat{\theta}_{n N_n}$$. Then, the risk corresponding to $$\hat{\theta}_{n N_n}$$ is

$$R_n(a, \theta) = a_0 E(\hat{\theta}_{n N_n} - \theta)^2 + a_1 n + a_2 E V_{n N_n}.$$  

We may recall that by definition,

$$k_n = k_n(a, \theta) = \begin{cases} \text{smallest } k(1 \leq k \leq n-1) \text{ for which } k(k+1) \leq \frac{\theta a_0}{a_2}, \\ n \text{ if } n(n-1) < \frac{\theta a_0}{a_2}. \end{cases}$$

Let then

$$R_n^{0}(a, \theta) = R_{n k_n}(a, \theta).$$

Our primary interest centers around the behavior of $$(a) N_n/k_n$$ and
(b) $R^*_n(\theta, \theta)/R^0_n(\theta, \theta)$ when we impose some asymptotic considerations on $\theta$ and $n$.

In the classical sequential point estimation theory [c.f. Robbins (1959) and others], $a_2 = 0$, $L_n = a_0(\hat{\theta}_{nn} - \theta)^2 + a_1 n$ and the problem is to choose $n$ in such a way that the corresponding risk is minimized. In this context, one lets $a_1 \to 0$ and, in this asymptotic sense, one obtains some optimal results. In our case, however, for a given $n$, the stopping number $N_n$ depends on $a_0$ and $a_2$, but not on $a_1$, and we let $a_2/a_0 \to 0$ or, simply, $a_2 \to 0$, keeping $a_0$ fixed. Note that our main interest lies in the case where $k_n$ in (2.4) is $< n$ and in this case, $a_2^2 a_2 n(1 - a_2^2) \geq 0a_0 > 0$. We assume that the sample size $n = n(a_2)$ depends on $a_2$ in such a way that

$$\lim_{a_2 \to 0} a_2[n(a_2)]^2 = a^*; \quad 0 < a^* < \infty.$$  

(2.6)

We may note that by (1.3), $R_n^{(k)}(\theta, \theta) = R_n^{(k)}(\theta, \theta) + a_1(n' - n) \geq R_n(\theta, \theta), \forall n' > n$, and hence, there is no point in increasing $n(a_2)$ indefinitely even when we allow $a_2 \to 0$, so that the restriction that $a^*$ in (2.6) is $< \infty$ is of no loss of generality. Secondly, we note that for $\{n\}$ satisfying (2.6), by (2.4),

$$\lim_{n \to \infty} k_n/n = \gamma = (\theta a_0/a_2)^{1/2}$$  

(2.7)  
and we assume that $0 < \gamma < 1$.

In terms of (2.6), (2.7) demands that $a^* > \theta a_0$. Finally, as in the classical sequential point estimation case, we assume that $a_1 \to 0$.

More explicitly, we let

$$a_1 = \rho a_2,$$  

(2.8)  
where $\rho > 0$, and allow $a_2 \to 0$.

Then, we have the following
Theorem 1. Under (2.6) and (2.7),

\[(2.9) \quad \frac{N_n}{k_n} \to 1 \text{ almost surely (a.s.) as } a_2 \to 0.\]

Moreover, for every real \(x (-\infty < x < \infty)\), under (2.6) and (2.7),

\[(2.10) \quad \lim_{n \to \infty} \Pr(2(N_n - k_n)/((n\gamma)^{1/2}) \leq x) = (2\pi)^{-1/2} \int_{-\infty}^{x} e^{-t^2/2} dt.\]

Theorem 2. Under (2.6), (2.7) and (2.8),

\[(2.11) \quad \lim_{a_2 \to 0} \frac{R_n^*(g, \theta)}{R_n^0(g, \theta)} = 1.\]

We may remark that by (2.2), \(N_n = N_n(g)\) is \(\downarrow\) in \(a_2\) and for any given \(n\), there exists an \(a_2(n)(>0)\), such that \(N_n = n, \forall 0 < a_2 \leq a_2(n)\). Also, \(N_n \leq n\), with probability 1, so that (2.7) and (2.9) insure that \(EN_n/k_n \to 1\) as \(n \to \infty\). Further, (2.11) holds even when in (2.8), \(\rho = 0\). If \(a_1/a_2 \to \infty\) as \(a_2 \to 0\), then \(R_n^*\) or \(R_n^0\) are both dominated by \(a_1n\), and hence, (2.11) holds trivially.

3. PROOFS OF THEOREMS 1 AND 2

Let us denote by \(V_{n0} = 0\) and

\[(3.1) \quad Z_{nk} = V_{nk} - V_{nk-1} = (n-k+1)(X_{n,k} - X_{n,k-1}), \quad 1 \leq k \leq n\]

(where \(X_{n,0} = 0\)). Then \(Z_{n1}, \ldots, Z_{nn}\) are i.i.d.r.v. each having the d.f. \(F_\theta(x) = 1 - e^{-x/\theta}\). Also, note that for every \(n(\geq 1),\)

\[(3.2) \quad V_{nk} \text{ is } \downarrow \text{ in } k: 0 \leq k \leq n.\]

Further, note that for every \(\eta > 0,\)

\[(3.3) \quad \Pr(X_{m,1} < m^{-1-\eta}, \text{ for some } m \geq n) \leq \sum_{k=0}^{\infty} \Pr(X_{m,1} < (n2^{k+j})^{-1-\eta}, \text{ for some } 0 \leq j \leq n2^k) \leq \sum_{k=0}^{\infty} \Pr(X_{m,1} < (2^{kn})^{-1-\eta}) = \sum_{k=0}^{\infty} \Pr(1 - e^{-2(2^k n)^{-\eta}}) \leq \sum_{k=0}^{\infty} (2(n2^k)^{-\eta}) = 2n^{-\eta}(1 - 2^{-\eta})^{-1} \to 0 \text{ as } n \to \infty.\]
Thus, by (3.1) and (3.3), for every \( \eta > 0 \),
\[
V_{n1} > \eta^{-n} \quad \text{a.s., as } n \to \infty.
\]
Let us now choose a positive number \( \lambda \) such that
\[
\frac{1}{2} < \lambda < \frac{2}{3}, \quad \text{i.e., } \xi = \frac{2}{3} - \lambda > 0.
\]
Then, under (2.6) and (2.7), by using (2.4) and (3.2),
\[
P\{N_m \leq m^\lambda \quad \text{for some } m \geq n\}
\leq P\{ U \sum_{m \geq n} [V_{mk} \leq k^2(k+1)a_2/a_0, \text{ for some } k \leq m^\lambda] \}
\leq P\{ U \sum_{m \geq n} [V_{m1} \leq (a_2m^2/a_0)(m^2(m^\lambda + 1)/m^2)] \},
\]
where \( a_2m^2/a_0 \to a^*/a_0 (> 0) \) while by (3.5), \( m^2(m^\lambda + 1)/m^2 \sim m^{-\xi} \), so that by (3.4), the right-hand side (rhs) of (3.6) converges to 0 as \( n \to \infty \). Let us now denote by
\[
k_n^{(1)} = \max\{k: k(k+1) \leq (1-\varepsilon)k_n(k_n+1)\}, \quad 0 < \varepsilon < 1,
\]
where \( k_n \) is defined by (2.4). Also, we choose \( n \) so large that \( n^\lambda \leq k_n^{(1)} \). Then
\[
P\{N_m \leq k_n^{(1)} \quad \text{for some } m \geq n\}
\leq P\{N_m \leq m^\lambda \quad \text{for some } m \geq n\} +
P\{ U \sum_{m \geq n} [V_{mk} \leq k^2(k+1)a_2/a_0, \text{ for some } k: m^\lambda \leq k \leq k_m^{(1)}] \}.
\]
By (3.6) the first term on the rhs of (3.3) converges to 0 as \( n \to \infty \), while by (3.7), the second term is bounded by
\[
\sum_{m \geq n} P\{(V_{nk} - k\theta)/\theta < -\eta, \text{ for some } k: m^\lambda \leq k \leq k_m^{(1)}\},
\]
where \( \eta(> 0) \) depends on \( \varepsilon(> 0) \) in (3.7). By (3.1), for every \( n(\varepsilon > 1) \), \( \{(V_{nk} - k\theta)/\theta, \ 0 \leq k \leq n\} \) is a martingale, so that \( \{(V_{nk} - k\theta)^4/\theta^4, \ 0 \leq k \leq n\} \)
is a sub-martingale, and hence, by the Chow (1961) extension of the H"{a}jek-R"{e}nyi inequality, 

\begin{equation}
(3.10) \quad P\{ (V_{mk} - k\theta) / k\theta < -\eta, \text{ for some } m^\lambda \leq k \leq m_c \} \\
\leq P\{ (V_{mk} - k\theta)^4 / k^4 \theta^4 > \eta^4 \text{ for some } k: m^\lambda \leq k \leq m_c \} \\
\leq \sum_{k = [m^\lambda]}^{m_c} \eta^{-4} E(V_{mk} - k\theta)^4 / \theta^4 \left\{ k^{-4} - (k + 1)^{-4} \right\} \\
\leq \eta^{-4} \sum_{k \geq [m^\lambda]} \lfloor 0(k^{-4}) \rfloor = \eta^{-4} \cdot (m^{-2\lambda}),
\end{equation}

so that by (3.5) and (3.10), the second term on the rhs of (3.8) converges to 0 as $n \to \infty$. Thus, for every $\varepsilon > 0$,

\begin{equation}
(3.11) \quad N_n / k_n > 1 - \varepsilon \text{ a.s., as } n \to \infty.
\end{equation}

In a similar way, it follows that for every $\varepsilon > 0$,

\begin{equation}
(3.12) \quad N_n / k_n < 1 + \varepsilon \text{ a.s., as } n \to \infty,
\end{equation}

and (2.9) follows from (3.11) and (3.12).

To prove (2.10), we note that for every (fixed) $u \in (-\infty, \infty)$,

\begin{equation}
(3.13) \quad P\{ N_n \geq k_n + uv_n \} = P\{ V_{nk} > k^2(k+1)a_2/a_0, \forall k \leq k_n + uv_n \},
\end{equation}

and we choose $n$ so large that $k_n + uv_n > k_{n\varepsilon}$, where $k_{n\varepsilon}$ is defined by (3.7) and $k_n$ by (2.4). Then, by using (3.11), the rhs of (3.13) can be written as

\begin{equation}
(3.14) \quad P\{ V_{nk} > k^2(k+1)a_2/a_0, \forall k_{n\varepsilon} \leq k \leq k_n + uv_n \} + o(1)
\end{equation}

\[ = P\left\{ \frac{V_{nk} - k\theta}{\theta\sqrt{n}} > \frac{k}{\sqrt{n}} \left[ \frac{k(k+1)}{k_n(k_n+1)} - 1 \right], \forall k_{n\varepsilon} \leq k \leq k_n + uv_n \right\} + o(1). \]

Let us now consider a sequence $\{ W_n(t), t \in [0,1] \}$ of stochastic processes, where we let $W_n(t) = W_n(k_n(t))$, for $\frac{k}{n} \leq t < \frac{k+1}{n}$, $0 \leq k \leq n-1$ and $W_n(k/n) = (V_{nk} - k\theta) / \theta\sqrt{n}$, $k = 0, 1, \ldots, n$. Then by virtue of (3.1), the classical Donsker Theorem applies and we have

\begin{equation}
(3.15) \quad W_n \overset{D}{\to} W = \{ W(t), t \in [0,1] \},
\end{equation}
where \( W \) is a standard Wiener process on \([0, 1]\). As a corollary to (3.15), we have that for every \( \varepsilon' > 0 \) and \( \eta > 0 \) there exist a \( \delta : 0 < \delta < 1 \) and an \( n_0 \) such that

\[
(3.16) \quad P\{\sup\{|W_n(t) - W_n(s)| : 0 \leq s < t \leq s + \delta \leq 1\} > \varepsilon'\} < \eta, \quad \forall \ n \geq n_0.
\]

To make use of (3.15) and (3.16) in (3.14), we note that for \( k = k_n + [u\sqrt{n}] \)

\[n^{k}k(k+1)/n_k(k_n + 1) - 1] \sim 2u \text{.}
\]

Thus, the rhs of (3.14) can be expressed as

\[
(3.17) \quad P\left\{ \frac{V_{nk - k\theta}}{\sqrt{n} \theta} > \frac{k}{\sqrt{n} \theta} \left[ \frac{k(k+1)}{k_n(k_n + 1) - 1} \right] , \forall \ k \in \mathbb{N}, \ W_n(\gamma) > \frac{2u - \varepsilon}{\theta} \right\} + \]

\[
P\left\{ \frac{V_{nk - k\theta}}{\sqrt{n} \theta} > \frac{k}{\sqrt{n} \theta} \left[ \frac{k(k+1)}{k_n(k_n + 1) - 1} \right] , \forall \ k \in \mathbb{N}, \ W_n(\gamma) = \frac{2u - \varepsilon}{\theta} \right\} + o(1),
\]

where \( \varepsilon > 0 \). The second term is bounded by

\[P\{W_n(n^{-1}k_n + un^{-1/2}) - W_n(\gamma) > \varepsilon/0\}\]

and, by (3.16), it converges to 0 as \( n \to \infty \) (or \( a_2 \to 0 \)). Similarly,

the first term is convergent-equivalent to

\[
(3.18) \quad P\{W_n(\gamma) > (2u - \varepsilon)/\theta\} = P\{W(\gamma) > (2u - \varepsilon)/\theta\}
\]

\[= P\{W(1) > (2u - \varepsilon)/\theta \sqrt{n}\} = \left(2\pi\right)^{-1/2} \int_{(2u - \varepsilon)/\theta \sqrt{n}}^{\infty} \exp\left(-\frac{1}{2} t^2\right) dt.
\]

Thus, (2.10) follows from (3.13), (3.14), (3.17) and (3.18) by letting

\[u = \theta \sqrt{n}/2 \quad \text{and} \quad \varepsilon \to 0 .
\]

This completes the proof of Theorem 1.

To prove Theorem 2, we first note that under (2.6), (2.7) and (2.8),

\[
(3.19) \quad \left( a_n^* / a_2 \right)^{k^0_n} \sim \theta_n^0(a_2, \theta), \quad \gamma^{a_n^0} / \gamma + pa_* + a_*^\gamma \theta
\]

Also, recalling that \( n^{-1}N_n \leq 1 \), with probability 1, we have by (2.9),

\[
(3.20) \quad \lim_{n \to \infty} \text{E}(n^{-1}N_n)^m = \gamma^m(< 1), \quad \forall \ m = 1, 2, \ldots
\]

Further, by (2.2), (3.1), and the fact that \( Z_n \) is \( \geq 0 \), \( \forall \ k \geq 1 \), we have

\[
(3.21) \quad N_n(N_n - 1)^2 a_2 / a_0 < V_n N_n^{-1} < V_{n, n} \leq N_n^2(N_n + 1) a_0 a_2 I(n < n) + V_{n, n} I(n = n).
\]
Note that by (2.6) and (2.7),

\begin{equation}
(3.22) \quad P\{N_n = n\} = P\{V_{nk} > k^2 (k+1)a_2 / a_0, \quad \forall \; 1 \leq k \leq n - 1\}
\end{equation}

\begin{equation}
\leq P\{V_{nn-1} > (n-1)^2 a_2 / a_0\} = P\{V_{nn-1} - (n-1)\theta > (n-1)[n(n-1)a_2 / a_0 - \theta]\}
\end{equation}

\begin{equation}
\leq \theta^2 / (n-1) [n(n-1)a_2 / a_0 - \theta]^2 \sim \theta^2 / [(n-1)[a^*/a_0 - \theta]^2] = o(n^{-1}).
\end{equation}

Also, \(E(V_{nn}^2) = n(n+1)\theta^2\), so that by the Schwarz inequality and (3.22),

\begin{equation}
(3.23) \quad \left| E\{V_{nn} I(N_n = n)\} \right| \leq \theta^2 \sqrt{n(n+1)} \{o(n^{-\frac{1}{2}})\} = o(n^{\frac{1}{2}}) = o(a_2^{-\frac{1}{2}}), \quad \text{by (2.6)}.
\end{equation}

Further, by (3.21)

\begin{equation}
(3.24) \quad a_2 |V_{nn} - a_0^{-1} a_2 N_n^3| \leq \frac{a_2}{a_0} 2N_n^2 \{I(N_n < n) + a_2 |V_{nn} - a_0^{-1} a_2 N_n^3| I(N_n = n)\},
\end{equation}

where by (2.6), \(a_0^{-1} a_2 N_n^3 \to a_0^{-1} \sqrt{a_2}(a^*)^{\frac{3}{2}}\), while \(a_2^2 N_n^2 I(N_n < n) \leq a_2 n^2 \to a_2 a^*\).

Hence, by (3.22), (3.23) and (3.24), we have

\begin{equation}
(3.25) \quad (a^*/a_2)^{\frac{1}{2}} E\{a_2 |V_{nn} - a_0^{-1} a_2 N_n^3|\} = o(a_2^{\frac{1}{2}}) \to 0 \quad \text{as} \quad a_2 \to 0.
\end{equation}

On the other hand, by (3.20) and (2.4) - (2.7),

\begin{equation}
(3.26) \quad (a^*/a_2)^{\frac{1}{2}} E(a_2^2 N_n^3) / a_0 \to \gamma a^* \quad \text{as} \quad a_2 \to 0.
\end{equation}

Also, by (2.8),

\begin{equation}
(3.27) \quad (a^*/a_2)^{\frac{1}{2}} a_1 n \to \rho a^*, \quad \text{as} \quad a_2 \to 0.
\end{equation}

Hence, by (2.3), (3.19), (3.26) and (3.27), for (2.11), it suffices to show that

\begin{equation}
(3.28) \quad \lim_{a_2 \to 0} (a^*/a_2)^{\frac{1}{2}} E(N_n^{-1} V_{nn} - \theta)^2 = \theta^2 / \gamma.
\end{equation}

Note that

\begin{equation}
(3.29) \quad (a^*/a_2)^{\frac{1}{2}} E(N_n^{-1} V_{nn} - \theta)^2 = (a^*/a_2)^{\frac{1}{2}} k_n^{-2} E(V_{nn} - \theta N_n^2)^2 +
\end{equation}

\begin{equation}
(a^*/a_2)^{\frac{1}{2}} k_n^{-2} E\{V_{nn} - \theta N_n^2\}^2 (k_n / N_n)^2 - 1\}.
\end{equation}

Now, for every \(n(\geq 1), \{V_{nk} - k\theta = \sum_{j=1}^{k} (Z_{nj} - \theta), \; 1 \leq k \leq n\} \) is a martingale,
\[ E(z_{nj} - \theta)^2 = \theta^2 \] and \( \frac{\text{EN}}{n} < \infty \). Hence, by the Wald second lemma [viz. Theorem 2 of Chow, Robbins and Teicher (1965)], we have \( E(V_{nN_n} - \Theta_{n})^2 = \theta^2 \frac{\text{EN}}{n} \), so that by (2.6), (2.7) and (3.20), the first term on the rhs of (3.29) is equal to

\[ (a^*/a_2) \frac{k}{n^2} \theta^2 \frac{\text{EN}}{n} \theta^2 / \gamma \quad \text{as} \quad a_2 \to 0. \]

Thus, we need to show that the second term on the rhs of (3.29) converges to 0 as \( a_2 \to 0 \). Now, by the same technique as in (3.24) - (3.25), it follows that

\[ (a^*/a_2) \frac{k}{n^2} E\{ |V_{nN_n} - a_0^{-1} a_1 N_n^{-1/2} | (k_n/N_n)^2 - 1 | \}
= 0(n^{-1}) = 0(\sqrt{a_2}) \to 0 \quad \text{as} \quad a_2 \to 0. \]

On the other hand, by (2.4), (2.6) and (2.7),

\[ (a^*/a_2) \frac{k}{n^2} E\{ |a_0^{-1} a_2 N_n^{-1} - N_n \theta | (k_n/N_n)^2 - 1 | \}
= \sqrt{a^*} a_2^{3/2} \frac{k}{n^2} E\{ (N_n^2 - \theta a_0/a_2)^2 | 1 - (N_n/k_n)^2 | \}
= \sqrt{a^*} a_2^{3/2} \frac{k}{n^2} E\{ (N_n^2 - k_n^2)^2 | 1 - (N_n/k_n)^2 | \} + O(\sqrt{a_2})
= \sqrt{a^*} a_2^{3/2} \frac{k}{n^2} E\{ N_n^2 - k_n^2 | 1 - k_n^2 |^3 \} + O(\sqrt{a_2})
\leq \sqrt{a^*} a_2^{3/2} 8n^{-2} k_n^{-2} E|N_n - k_n|^3 + O(\sqrt{a_2}) . \]

Thus, it suffices to show (by virtue of (2.6) - (2.7)) that

\[ \lim_{n \to \infty} \frac{k}{n^2} E|N_n - k_n|^3 = 0 \quad \text{as} \quad a_2^{3/2} n^3 \to (a^*)^{3/2} . \]

Define \( \lambda \) as in (3.5). Then

\[ k_n^{-2} E|N_n - k_n|^3 = k_n^{-2} E(|N_n - k_n|^3 I_{|N_n - k_n| \leq \lambda}) + k_n^{-2} E(|N_n - k_n|^3 I_{|N_n - k_n| > \lambda}), \]

where by (2.7), the first term on the rhs of (3.34) converges to 0 as \( n \to \infty \). On the other hand, the second term is bounded by

\[ n^3 k_n^{-2} P(|N_n - k_n| > \lambda) . \]
Let $b_1 = 1/3 - \epsilon, \epsilon > 0$. Then

$$
(3.36) \quad P(N_n \leq b_1) = P(V_{nk} \leq k^2(k+1)a_2/a_0 \text{ for some } k \leq b_1)
\leq P(V_{n1} \leq \frac{a_2}{a_0} \cdot \frac{2b_1}{b_1} \cdot (n_1+1)) = P(V_{n1} \leq 0(n^{-2+3b_1}))
= 0(n^{-2+3b_1}) = 0(n^{-1-3\epsilon}).
$$

Also, let $k_{n\epsilon}$ be defined as in (3.7). Then proceeding as in (3.10) but using the 8th order moment of $(V_{nk} - k\theta)$, we obtain that

$$
(3.37) \quad P(n_1 \leq N_n < k_{n\epsilon}) = 0([n_1]^{-4}) = 0(n^{-4/3-4\epsilon}).
$$

Finally, let $k^* = k_n - n^\lambda$ and assume $n$ so large that $k^* > k_{n\epsilon}$. Then

$$
(3.38) \quad P\{k_{n\epsilon} < N_n \leq k^*\} \leq \frac{k^*}{k_{n\epsilon}} P\{\frac{1}{k} (V_{nk} - k\theta) < 0(k(k+1) - 1)\}
\leq \frac{k^*}{k_{n\epsilon}} \sum_{k=k_{n\epsilon}}^{k^*} k^{-2r} E(V_{nk} - k\theta)^{2r} / 0^{2r} [k(k+1) - 1]^{2r},
$$

for any $r > 0$. Now,

$$
(3.39) \quad E(V_{nk} - k\theta)^{2r} = 0(k^r), \text{ for every } r = 2, 3, 4, \ldots.
$$

Also, for $k_{n\epsilon} \leq k \leq k^*$,

$$
(3.40) \quad |k(k+1)/k_n (k_n + 1) - 1| = 0(\frac{k_n - k}{k_n}),
$$

so that the rhs of (3.38) is

$$
(3.41) \quad 0(\sum_{k=k_{n\epsilon}}^{k^*} k^{-r} k_n^{-2r} (k_n - k)^{-2r})
= 0(n^r) \sum_{k=k_{n\epsilon}}^{k^*} (k_n - k)^{-2r}
= 0(n^r) \cdot 0(n^{-\lambda(2r-1)}) = 0(n^{-(2\lambda-1)r+\lambda}).
$$

Since (3.38) and (3.39) hold for every positive integer $r$ and $\lambda > \frac{1}{2}$, we may choose $r$ so large that $(2\lambda - 1)r - \lambda > 1$, and this leads to the rhs of (3.41) as $0(n^{-1-\eta})$, for some $\eta > 0$. A similar treatment holds for the case of $N_n \geq k_n + n^\lambda$. Thus, $P\{|N_n - k_n| > n^\lambda\} = 0(n^{-1-\eta})$ for some $\eta > 0$, and this proves that (3.35) converges to 0 as $n \to \infty$. Q.E.D.
REFERENCES


