SIMULTANEOUS M-ESTIMATOR OF THE COMMON LOCATION AND THE SCALE-RATIO IN THE TWO-SAMPLE PROBLEM

by

Jana Jurečková and Pranab Kumar Sen

Department of Biostatistics
University of North Carolina at Chapel Hill

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JANA JUREČKOVÁ[1] and PRANAB KUMAR SEN[2]

Summary. For the two-sample problem with the common location and possibly different scale parameters, a general class of M-estimators for the common location and the scale-ratio is considered. Along with a general linearity theorem, the asymptotic distribution of the estimators is derived. An algorithm for the computation of the estimators is also suggested.

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[1] Charles University, Department of Probability & Statistics, Sokolovská 83, 18600 Prague 8, Czechoslovakia

[2] University of North Carolina, Department of Biostatistics, School of Public Health, 201H. Chapel Hill, N.C. 27514. U.S.A.

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1. Introduction

Let $X_1,\ldots, X_m$ be m independent and identically distributed random variables (i.i.d.r.v.), distributed according to a continuous distribution function (d.f.) F_1, defined on the real line $R = (-\infty, \infty)$. Also, let $Y_1,\ldots, Y_n$ be an independent set of i.i.d.r.v.'s with a continuous d.f. F_2, defined on R. It is assumed that

$$F_1(x) = F((x - \mu)/\nu) \text{ and } F_2(x) = F(x - \mu), \; x \in R \tag{1.1}$$

where F is an unspecified d.f. (symmetric about the origin), μ is the *common location* and ν is the *ratio of the scale parameters* of the two d.f.'s. We are interested in *robust estimation* of (μ, ν).

Some attempts have been made (mostly, in the one-sample case) to provide simultaneous M-estimators of location and scale parameters [viz., Huber (1964), Maronna (1976) and Carroll (1978), where other references are cited]. However, unlike the location case, where the parameter is uniquely defined by restricting to symmetric distributions, there is no natural and unique scale parameter over a large class of distributions. In the two-sample case, if both the distributions have the same shape, the ratio of the scale parameters can be defined uniquely and we may try as well to estimate the same. The model (1.1) is not uncommon in practice. For example, in combining the estimators of location from two independent samples, we may be aware of the differences in the variability of the observations (due to different instruments/investigators or other plausible sources of variations), and hence, it may be more meaningful to eliminate the same. Another point in which our approach differs from the previous ones, mentioned before, is that we use a (rather natural) *reparameterization*

$$(\mu, \nu)' \to \underset{\sim}{\theta} = (\theta_1, \theta_2)' = (\mu, \log \nu)' \tag{1.2}$$

and suggest a general class of extimators of $\underset{\sim}{\theta}$. This reparameterization enables us to prove a basic *asymptotic linearity theorem* pertaining to the M-estimators of $\underset{\sim}{\theta}$, which is incorporated in the study of the asymptotic distribution of these estimators. This, in turn, provides the asymptotic distribution of the estimators of the original parameter (μ, ν).

Along with the preliminary notions, the proposed M-estimators are introduced in Section 2. The basic linearity theorem is formulated in Section 3. Asymptotic distribution theory of the estimators is then considered in Section 4. In the concluding section, the results of Section 3 are utilized in providing an algorithm for (an iterative) conputation of the estimators.

2. The proposed estimators

Referred to the model (1.1), we assume that the d.f. F has an absolutely continuous density function f which has finite Fisher's information with respect to both location and scale, i.e.,

$$I(f) = \int_R (f'(x)/f(x))^2 dF(x) < \infty \tag{2.1}$$

and

$$I_1(f) = -1 + \int_R x^2 (f'(x)/f(x))^2 dF(x) < \infty \tag{2.2}$$

where $f'(x) = (d/dx)f(x)$. Note that by the assumed symmetry of F,

$$f(-x) = f(x) \text{ and } F(x) + F(-x) = 1, \ \forall\ x \in R . \tag{2.3}$$

Let $N = m + n$, $\lambda_N = m/N$ and assume that

$$\lim_{N\to\infty} \lambda_N = \lambda \text{ exists, where } 0 < \lambda < 1 . \tag{2.4}$$

Let $\psi(x)$: $R \to R$ be a non-constant, non-decreasing, skew-symmetric and absolutely continuous function, such that

$$\int_R \psi^2(x)dF(x) < \infty \text{ and } \int_R x^2\psi^2(x)dF(x) < \infty . \tag{2.5}$$

Moreover, let $\xi(x)$: $R \to R$ be a symmetric, non-constant, absolutely continuous and convex function, such that

$$\int_R x\xi'(x)dF(x) < \infty \text{ and } \int_R \xi^2(x)dF(x) < \infty . \tag{2.6}$$

Note that $\psi(-x) = -\psi(x)$ and $\xi(-x) = \xi(x)$, $\forall\, x \in R$, so that by (2.3), (2.5) and (2.6),

$$\int_R \psi(x)dF(x) = 0 \text{ and } \int_R x\xi(x)dF(x) = 0 . \tag{2.7}$$

Denote by $||\underset{\sim}{x}|| = \max\{|x_1|,|x_2|\}$, $\underset{\sim}{x} \in R^2$. Let $\underset{\sim}{\tilde{\theta}}_N$ be any estimator of $\underset{\sim}{\theta}$, such that

$$N^{1/2}||\underset{\sim}{\tilde{\theta}}_N - \underset{\sim}{\theta}|| = O_p(1) . \tag{2.8}$$

[For example, if Z_N be the combined sample median and if $\hat{\nu}_m^{(1)}$ and $\hat{\nu}_n^{(2)}$ be, respectively, the first and second sample inter-quartile ranges, then, we may set

$$\underset{\sim}{\tilde{\theta}}_N = (Z_N, \log(\hat{\nu}_m^{(1)}/\hat{\nu}_n^{(2)}))' . \tag{2.9}$$

If F has a positive and continuous density at the population median and the quartiles, then (2.8) holds for (2.9).] Consider the system of equations:

$$\left.\begin{aligned} e^{-t_2}\sum_{i=1}^{m}\psi((X_i - t_1)/e^{t_2}) + \sum_{j=1}^{n}\psi(Y_j - t_1) = 0 \\ \frac{1}{m}\sum_{i=1}^{m}\xi((X_i - t_1)/e^{t_2}) - \frac{1}{n}\sum_{j=1}^{n}\xi(Y_j - t_1) = 0 \end{aligned}\right\} \tag{2.10}$$

The estimator $\underset{\sim}{T}_N = (T_N^{(1)}, T_N^{(2)})'$ of $\underset{\sim}{\theta} = (\theta_1, \theta_2)'$ is defined as the root of the system (2.10) nearest to $\underset{\sim}{\tilde{\theta}}_N$ in the sense of $||\cdot||$ (and that with the larger components if there are more roots equally distant from $\underset{\sim}{\tilde{\theta}}_N$). For definiteness, let $\underset{\sim}{T}_N = \underset{\sim}{0}$ if there is no root.

In particular, if we put $\psi(x) \equiv -(f'(x)/f(x))$ and $\xi(x) = -1 - x(f'(x)/f(x))$, $x \in R$, then (2.10) reduces to the system of likelihood equations corresponding to the situation of F being known. In our setup, by an appropriate choice of ψ and ξ, we provide a class of estimators with finite asymptotic variances over a general class of F's.

On the other hand, like the likelihood equations, (2.10) has to be generally solved by an iterative procedure. This is presented in Section 5. In this context and for the study of the asymptotic properties of $N^{1/2}(\underset{\sim}{T}_N - \underset{\sim}{\theta})$, a linearity theorem, considered first in Section 3, plays a fundamental role. The theorem may have also other interesting applications.

3. A basic linearity theorem

For every $\underset{\sim}{t} = (t_1, t_2)' \in R^2$, we define $M_N(\underset{\sim}{t}) = (M_{N1}(\underset{\sim}{t}), M_{N2}(\underset{\sim}{t}))'$ by letting

$$M_{N1}(\underset{\sim}{t}) = N^{-1/2}\left\{e^{-t_2}\sum_{i=1}^{m}\psi\left(\frac{X_i - t_1}{e^{t_2}}\right) + \sum_{j=1}^{n}\psi(Y_j - t_1)\right\} \tag{3.1}$$

$$M_{N2}(t) = N^{1/2}\left\{\frac{1}{m}\sum_{i=1}^{m}\xi\left(\frac{X_i - t_1}{e^{t_2}}\right) - \frac{1}{n}\sum_{j=1}^{n}\xi(Y_j - t_1)\right\} \tag{3.2}$$

Also, let

$$\begin{aligned}\gamma_1 = \gamma_1(\psi, F) &= \int\psi(x)(-f'(x)/f(x))dF(x) \\ &= \int\psi'(x)dF(x)\end{aligned} \tag{3.3}$$

$$\gamma_2 = \gamma_2(\xi,\ F) = \int x\xi'(x)dF(x)\ . \tag{3.4}$$

Note that by (2.5) and by (2.6), both γ_1 and γ_2 are non-negative and finite. Denote

$$\gamma_1^* = (\lambda e^{-2\theta_2} + (1-\lambda))\gamma_1\ ,\ \gamma_2^* = \gamma_2 \tag{3.5}$$

and assume that

$$\boldsymbol{\Gamma} = \text{Diag}\ (\gamma_1^*,\ \gamma_2^*) \text{ is positive definite.} \tag{3.6}$$

Then, we have the following

<u>Theorem 3.1</u>. *Under the regularity conditions of Section* 2 *and under* (3.6), *when* $\boldsymbol{\theta}$ *holds, for any* (*fixed*) $C, 0 < C < \infty$, *as* $N \to \infty$,

$$\sup\{||\mathbf{M}_N(\mathbf{t}) - \mathbf{M}_N(\boldsymbol{\theta}) + N^{1/2}\boldsymbol{\Gamma}(\mathbf{t} - \boldsymbol{\theta})||:\ \ N^{1/2}||\mathbf{t} - \boldsymbol{\theta}|| \le C\} \xrightarrow{P} 0 \tag{3,7}$$

<u>Proof</u>. Note that by (3.1) and (3.2), for a given t_2, $M_{N1}(t_1,\ t_2)$ is $\searrow$ in t_2 and for given t_1, $M_{N2}(t_1,\ t_2)$ is $\searrow$ in t_2. Also, for a given t_1, $M_{N1}(t_1,\ t_2)$ can be expressed as a difference of two terms, each of which is $\searrow$ in t_2 and a similar case holds for $M_{N2}(t_1,\ t_2)$ when t_2 is fixed. These properties provide the necessary tool for replacing the "sup" by "max" over a finite number of grid-points which, in turn, enables one to use a simpler way to prove (3.6). We only prove the case of $M_{N1}(\mathbf{t})$, i.e.

$$\sup\{|M_{N1}(\mathbf{t}) - M_{N1}(\boldsymbol{\theta}) + N^{1/2}(t_1 - \theta_1)\gamma_1^*|:\ \ N^{1/2}||\mathbf{t} - \boldsymbol{\theta}|| \le C\} \xrightarrow{P} 0\ , \tag{3.8}$$

as a very similar proof holds for $M_{N2}(\mathbf{t})$.

Let us first assume that $\boldsymbol{\theta} = \mathbf{0}$. Then $\gamma_1^* = \gamma_1$ and we may write

$$M_{N1}(\underset{\sim}{t}) - M_{N1}(\underset{\sim}{0}) + N^{1/2}t_1\gamma$$

$$= [M_{N1}(\underset{\sim}{t}) - M_{N1}(t_1, 0)] + \{M_{N1}(t_1, 0) - M_{N1}(\underset{\sim}{0}) + N^{1/2}t_1\gamma_1\} \qquad (3.9)$$

$$= M^*_{N1}(\underset{\sim}{t}) + M^{**}_{N1}(\underset{\sim}{t}) \text{ , say}$$

Since $M^{**}_{N1}(\underset{\sim}{t})$ does not depend on t_2 and $M_{N1}(t_1, 0)$ is $\searrow$ in t_1, it readily follows along the lines of Jurečková (1977) that

$$\sup\{|M^{**}_{N1}(\underset{\sim}{t})|: \ N^{1/2}||\underset{\sim}{t}|| \le C\} \xrightarrow{p} 0 \text{ , as } N \to \infty \qquad (3.10)$$

Thus, to prove (3.8), it suffices to show that

$$\sup\{|M^{*}_{N1}(\underset{\sim}{t})|: \ N^{1/2}||\underset{\sim}{t}|| \le C\} \xrightarrow{p} 0 \text{ , as } N \to \infty \qquad (3.11)$$

Let us fix an $\eta > 0$ and let $\underset{\sim}{u}$ and $\underset{\sim}{v}$ be two points such that $N^{1/2}||\underset{\sim}{u}|| \le C$, $N^{1/2}||\underset{\sim}{v}|| \le C$, and $N^{1/2}||\underset{\sim}{v} - \underset{\sim}{u}|| < \eta$, $\underset{\sim}{u} \le \underset{\sim}{v}$.
Also, let

$$\tilde{M}^*_{N1}(\rho;\ u_2, v_2)$$

$$= N^{-1/2}\sum_{i=1}^{m}\{e^{-v_2}\psi(e^{-v_2}(X_i - \rho)) - e^{-u_2}\psi(e^{-u_2}(X_i - \rho))\}I(X_i < \rho) \qquad (3.12)$$

for every ρ: $N^{1/2}\ |\rho| \ \le C$ and u_2, v_2 as defined before. Then, for any $\underset{\sim}{t}$, $\underset{\sim}{u} \le \underset{\sim}{t} \le \underset{\sim}{v}$, by the monotomicity properities mentioned above,

$$\begin{aligned} M^*_{N1}(\underset{\sim}{t}) &\le M^*_{N1}(u_1, t_2) \\ &\le M^*_{N1}(u_1, u_2) + \tilde{M}^*_{N1}(u_1;\ u_2, v_2); \end{aligned} \qquad (3.13)$$

$$\begin{aligned} M^*_{N1}(\underset{\sim}{t}) &\ge M^*_{N1}(v_1, t_2) \\ &\ge M^*_{N1}(v_1, v_2) - \tilde{M}^*_{N1}(v_1;\ u_2, v_2) \end{aligned} \qquad (3.14)$$

so that by (3.13) and (3.14),

$$\sup_{\underset{\sim}{u} \le \underset{\sim}{t} \le \underset{\sim}{v}} |M^*_{N1}(\underset{\sim}{t})| \le \max_{\underset{\sim}{t} = \underset{\sim}{u}, \underset{\sim}{v}} |M^*_{N1}(t)|$$

$$+ \max\{|\tilde{M}^*_{N1}(u_1;\ u_2,\ v_2)|, |\tilde{M}^*_{N1}(v_1;\ u_2,\ v_2)|\}. \tag{3.15}$$

For any $C > 0$ and $\eta > 0$, there exists a finite number $K\ (= K(\eta, C))$ of grid-points $\underset{\sim}{u}_i$, $i = 1,\ldots,K$, such that any two consecutive grid-points are not apart by more than η. Using then (3.15) for each grid, to prove (3.11), it then suffices to show that under $\underset{\sim}{\theta} = \underset{\sim}{0}$ and the regularity conditions of Section 2, as $N \to \infty$,

$$M^*_{N1}(\underset{\sim}{t}) \overset{p}{\to} 0 \text{ uniformly in } \underset{\sim}{t}:\ N^{1/2}||\underset{\sim}{t}|| \le C \tag{3.16}$$

$$\tilde{M}^*_{N1}(\rho;\ t_1,\ t_2) \overset{p}{\to} 0 \text{ uniformly in } \rho:\ N^{1/2}\ |\rho| \le C,\ N^{1/2}|t_2 - t_1| < \eta \tag{3.17}$$

Note that by (3.1) and (3.9),

$$M^*_{N1}(\underset{\sim}{t}) = N^{-1/2} \sum_{i=1}^{m} \{e^{-t_2}\psi(e^{-t_2}(X_i - t_1)) - \psi(X_i - t_1)\} \tag{3.18}$$

is a sum of independent random variables, $m/N \to \lambda$: $0 < \lambda < 1$ and the variables $m^{-1/2}\{e^{-t_2}\psi(e^{-t_2}(X_i - t_1)) - \psi(X_i - t_1)\}$, $1 \le i \le m$ satisfy the condition of uniform asymptotic negligibility for any $\underset{\sim}{t}$: $N^{1/2}||\underset{\sim}{t}|| \le C$. Hence, (3.16) follows directly by verifying the degenerate convergence criterion of the classical central limit theorem for a triangular array of row-wise independent r.v.'s.

The proof of (3.17) also follows by using the degenerate convergence criterion (of central limit theorem) and (3.12), where, for any $(\rho;\ t_1,\ t_2)$: $N^{1/2}|\rho| \le C$, $N^{1/2}|t_j| \le C$, $j = 1,2$ and $N^{1/2}|t_2 - t_1| < \eta$, the independent summands in (3.11) satisfy the needed regularity conditions.

If $(\theta_1,\ \theta_2) \neq \underset{\sim}{0}$, then, using the transformation $X_i = e^{\theta_2} X_i^0 + \theta_1$, $1 \le i \le m$; $Y_j = Y_j^0 + \theta_1$, $1 \le j \le n$ (where X_i^0's and Y_j^0's correspond to zero parameter, we prove (3.8) with the aid of part one of the present proof. Q.E.D.

4. Asymptotic properties of $\underset{\sim}{T}_N$

We shall employ Theorem 3.1 in the study of the asymptotic properties of $\underset{\sim}{T}_N$. The asymptotic distribution of $\underset{\sim}{T}_N$ is given in the following theorem.

Theorem 4.1. *Under the regularity conditions of Section 2, if* $\underset{\sim}{\Gamma}$ *is positive definite,*

$$N^{1/2}(\underset{\sim}{T}_N - \underset{\sim}{\theta}) \xrightarrow{\mathcal{D}} N(\underset{\sim}{0}, \underset{\sim}{\Sigma}) \tag{4.1}$$

where

$$\Sigma = \mathrm{Diag}\left(\frac{\int\psi^2(x)dF(x)}{(\lambda e^{-2\theta_2} + (1-\lambda))\lambda_1^2}, \frac{\int\xi^2(x)dF(x)}{\lambda(1-\lambda)\lambda_2^2}\right) \tag{4.2}$$

with γ_1 *and* γ_2 *given in* (3.3) *and* (3.4).

Corollary. *The estimator* $\underset{\sim}{T}_N^* = (T_{N1}, \exp T_{N2})$ *of the original parameter* $\underset{\sim}{\theta}^* = (\mu, \nu)$ *is asymptotically normally distributed,* i.e.,

$$N^{1/2}(\underset{\sim}{T}_N^* - \underset{\sim}{\theta}^*) \xrightarrow{\mathcal{D}} N_2(\underset{\sim}{0}, \underset{\sim}{\Sigma}^*) \tag{4.3}$$

where

$$\underset{\sim}{\Sigma}^* = \mathrm{Diag}\left(\frac{\int\psi^2(x)dF(x)}{(\lambda\nu^{-2} + (1-\lambda))\lambda_1^2}, \nu^2\frac{\int\xi^2(x)dF(x)}{\lambda(1-\lambda)\lambda_2^2}\right) \tag{4.4}$$

Proof of Theorem 4.1. First, we may note that by (3.1), (3.2) and the central limit theorem, when θ holds,

$$\underset{\sim}{M}_N(\underset{\sim}{\theta}) \xrightarrow{\mathcal{D}} N_2(\underset{\sim}{0}, \underset{\sim}{S}), \tag{4.5}$$

where $\underset{\sim}{S} = \mathrm{Diag}(\sigma_{11}, \sigma_{22})$ and

$$\sigma_{11} = (\lambda e^{-2\theta_2} + (1 - \lambda))\int\psi^2(x)dF(x) \tag{4.6}$$

$$\sigma_{22} = [\lambda(1 - \lambda)]^{-1}\int\xi^2(x)dF(x) \tag{4.7}$$

and by (2.5) and (2.6), both σ_{11} and σ_{22} are finite.

Secondly, (4.5) insures that under θ,

$$||\underset{\sim}{M}_N(\underset{\sim}{\theta})|| = 0_p(1), \tag{4.8}$$

so that by Theorem 3.1, there exists (in probability) a solution $\underset{\sim}{t}_N$ of $\underset{\sim}{M}_N(\underset{\sim}{t}) = 0$ for which

$$(\underset{\sim}{t}_N - \underset{\sim}{\theta}) = N^{-1/2}\underset{\sim}{\Gamma}^{-1}\underset{\sim}{M}_N(\underset{\sim}{\theta}) + o_p(N^{-1/2})(= 0_p(N^{-1/2})). \tag{4.9}$$

Let $\underset{\sim}{T}_N$ be the estimator suggested in Section 2. Then by (2.8), (4.8) and (4.9),

$$||\underset{\sim}{T}_N - \underset{\sim}{\tilde{\theta}}_N|| \le ||\underset{\sim}{t}_N - \underset{\sim}{\tilde{\theta}}|| \le ||\underset{\sim}{t}_N - \underset{\sim}{\theta}|| + ||\underset{\sim}{\tilde{\theta}}_N - \underset{\sim}{\theta}|| = 0_p(N^{-1/2}) \tag{4.10}$$

which implies $N^{1/2}||\underset{\sim}{T}_N - \underset{\sim}{\theta}|| = 0_p(1)$ and it follows from Theorem 3.1 that

$$N^{1/2}(\underset{\sim}{T}_N - \underset{\sim}{\theta}) - \underset{\sim}{\Gamma}^{-1}\underset{\sim}{M}_N(\underset{\sim}{\theta}) \xrightarrow{P} \underset{\sim}{0} \tag{4.11}$$

and this together with (4.5) implies (4.1). Q.E.D.

5. An algorithm for the computation of $\underset{\sim}{T}_N$

In the general situation, we could hardly find the estimator $\underset{\sim}{T}_N$ by the direct solution of the system (2.10). The estimator should be obtained by an iterative procedure. One possible algorithm we provide in the present section.

Let $\underset{\sim}{\tilde{\theta}}_N$ be any initial estimator of $\underset{\sim}{\theta}$ such that

$$N^{1/2}||\underset{\sim}{\tilde{\theta}}_N - \underset{\sim}{\theta}|| = 0_p(1) \tag{5.1}$$

(e.g. the estimator of (2.9)). For any arbitrary (but fixed) $\underset{\sim}{a} \neq \underset{\sim}{0}$, let then

$$D_N(\underset{\sim}{\tilde{\theta}}_N; \underset{\sim}{a}) = \text{Diag}\{[M_{Nj}(\underset{\sim}{\tilde{\theta}}_N + N^{-1/2}\underset{\sim}{a}) - M_{Nj}(\underset{\sim}{\tilde{\theta}}_N)]/a_j, \ j = 1, 2\}. \qquad (5.2)$$

Note that by (5.1), Theorem 3.1 and (4.3),

$$\underset{\sim}{D}_N(\underset{\sim}{\tilde{\theta}}_N; \underset{\sim}{a}) \xrightarrow{P} \underset{\sim}{\Gamma}, \text{ as } N\to\infty .$$

Let then

$$\underset{\sim}{T}_N^0 = \underset{\sim}{\tilde{\theta}}_N, \ \underset{\sim}{T}_N^{(j)} = \underset{\sim}{T}_N^{(j-1)} + N^{-1/2}\underset{\sim}{\hat{\Gamma}}_{N,\ j-1} \cdot \underset{\sim}{M}_N(\underset{\sim}{T}_N^{(j-1)}), \ j \geq 1 \qquad (5.3)$$

where $\underset{\sim}{\hat{\Gamma}}_{N,\ j} = \underset{\sim}{D}_N(\underset{\sim}{T}_N^{(j)}; \underset{\sim}{a}), \ j \geq 0.$

By (2.10), (3.1), (3.2), Theorem (3.1) and (5.1) - (5.3), we conclude that $\{\underset{\sim}{T}_N^{(k)}, \ k \geq 1\}$ converges to $\underset{\sim}{T}_N$, in probability. In actual practice, we may stop the iteration in (5.3) when for some specified (small) $\varepsilon(>0)$, $||\underset{\sim}{T}_N^{(k)} - \underset{\sim}{T}_N^{(k-1)}|| < \varepsilon$. The procedure terminates, in probability.

References

1. CARROLL, R.J. (1978). On the asymptotic distribution of multivariate M-estimates. *J. Multivariate Analysis* 8, 361-371.

2. HUBER, P.J. (1964). Robust estimation of location. *Ann. Math. Statist.* 35, 73-101.

3. HUBER, P.J. (1967). The behavior of maximum likelihood estimates under nonstandard conditions. *Proc. 5th Berkeley Symp.* 1, 221-233.

4. HUBER, P.J. (1977). Robust methods of estimation of regression coefficients. *Math. Operations for sch. Statist., Ser. Statistics* 8, 41-53.

5. JUREČKOVÁ, J. (1977). Asymptotic relations of M-estimates and R-estimates in linear regression model. *Ann, Statist.* 5, 464-472.

6. MARONNA, R.A. (1976). Robust M-estimators of multivariate location and scatter. *Ann. Statist.* 4, 51-67.

7. PORTNOY, S.L. (1977). Robust estimation in dependent situations. *Ann. Statist.* 5, 22-43.