

ON THE DISTRIBUTION OF QUANTILES OF
RESIDUALS IN A LINEAR MODEL

by

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ABSTRACT

The representation for quantiles discussed by Bahadur (1966) and Ghosh (1971) is extended to quantiles from the residuals in a linear model. As an application, a robust test for symmetry is proposed.

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I. Introduction

Let y_1, \dots, y_n be independent and identically distributed with common distribution F , and let $F(\xi) = p$. Let k_n be a sequence of integers $k_n = np + o(n^{1/2} \log n)$, and let V_n be the k_n^{th} sample quantile. Bahadur (1966) and Ghosh (1971) have shown that if F_n is the empirical distribution function of y_1, \dots, y_n , then

$$(1) \quad n^{1/2} (V_n - \xi - (k_n/n - F_n(\xi))/F'(\xi)) \xrightarrow{P} 0.$$

Actually, Bahadur's result is an almost sure result, but (1) suffices for many statistical applications. Our goal is to extend (1) to the quantiles of the residuals in a simple linear model, using this to conduct a quick test for symmetry.

We consider the simple linear model $z_i = x_i' \underline{\beta} + y_i$, where y_1, \dots, y_n are i.i.d. with distribution F continuous and strictly increasing in a neighborhood of ξ , with $F(\xi) = p$.

If $\underline{\beta}_n$ is an estimate of $\underline{\beta}$ (possibly the least squares estimate or one of the new robust estimates), the residuals are

$r_{in} = z_i - x_i' \underline{\beta}_n = y_i - x_i' (\underline{\beta}_n - \underline{\beta})$. Let $E_n(F_n)$ be the empirical distribution function of the residuals (errors), and choose

$a_n = n^{-1/2} \log n$, $b_n = (\log n)^2$. Let k_n be as above, let $I_n = (\xi - a_n, \xi + a_n)$, and define V_n to be the k_n^{th} order statistic of the residuals. We make the following assumptions:

(A1) F has two bounded derivatives in a neighborhood of ξ , and $F'(\xi) = f(\xi) > 0$.

(A2) $a_n^{-1} (\underline{\beta}_n - \underline{\beta}_n) \rightarrow 0$ (a.s.)

(A3) For some $\delta > 0$, $a_n^{1/2 - \delta} \max\{|x_i| : i=1, \dots, n\} \rightarrow 0$.

(A4) There exists \underline{x}_0 finite with $(\log n)n^{-1} \sum_1^n (x_i - \underline{x}_0) \rightarrow 0$.

Theorem Under conditions (A1) - (A4),

$$(2) \quad n^{1/2} |(V_n - \xi) - (k_n/n - F_n(\xi))/f(\xi) + \underline{x}'_0 (\underline{\beta}_n - \underline{\beta})| \rightarrow 0 \text{ (a.s.)}$$

Remarks Condition (A1) is the same as Bahadur's, while (A2) is quite reasonable. Condition (A3) guarantees that the design elements will not vary too wildly. The usual condition on the design matrix X is that $n^{-1}X'X \rightarrow \Sigma$ (Drygas (1976)), and this implies that if there is an intercept term there is an \underline{x}_0 with $n^{-1} \sum_1^n (x_i - \underline{x}_0) \rightarrow 0$, so that (A4) is only a slight strengthening of a standard condition, which in fact is reduced to $n^{-1} \sum_1^n (x_i - \underline{x}_0) \rightarrow 0$ at the end of Section 2. Note that (A3) and (A4) are guaranteed in the problem considered by Bahadur and Ghosh.

2. Proof of the Theorem

We first consider simple linear regression without an intercept $z_i = x_i \beta + y_i$. The problems of simple and multiple linear regression with an intercept are easy consequences of the proof presented here.

Define the following processes:

$$G_n(x) = n^{-1} \sum_{i=1}^n \{I(r_i \leq x) - I(y_i \leq \xi) - (F(x+x_i(\beta_n - \beta)) - F(x))\}$$

$$H_n = n^{1/2} \sup\{|G_n(x)| : x \in I_n\}$$

$$W_n(s, \cdot) = n^{-1/2} \sum_{i=1}^n \left\{ \begin{array}{l} I(y_1 \leq \xi + a_n s + a_n t x_i) - I(y_1 \leq \xi) \\ -F(\xi + a_n s + a_n t x_i) + F(\xi) \end{array} \right\}$$

where $I(A)$ is the indicator of the event A . We note that from

(A1) - (A4),

$$(3) \quad n^{-1} \sum_{i=1}^n \{F(\xi + sa_n) - F(\xi + sa_n + ta_n x_i) + ta_n x_i f(\xi)\} = O(n^{-1/2})$$

uniformly for $0 \leq |s|, |t| \leq 1$.

Lemma 1 Under (A1) - (A4), $H_n \rightarrow 0$ (a.s.) .

Proof of Lemma 1 Since $r_{in} = y_i - x_i(\beta_n - \beta)$, by (A2) it suffices to show that $\sup \{|W_n(s,t)| : 0 \leq s, t, \leq 1\} \rightarrow 0$ (a.s.) .

For this, we may consider only $x_i \geq 0$ by breaking the sum defining W_n into two parts, one with $x_i \geq 0$ and the other over the terms $x_i < 0$. By (A1) as in Bahadur's proof, it also suffices to consider

points $s, t = \eta_{r,n} = r/b_n$ ($r = 0, \dots, b_n$). To see this, note that if $|s - \eta_{r,n}| \leq b_n$, $|t - \eta_{p,n}| \leq b_n$, then

$$\begin{aligned} & \left| n^{-1} \sum_{i=1}^n \{F(\xi + sa_n + ta_n x_i) - F(\xi + \eta_{r,n} a_n + \eta_{p,n} a_n x_i)\} \right| \\ & = O(a_n b_n^{-1}) = O(n^{-1/2}). \end{aligned}$$

Now, by Bernstein's inequality (Hoeffding, eq. 2.13),

$$P_r \{|W_n(s,t)| > \epsilon\} \leq c_0 \exp\{-c, n^{1/4}\}$$

for some constants c_0, c depending on ϵ . Thus,

$$\begin{aligned} & \Pr\{|W_n(s,t)| > \epsilon \text{ for some } n \geq n_0, (s,t) = (n_{rn}, n_{pn})\} \\ & \leq \sum_{n=n_0}^{\infty} c_0 b_n^2 \exp\{-c_1 n^{1/4}\} \rightarrow 0 \text{ as } n_0 \rightarrow \infty. \end{aligned}$$

This completes the proof of the Lemma. \square

Lemma 2 Almost surely as $n \rightarrow \infty$, $V_n \in I_n$.

Proof of Lemma 2 We have

$$\Pr\{V_n \geq \xi + a_n\} = \Pr\left\{\sum_1^n I(y_i \leq \xi + a_n + x_i(\beta_n - \beta)) \leq k_n\right\}.$$

It suffices to show the existence of an $\eta > 0$ for which

$Q_{n_0}(\eta) \rightarrow 0$, where

$$\begin{aligned} Q_{n_0}(\eta) &= \Pr\left\{\sum_1^n I(y_i \leq \xi + a_n + t a_n x_i) \leq k_n \text{ for some } 0 \leq t \leq \eta, n \geq n_0\right\} \\ &= \Pr\left\{F_n(\xi + a_n) \leq k_n/n + n^{-1} \sum_1^n (F(\xi + a_n) - F(\xi + a_n + t a_n x_i)) \right. \\ & \quad \left. - n^{-1/2} W_n(1,t) \text{ for some } 0 \leq t \leq \eta, n \geq n_0\right\}. \end{aligned}$$

By Lemma 1 and using equation (3),

$$Q_{n_0}(\eta) = \Pr\left\{F_n(\xi + a_n) - F(\xi + a_n) \leq -a_n f(\xi)(1 + t x_0) + o(a_n) \right. \\ \left. \text{for some } 0 \leq t \leq \eta, n \geq n_0\right\} + o(1).$$

Choosing $\eta > 0$ small, $Q_{n_0}(\eta) \rightarrow 0$ as in the proof of Bahadur's Lemma 2. \square

Proof of the Theorem Since $E_n(k_n) = k_n/n$, Lemmas 1 and 2 may be applied to show that

$$n^{1/2} |F(V_n) - F(\xi) - (k_n/n - E_n(\xi))| \rightarrow 0 \text{ (a.s.)}.$$

Further, $F(V_n) - F(\xi) = (V_n - \xi)f(\xi) + o(n^{-1/2})$ (a.s.) and

$$E_n(\xi) - F(\xi) = F_n(\xi) - F(\xi) + W_n(0, a_n^{-1}(\beta_n - \beta)) + n^{-1} \sum_{i=1}^n \{F(\xi + x_i(\beta_n - \beta)) - F(\xi)\}.$$

Since $A_n^{-1}(\beta_n - \beta) \rightarrow o$ (a.s.), the proof is complete by Lemma 1 and (A4). □

3. An Application

Hinkley (1975) suggested a method based on order statistics for estimating the parameter of a power transformation; his method is most applicable in the location parameter case. The Theorem can be used to construct a quick test for symmetry in more complicated models. Let $p \neq 1/2$ and define $V_n^{(p)}$ to be the $[np]^{\text{th}}$ order residual. For $F(\xi_p) = p$,

$$(4) \quad n^{1/2}(V_n^{(p)} + V_n^{(1-p)} - 2V_n^{(1/2)} - (\xi_p + \xi_{1-p} - 2\xi_{1/2}))$$

has a limiting normal distribution (with mean zero and variance σ^2) independent of the limit distribution of β_n . Under the hypothesis of symmetry, $\xi_p + \xi_{1-p} = 2\xi_{1/2}$, so one rejects the hypothesis if

$$(5) \quad n^{1/2}|V_n^{(p)} + V_n^{(1-p)} - 2V_n^{(1/2)}|/\sigma$$

is too large. To estimate σ one needs estimates of the density at the quantiles $\xi_p, \xi_{1-p}, \xi_{1/2}$, which can be accomplished as in Bloch and Gastwirth (1968) by using the Theorem.

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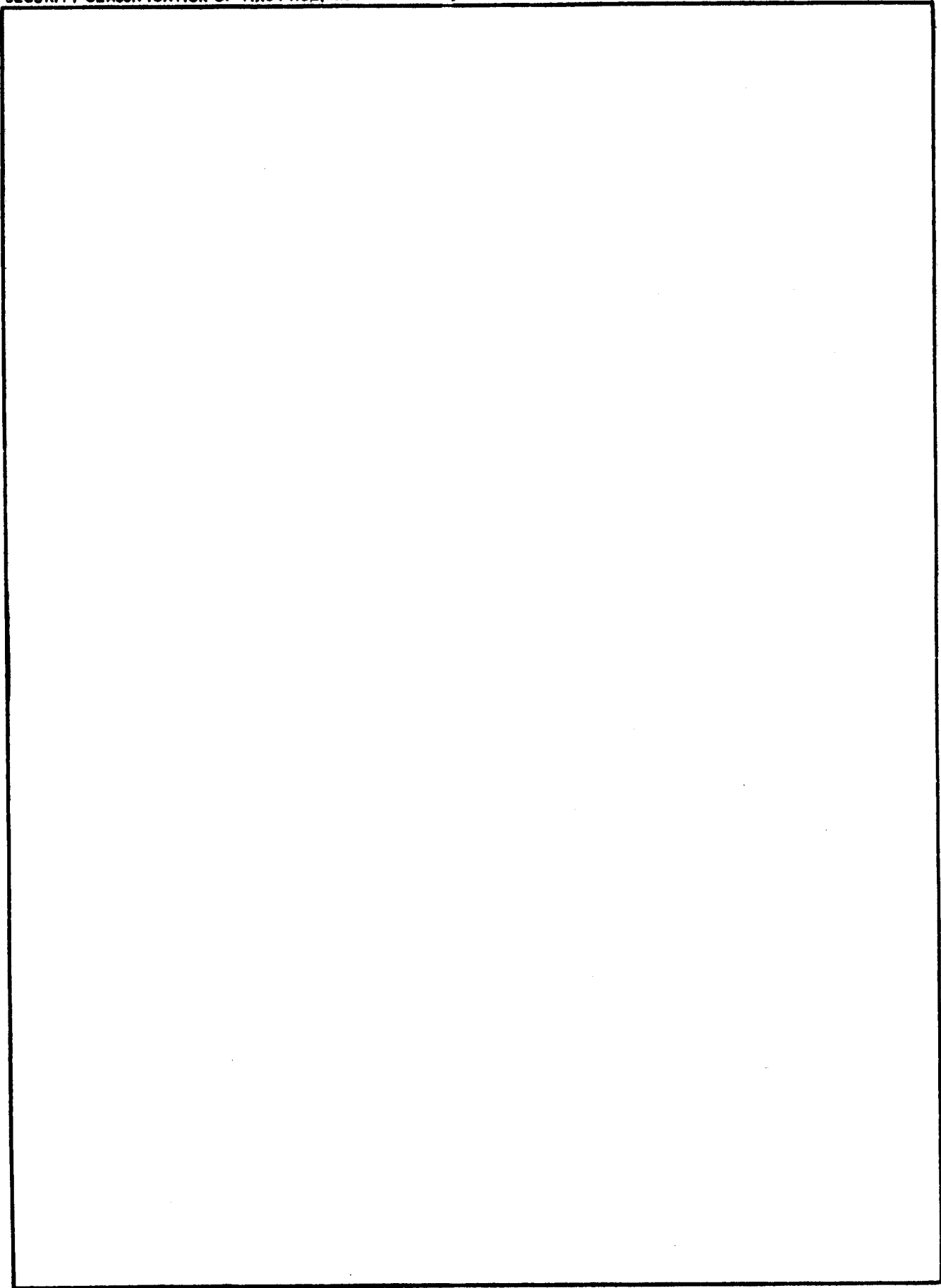
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