

A CONSERVATIVE SEMI-LAGRANGIAN METHOD FOR MULTI-DIMENSIONAL FLUID DYNAMICS APPLICATIONS*

J. S. SCROGGS[†] AND F. H. M. SEMAZZI[‡]

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Abstract. We describe a finite volume semi-Lagrangian method for the numerical approximation of conservation laws arising in fluid-dynamic applications. A discrete conservation relation is satisfied by using conservative interpolation for the material (or property) being conserved. The method was developed with a view to application in climate prediction.

1. Introduction. Semi-Lagrangian methods [16] have gained attention as a treatment of the advection in numerical weather simulations. The central idea is to develop a finite difference method that treats advection by following the paths of the particles that pass through the fixed points of an Eulerian grid on each time step. The same idea has also been applied to the finite element approach [2, 4, 7, 10, 13, 14]. A primary difficulty with the existing semi-Lagrangian (and related) methods is the satisfaction of a discrete conservation relation. (This relation is described in [6].) The discrete conservation relation can be expressed as an algebraic condition, and standard semi-Lagrangian methods can be modified to satisfy this relation [11]. This conservation relation can be satisfied locally, for example, by forming a hybrid method from the piecewise parabolic method [9] with the semi-Lagrangian approach, such as was studied in [12]. In comparison with [12], our formulation of the discrete problem is based on satisfying the physical conservation constraint, not simply imposing an algebraic constraint. In this way, higher-order schemes can be developed in a straightforward manner. In §2, we derive the scheme with a finite volume approach, where the conservation is satisfied by construction.

The conservative semi-Lagrangian method presented here will be developed for the following two-dimensional conservation law

$$(1) \quad \frac{\partial(\phi)}{\partial t} + \frac{\partial(u\phi)}{\partial x} + \frac{\partial(v\phi)}{\partial y} = Q$$

where ϕ is a vector-valued function, Q represents external forces and sources, and the flow field has x -velocity u , and y -velocity v (there may be a dependence of the flow field on ϕ).

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[†] Center for Research in Scientific Computing, Mathematics Department, Box 8205, North Carolina State University, Raleigh, NC 27502 (na.scroggs@na-net.ornl.gov)

[‡] Department of Marine, Earth, and Atmospheric Sciences, Center for Research in Scientific Computing, Box 8208, Mathematics Department, North Carolina State University, Raleigh, NC 27502.

Here we assume that there is no variation in the vertical direction; however, the results can be extended to three-dimensions.

In general, semi-Lagrangian methods incorporate a treatment of the advection (or convection) by using a particle-following transformation that satisfies

$$(2) \quad \frac{d\tau(t)}{dt} = 1, \quad \frac{d\xi(t)}{dt} = u(\tau, \xi(t), \eta(t)), \quad \frac{d\eta(t)}{dt} = v(\tau, \xi(t), \eta(t)),$$

where τ is the temporal variable, ξ and η are the spatial variables. After applying a transformation that satisfies Eqn. (2), the equation that governs ϕ becomes

$$(3) \quad \frac{d\phi(\tau, \xi, \eta)}{d\tau} = \hat{Q},$$

where

$$(4) \quad \hat{Q}(\tau, \xi, \eta) = Q(t, x, y) - \left(\frac{\partial u(t, x, y)}{\partial x} + \frac{\partial v(t, x, y)}{\partial y} \right) \phi(t, x, y).$$

The emphasis in this presentation is on the conservation; thus, we restrict our attention to the ideas surrounding the development of a conservative method. In §2, we develop the numerical method. First-order approximations are used for the sake of simplicity, however the techniques can be extended to higher-order methods. A necessary condition for the use of the method, similar to the familiar stability conditions, is presented in §3. The method is then demonstrated on a two-dimensional advection problem in §4.

2. Derivation of the Numerical Method. In general, the velocity components u and v depend on ϕ ; thus, (2) is strongly coupled with (3). Our numerical treatment does not explicitly treat this coupling; however, techniques such as those in [5] could be used here. First, we present our numerical treatment of (2).

The solution of the system (2) subject to

$$(5) \quad \tau(t^{n+1}) = t^{n+1}, \quad \xi(t^{n+1}) = x_{i+\frac{1}{2}, j+\frac{1}{2}}, \quad \text{and} \quad \eta(t^{n+1}) = y_{i+\frac{1}{2}, j+\frac{1}{2}}$$

describes the path of the particle that passes through the grid location $(x, y) = X_{i+\frac{1}{2}, j+\frac{1}{2}}$ at time $t = t^{n+1}$. The following one-step method is used to approximate the solution to this problem

$$(6) \quad x_{i+\frac{1}{2}, j+\frac{1}{2}}^{*n} = x_{i+\frac{1}{2}, j+\frac{1}{2}} - \frac{\Delta t}{4} (u_{i,j}^n + u_{i+1,j}^n + u_{i,j+1}^n + u_{i+1,j+1}^n)$$

and

$$(7) \quad y_{i+\frac{1}{2}, j+\frac{1}{2}}^{*n} = y_{i+\frac{1}{2}, j+\frac{1}{2}} - \frac{\Delta t}{4} (v_{i,j}^n + v_{i+1,j}^n + v_{i,j+1}^n + v_{i+1,j+1}^n),$$

where subscripts indicate (spatial) grid locations, and superscripts indicate the time step. Notice that the conserved variable ϕ with the velocity components u and v in this formulation

are all cell centered values (in contrast to the Arakawa grid, where some variables are given at different locations in the same grid). The cell-centered assumption is made to simplify this presentation. For example, we have demonstrated this approach with a modified Arakawa grid as well [15]. This is a single-step method using a linearization of the velocity in space and a forward Euler in time.

The numerical treatment of (3) involves the conservation of ϕ inside each cell $C_{i,j}$, where the cell is determined by the four grid points $\mathbf{X}_{i-\frac{1}{2},j-\frac{1}{2}}$, $\mathbf{X}_{i+\frac{1}{2},j-\frac{1}{2}}$, $\mathbf{X}_{i-\frac{1}{2},j+\frac{1}{2}}$, and $\mathbf{X}_{i+\frac{1}{2},j+\frac{1}{2}}$ (see Fig. 1). (Here, $(x_{i+\frac{1}{2},j+\frac{1}{2}}, y_{i+\frac{1}{2},j+\frac{1}{2}}) = \mathbf{X}_{i+\frac{1}{2},j+\frac{1}{2}} \in \mathbf{R}^2$.) Particles that arrive at the four grid points at time $t = t^{n+1}$ were located at corners of cell $C_{i,j}^{*n}$, which are the *departure* points $\mathbf{X}_{i-\frac{1}{2},j-\frac{1}{2}}^{*n}$, $\mathbf{X}_{i+\frac{1}{2},j-\frac{1}{2}}^{*n}$, $\mathbf{X}_{i-\frac{1}{2},j+\frac{1}{2}}^{*n}$, and $\mathbf{X}_{i+\frac{1}{2},j+\frac{1}{2}}^{*n}$ at time $t = t^n$. The departure point for

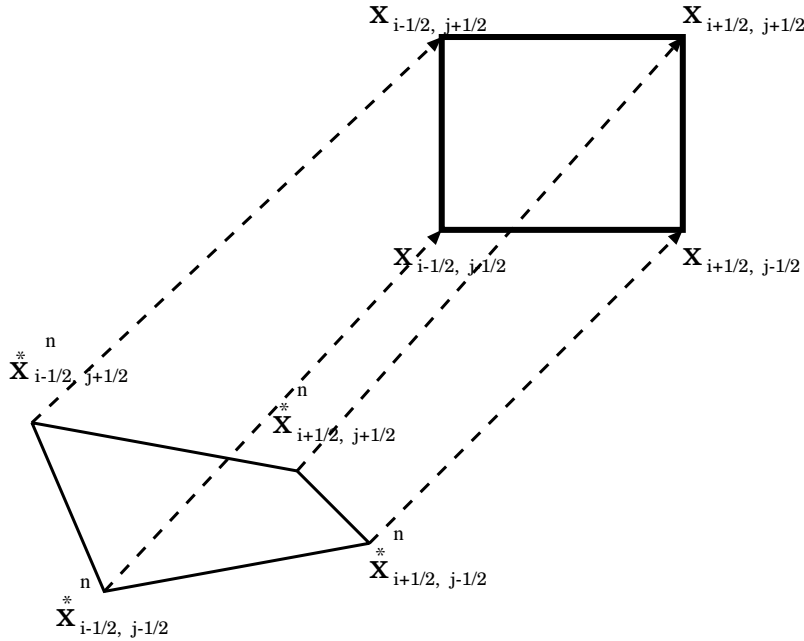


FIG. 1. Arrival points for cell $C_{i,j}$ and departure points for cell $C_{i,j}^{*n}$

the trajectory passing through $X_{i+\frac{1}{2},j+\frac{1}{2}}$ at time $t = t^{n+1}$ is the solution to (2) and (5).

Using a two-time-level scheme, the discretization of Eqn. (3) is

$$(8) \quad \phi_{i,j}^{n+1} = \phi_{i,j}^{*n} + \Delta t \hat{Q}$$

We treat the dependent variables (e.g. ϕ , u and v) as representing cell-centered values. That is, the discrete value $\phi_{i,j}^{n+1}$ is the average of ϕ in cell $C_{i,j}$ at time $t = t^{n+1}$. Similarly, $\phi_{i,j}^{*n}$ is the average value of ϕ in cell $C_{i,j}^{*n}$. For a fixed time step, we begin by knowing $\phi_{i,j}^n$ for all i, j , then, given the departure points, we must interpolate from the known values to obtain $\phi_{i,j}^{*n}$.

We determine $\phi_{i,j}^{*n}$ from the following weighted sum of the values of ϕ^n in the regions that overlap with cell $C_{i,j}^{*n}$ (see Fig. 2)

$$(9) \quad \phi_{I,J}^{*n} = \sum_{i,j \in Z} \omega_{i,j}^{I,J} \phi_{i,j}^n,$$

where $\omega_{i,j}^{I,J}$ is the area that is in both $C_{i,j}^{*n}$ and $C_{I,J}$, and Z is the set of indices of all the points in the computational domain. If the boundaries of cell $C_{I,J}$ remain inside the computation

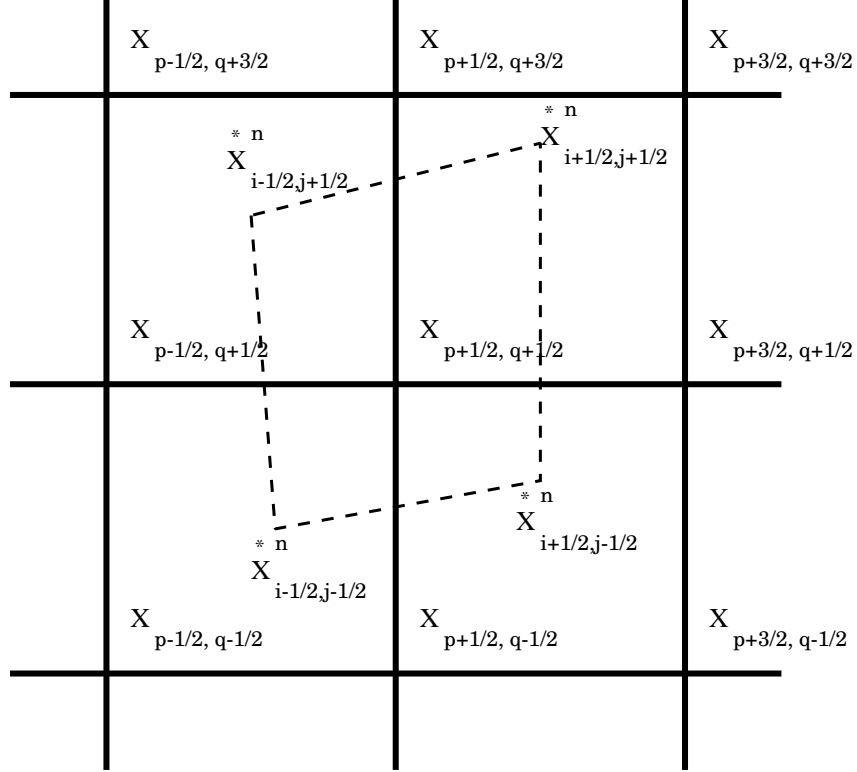


FIG. 2. *Overlapping*

domain from time step n to step $n + 1$, then conservation is satisfied because

$$\sum_{i,j \in Z} \omega_{i,j}^{I,J} = \text{area of } C_{I,J}.$$

If, however, some portion of $C_{I,J}$ leaves the computational domain, then some portion of ϕ is carried out of the domain. Similarly, fluid that enters the domain will carry some amount of ϕ into the domain. This results in a net flux F^{n+1} of ϕ into (or out of) the computational domain, and can be expressed as

$$\sum_{i,j \in Z} \phi_{i,j}^{*n} = \sum_{i,j \in Z} \phi_{i,j}^{n+1} + F.$$

This is the discrete conservation relation.

3. Necessary Condition. Since we are propagating fluid properties in a cell, we have the condition that the line segments forming the cell do not cross as we traverse the trajectories backwards in time to the departure points. Thus, a cell $C_{i,j}^{*n}$ such as that in Fig. 3 is unacceptable.

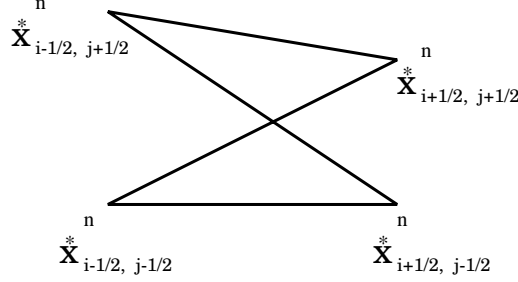


FIG. 3. Unacceptable cell formed by departure points

Here, we will develop a sufficient condition in the special case where cell boundaries on the fixed grid are alligned with the coordinate axis. This means that $x_{i+\frac{1}{2}, j+\frac{1}{2}} = x_{i+\frac{1}{2}}$ for all j , and $y_{i+\frac{1}{2}, j+\frac{1}{2}} = y_{j+\frac{1}{2}}$ for all i . The unacceptable cell is prevented if we prohibit the x and y values of $\mathbf{X}_{i+\frac{1}{2}, j+\frac{1}{2}}^{*n}$ from overtaking the corresponding values of adjacent points. This can be enforced with the following conditons

$$(10) \quad x_{i+\frac{1}{2}, j+\frac{1}{2}}^{*n} - x_{i-\frac{1}{2}, j+\frac{1}{2}}^{*n} > 0, \quad J = j-1, j, \text{ and } j+1,$$

$$(11) \quad x_{i+\frac{3}{2}, j+\frac{1}{2}}^{*n} - x_{i+\frac{1}{2}, j+\frac{1}{2}}^{*n} > 0, \quad J = j-1, j, \text{ and } j+1,$$

$$(12) \quad y_{i+\frac{1}{2}, j+\frac{1}{2}}^{*n} - y_{i+\frac{1}{2}, j-\frac{1}{2}}^{*n} > 0, \quad I = i-1, i, \text{ and } i+1,$$

and

$$(13) \quad y_{i+\frac{1}{2}, j+\frac{3}{2}}^{*n} - y_{i+\frac{1}{2}, j+\frac{1}{2}}^{*n} > 0, \quad I = i-1, i, \text{ and } i+1.$$

Since we are using (6-7) to determine the departure points, conditions (10-13) are satisfied when

$$(14) \quad \Delta t < 8 \min_{i,j \in \mathbb{Z}} \left(\frac{1}{U_{i,j}}, \frac{1}{V_{i,j}} \right),$$

where

$$\bar{U}_{i,j} = \max_{\substack{I1,I2 \in \{i-1,i,i+1\} \\ J1,J2 \in \{j-1,j,j+1,j+2\}}} \frac{|u_{I1,J1} - u_{I2,J2}|}{x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}}, \quad \text{and} \quad \bar{V}_{i,j} = \max_{\substack{I1,I2 \in \{i-1,i,i+1,i+2\} \\ J1,J2 \in \{j-1,j,j+1\}}} \frac{|v_{I1,J1} - v_{I2,J2}|}{y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}}}.$$

Notice that in the case where the velocity is constant, condition (14) states that there is no restriction on the size of Δt . If the velocity is divergence-free, the unacceptable cell will not form. These conditions are over-restrictive in the sense that they do not allow for rotation of the cells, so it is expected that a less restrictive condition could be developed.

4. Numerical Experiments. Here, we compare the conservative semi-Lagrangian method described in this paper with a non-conservative semi-Lagrangian method. The non-conservative method is obtained by computing weights based on standard bilinear interpolation from the four cells that are closest to the center of each cell. The center of cell $C_{I,J}^{*n}$ is defined as the point

$$x = (x_{i,j}^{*n} + x_{i+1,j}^{*n} + x_{i,j+1}^{*n} + x_{i+1,j+1}^{*n})/4$$

and

$$y = (y_{i,j}^{*n} + y_{i+1,j}^{*n} + y_{i,j+1}^{*n} + y_{i+1,j+1}^{*n})/4.$$

We choose a model problem with the divergence-free velocity field $(u, v) = (-y, x)$ and a scalar conserved quantity ϕ . The initial condition $\phi(0, x, y) = 1$ when $(x - .25)^2 + (y - .25)^2 < .02$, and $\phi(0, x, y) = 0$ otherwise is shown in Fig. 4. The homogeneous dirichlet condition $\phi(t, x, y) = 0$ is used for x and y on the boundary of the computational domain $[-1, 1] \times [-1, 1]$. The solution at time $t = 5.0$ is shown in Fig. 5. Both of these methods are first-order in space and time, thus they both exhibit a large amount of numerical diffusion. The primary difference is that the maximum value computed by the conservative method is much greater than that computed by the non-conservative method. The dependent variable ϕ is being conserved and there is no net flux into the domain, thus we should have

$$\sum_{i,j \in Z} \phi_{i,j}^0 = \sum_{i,j \in Z} \phi_{i,j}^n$$

for all values of n . As a comparison of the conservation of the two methods, we present the total of ϕ computed by the numerical methods in Table 4. We used a computational grid of 50×50 , with $\Delta t = .05$. Notice that the non-conservative method loses approximately 22% of ϕ , while the conservative scheme loses only 0.18%.

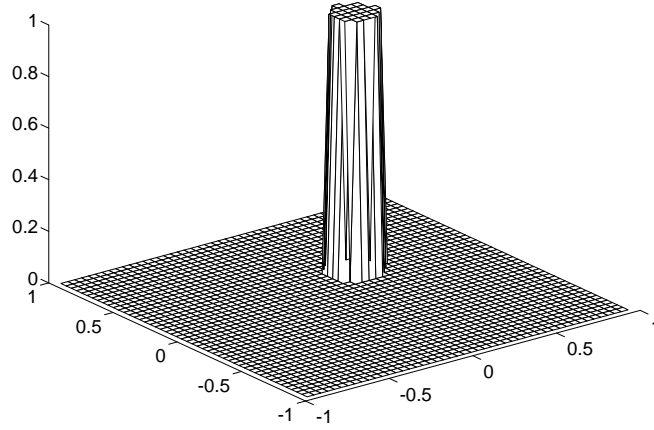
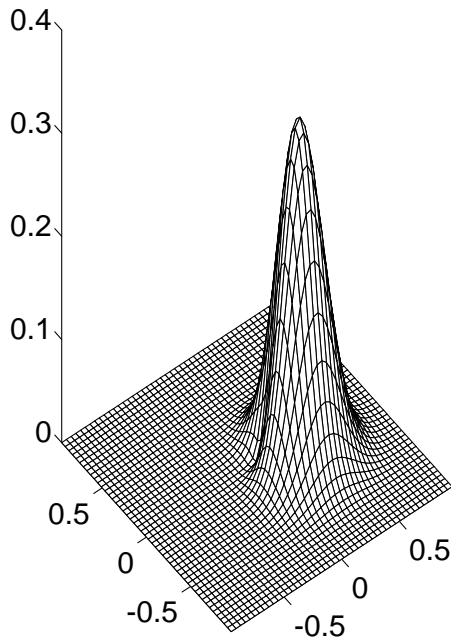


FIG. 4. *Numerical Solutions at $t = 0.0$*

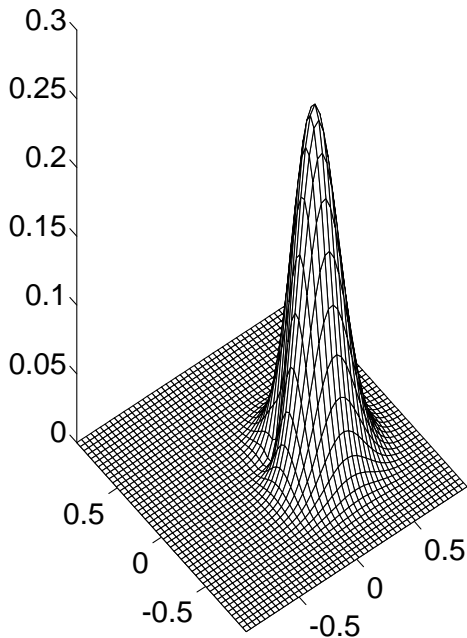
Time	Conservative Scheme	Nonconservative Scheme
0	37.00	37.00
1	36.99	35.19
2	36.98	33.47
3	36.96	31.76
4	36.95	30.21
5	36.93	28.73

TABLE 1
Values of $\sum \phi$

Conservative Method



Nonconservative Method

FIG. 5. Numerical Solutions at $t = 5.0$

5. Concluding Remarks. We have presented a first-order finite-volume method with a simple necessary condition. The conservative semi-Lagrangian method depends only on the conservative interpolation to obtain values of conserved quantities inside the cells determined by departure points. Extension of this method to a higher-order method can utilize techniques already demonstrated, such as [1, 8]. In addition, general higher-order interpolation has been presented [3].

REFERENCES

- [1] J. BATES AND A. McDONALD, *multiply-upstream, semi-Lagrangian advective schemes: analysis and application to a multi-level primitive equation model*, Monthly Weather Review, 110 (1982), pp. 1831–1842.
- [2] C. JOHNSON AND A. SZEPESSY, *On the convergence of a finite element method for a nonlinear hyperbolic conservation law*, Mathematics of Computation, 49 (1987), pp. 427–444.
- [3] J. HYMAN, *Accurate monotonicity preserving cubic interpolation*, SIAM J. Sci. Stat. Comput., 4 (1983), pp. 645–654.
- [4] J. DOUGLAS, JR. AND T. F. RUSSELL, *Numerical methods for convection-dominated diffusion problems*

- based on combining the method of characteristics with finite element or finite difference procedures*, SIAM J. Numer. Anal., 19 (1982).
- [5] H.-C. KUO AND R. WILLIAMS, *Semi-Lagrangian solutions to the inviscid burgers equation*, Monthly Weather Review, 118 (1990), pp. 1278–1288.
 - [6] R. J. LEVEQUE, *Numerical Methods for Conservation Laws*, Birkhauser, Boston, 1990.
 - [7] M. BERCOVIER, O. PIRONNEAU, AND V. SASTRI, *Finite elements and characteristics for some parabolic-hyperbolic problems*, Appl. Math. Modeling, 7 (1983).
 - [8] A. McDONALD, *accuracy of multiply-upstream, semi-Lagrangian advective schemes II*, Monthly Weather Review, 115 (1986), pp. 1446–1450.
 - [9] P. COLELLA AND P. R. WOODWARD, *The piecewise parabolic method (ppm) for gas-dynamical simulations*, J. Comp. Phy., 54 (1984), pp. 174–201.
 - [10] P. DEGOND AND S. MAS-GALLIC, *The weighted particle method for convection-diffusion equations*, Tech. Rep. 180, Centre de Mathematiques Appliquees, Unite de Recherche Associee au CNRS-756, 91128 Palaiseau Cedex, France, June 1988.
 - [11] A. PRIESTLEY, *A quasi-conservative version of the semi-Lagrangian advection scheme*, Monthly Weather Review, 121 (1993), pp. 621–629.
 - [12] M. RANCIC, *Semi-Lagrangian piecewise biparabolic scheme for two-dimensional horizontal advection of a passive scalar*, Monthly Weather Review, 120 (1992), pp. 1394–1405.
 - [13] R.E. EWING, T.F. RUSSELL, AND M.F. WHEELER, *Simulation of Miscible Displacement Using Mixed Methods and a Modified Method of Characteristics*, SPE, Dallas, TX, 1983.
 - [14] S. MAS-GALLIC AND P. A. RAVIART, *Particle approximation of convection-diffusion problems*, Tech. Rep. R86013, Laboratoire D'Analyse Numerique, Universite Pierre et Marie Curie, Laboratoire d'Analyse Numerique (U.A. 189), Tour 55-65 - 5eme etage, 4, place Jussieu, 75252 Paris Cedex 05, 1986.
 - [15] F. SEMAZZI, J. SCROGGS, AND G. POULIOT, *A locally conserving semi-lagrangian numerical scheme*, in Proceedings of the Tenth Conference on Numerical Weather Prediction, Portland Oregon, July 1994, pp. 60–62.
 - [16] A. STANFORTH AND J. COTE, *Semi-Lagrangian integration schemes for atmospheric models—a review*, Monthly weather review, 119 (1991), pp. 2206–2223.