On Tournaments Having
A Unique Hamiltonian Circuit*

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ABSTRACT

A tournament is a directed complete graph. A Hamiltonian circuit is a circuit which passes through each vertex of the graph once and only once. This note examines the family $T_n$ of all nonisomorphic tournaments on $n$ vertices which have a unique Hamiltonian circuit. We let $T_n = |T_n|$. In [1] Douglas gives a graphical characterization of the family $T_n$, and from this characterization obtains an involved formula for calculating the values $T_n$. In particular

$$
\begin{array}{cccccccccc}
  n = & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
  T_n = & 1 & 1 & 3 & 8 & 21 & 55 & 144 & 377 \\
\end{array}
$$

This note presents a graphical construction of the family $T_{n+1}$ from the family $T_n$ and, using this construction, obtains a proof of the recurrence

$$
T_{n+2} = 3 T_{n+1} - T_n, \quad (n \geq 4).
$$
1. **Preliminaries.**

The *outdegree* of a vertex $v$ is the number of edges incident with $v$ and directed away from it. The *indegree* of $v$ is the number of incident edges directed into it.

**THEOREM 1:**

(i) If $T \in \mathcal{T}_n$, then $T$ has at least one, and at most two, vertices with maximal outdegree, i.e., with outdegree $n-2$.

(ii) If $T \in \mathcal{T}_n$ has 2 vertices $v_1,v_2$ of maximal outdegree, then $v_1,v_2$ are successive vertices of the unique Hamiltonian circuit in $T$.

**PROOF:** Since $T \in \mathcal{T}_n$ has a Hamiltonian circuit, we first note that the maximal outdegree of any vertex is $n-2$.

(i) If $T \in \mathcal{T}_n$ then it follows from corollary 3 of Douglas [1] that at least one vertex must have outdegree $n-2$. Furthermore, if $v_1,v_2$ are vertices of maximal outdegree $n-2$, and $v$ any other vertex, then the indegree of $v$ must be at least 2. Consequently, the outdegree of $v$ is at most $(n-1)-2 = n-3$.

(ii) Let $v_1,v_2$ be of maximal outdegree $n-2$. Without loss of generality, assume that the edge between $v_1$ and $v_2$ is $v_1 \rightarrow v_2$. Then this is the only incoming edge for $v_2$. Hence it must be an edge of the Hamiltonian circuit.

Thus, we can divide the family $\mathcal{T}_n$ into disjoint families according to whether a tournament $T \in \mathcal{T}_n$ has 1 or 2 vertices of maximal outdegree. We define

$$
A_n = \{T \in \mathcal{T}_n : T \text{ has only 1 vertex of maximal outdegree}\}
$$

$$
B_n = \{T \in \mathcal{T}_n : T \text{ has 2 vertices of maximal outdegree}\}
$$
and

\[ A_n = |A_n| , \quad B_n = |B_n| . \]

Then clearly

\[ T_n = A_n \cup B_n \]

and

\[ T_n = A_n + B_n . \]

2. **Construction.**

For \( n \geq 3 \), if we are given the family \( T_n \) then we may use the following graphical construction to obtain the family \( T_{n+1} \). For each \( T \in T_n \) we will adjoin an \((n+1)\)-st vertex of maximal outdegree (i.e., of outdegree \( n-1 \)), while at the same time maintaining all edge orientations of \( T \), thus obtaining a tournament on \((n+1)\) vertices. This vertex may be adjoined at either 2 or 3 different positions (depending on whether \( T \in A_n \) or \( T \in B_n \)) with the resulting tournaments being nonisomorphic. The resulting tournaments will, of necessity, be in \( T_{n+1} \).

Conversely, we will show that every tournament in \( T_{n+1} \) can be gotten by using this construction. It will follow that \( T_{n+1} \) is precisely the family of all tournaments obtained by applying this construction to \( T_n \).

The graphical construction is as follows:

I. Suppose \( T \in A_n \). Let \( a \) be the vertex of maximal outdegree and the Hamiltonian circuit \( H \) be as shown.
(i) Adjoin to $T$ an $(n+1)$-st vertex $v$ of maximal outdegree $n-1$ and whose only incoming edge is from $b_0$. (All other edges of $T$ are retained with their same orientation.) We thus obtain a tournament $T'$ on $n+1$ vertices containing

We wish to show that $T' \in \mathcal{T}_{n+1}$. Obviously $T'$ has a Hamiltonian circuit; namely, the path (of $H$) from $a$ to $b_0$, along with the edges $b_0 \rightarrow v \rightarrow a$. Call this Hamiltonian circuit $H'$.

Furthermore, $H'$ will be the only Hamiltonian circuit of $T'$. Indeed any Hamiltonian circuit of $T'$ must contain the edges $b_0 \rightarrow v \rightarrow a$ since $v$ has only one incoming edge, and $a$ has only two incoming edges, one of which is from $b_0$. Thus any Hamiltonian circuit of $T'$ must look like
But then the path from $a$ to $b_0$ along with the edge $b_0 \rightarrow a$ would constitute a Hamiltonian circuit of $T$, and by assumption there is only one such circuit, namely $H$. Thus the circuit $H'$ is the only Hamiltonian circuit of $T'$, and hence $T' \in T_{n+1}$.

Note that $T' \in A_{n+1}$ since it has only one vertex of maximal outdegree, namely $v$.

We will denote this particular constructive technique (applied only to members of $A_n$) as $C_1$, and we write

$$T' = C_1(T).$$

(ii) Similarly, if we adjoin to $T$ an $(n+1)$-st vertex $v$ of maximal outdegree and whose only incoming edge is from $a$
then we will obtain another tournament \( T' \) in \( \mathcal{T}_{n+1} \). Note that in this case \( T' \in \mathcal{B}_{n+1} \) since \( a \) and \( v \) both have maximal outdegree \( n-1 \).

We denote this constructive technique (applied only to members of \( A_n \)) as \( C_2 \) and write

\[
T' = C_2(T).
\]

II. Suppose \( T \in \mathcal{B}_n \). Let \( a_0, a_1 \) be the vertices of maximal outdegree and the Hamiltonian circuit as shown.

\[
\begin{array}{c}
\text{II} \\
\begin{array}{c}
\text{H} \\
\text{H} \\
\text{H}
\end{array}
\end{array}
\]

(i) Adjoin an \((n+1)\)-st vertex \( v \) of maximal outdegree whose only incoming edge is from \( b_0 \). We obtain a tournament \( T' \) on \( n+1 \) vertices containing

Again the fact that \( T' \) will have one and only one Hamiltonian circuit follows from the fact that the same property holds for \( T \). Since \( v \) will be the only vertex of maximal degree, then \( T \in \mathcal{A}_{n+1} \). We call this procedure \( C_3 \) and let \( T' = C_3(T) \).
(ii) Adjoin a vertex \( v \) of maximal outdegree \( n-1 \) whose only incoming edge is from \( a_0 \). The resulting tournament \( T' \) on \( n+1 \) vertices contains

![Diagram]

As before, \( T' \) has one and only one Hamiltonian circuit. Further, \( T' \in \mathcal{B}_{n+1} \) since \( v \) and \( a_0 \) have maximal outdegree. This technique we call \( C_4 \) and we write

\[ T' = C_4(T) \]

in this case.

(iii) Adjoin a vertex \( v \) of maximal outdegree whose only incoming edge is from \( a_1 \). This tournament \( T' \) contains

![Diagram]

Again, \( T' \in \mathcal{T}_{n+1} \) and in fact, \( T' \in \mathcal{B}_{n+1} \) since \( v \) and \( a_1 \) both have maximal outdegree.
This technique we denote as $C_5$ and thus

$$T' = C_5(T) \in B_{n+1} \text{ for every } T \in B_{n}.$$ 

This completes the graphical construction procedure.

**Notation:** In the following, we let

$$C_i(T_n) = \{C_i(T): T \in A_{n}\} \text{ for } i = 1, 2$$

$$C_i(T_n) = \{C_i(T): T \in B_{n}\} \text{ for } i = 3, 4, 5$$

and

$$C = \bigcup_{i=1}^{5} C_i(T_n). \quad (1)$$

In other words $C$ is the family of all tournaments which can be constructed from members of $T_n$.

3. **Distinctness of Elements of $C$.**

We wish to show that all members of $C$ are nonisomorphic. Specifically, we will show that each element of $T_{n+1}$ can be constructed from one and only one member of $T_n$, and this must be done using a uniquely determined procedure $C_i$. Since each member of $C$ is in $T_{n+1}$, and no two members of $T_{n+1}$ are isomorphic, this will mean that no two different members of $C$ (i.e., constructed by different procedures $C_i$ or from different members of $T_n$) can be isomorphic.

**Lemma 1:** Let $T \in T_{n+1}$. Then there exists a unique $S \in T_n$ and a uniquely determined procedure $C_i$ such that
\[ T = C_1(S) . \]

**PROOF:** If \( T \in A_{n+1} \), then \( T \) has only one vertex \( v \) of maximal outdegree, so \( v \) must have been the vertex added in the constructive process. Therefore, the member of \( T_n \) from which \( T \) was constructed must have been the subgraph \( S \) of \( T \) gotten by deleting the vertex \( v \). If \( S \in A_n \), then the constructive procedure must have been \( C_1 \). If \( S \in B_n \), it must have been \( C_3 \).

If \( T \in B_{n+1} \) then let \( v_1, v_2 \) be the vertices of maximal outdegree. By Theorem 1 (ii), \( v_1 \) and \( v_2 \) must be adjacent so, without loss of generality, assume that the edge \( v_1 \rightarrow v_2 \) is in \( T \). From the description of the construction (and specifically procedures \( C_2, C_4, C_5 \)) we see that \( v_2 \) must have been the vertex which was added. Thus, again, the member of \( T_n \) from which \( T \) was constructed must have been the subgraph \( S \) of \( T \) gotten by deleting \( v_2 \). If \( S \in A_n \) then the constructive procedure must have been \( C_2 \). If \( S \in B_n \) and its vertices of maximal outdegree \( v_1, v_3 \) are joined by the edge \( v_1 \rightarrow v_3 \) then the procedure had to be \( C_4 \); if they are joined by \( v_3 \rightarrow v_1 \) then the procedure had to be \( C_5 \).

**THEOREM 2:** The members of \( C \) are nonisomorphic.

**PROOF:** Follows from lemma 1, since each member of \( C \) is also a member of \( T_{n+1} \).

4. **Equivalence of \( C \) and \( T_{n+1} \).**

Each object we construct from \( T_n \) is an element of \( T_{n+1} \). Since the constructed objects are nonisomorphic, then

\[ C \subseteq T_{n+1} . \]
We wish to show now that each member of $T_{n+1}$ may be gotten by applying the construction to some member of $T_n$.

**Theorem 3:** $T_{n+1} \subseteq C$.

**Proof:** Let $T \in T_{n+1}$.

Case I: If $T \in B_{n+1}$ with vertices $v_1, v_2$ of maximal outdegree and Hamiltonian circuit $H$, then $T$ contains the following structure,

![Diagram](image)

Consider the subgraph $S$ gotten by deleting vertex $v_2$ and all its incident edges. Then $S$ is a tournament with at least one Hamiltonian circuit, namely

$$C = a, v_1, b, \ldots, a.$$  

If we can show that $C$ is the only Hamiltonian circuit of $S$, then it follows immediately that $S \in T_n$ and that $T$ can be constructed from $S$.

Suppose $S$ has a second Hamiltonian circuit $C_0$ different from $C$.

![Diagram](image)
Then, for some vertex w, w \( \neq v_1 \), in S we must have the edge \( v_1 \to w \) in \( C_0 \). Now let \( C'_0 \) be the circuit in T which is obtained by replacing the edge \( v_1 \to w \) in \( C_0 \) by the two edges \( v_1 \to v_2 \to w \). (Recall that the edge between \( v_2 \) and w must be directed towards w.) Further, it is clear that \( C'_0 \) differing from \( C \) (in S) implies that \( C'_0 \) will differ from \( H \) (in T). But this is not possible since \( H \) is unique. Thus our supposition that S has two Hamiltonian circuits must be false and the desired results follow.

Case II: Let \( T \in A_{n+1} \) with \( v \) the vertex of maximum outdegree and \( H \) the unique Hamiltonian circuit.

Consider the subgraph S gotten by deleting \( v \) from T. Again S is a tournament. We must show that \( S \in T' \). Observe that if we knew that the edge \( a \to b \) were in T then we would obviously have a Hamiltonian circuit in S; namely, the circuit gotten by replacing the edges \( a \to v \to b \) in \( H \) by the edge \( a \to b \).

In fact, we will prove that the edge \( a \to b \) must be in T. Suppose otherwise, i.e., suppose that \( b \to a \) is an edge of T.
Let \( a_0 \) be the first vertex preceding \( a \) (in the Hamiltonian circuit \( H \)) for which the edge between \( a_0 \) and \( b \) is directed towards \( b \). Thus if \( a_0 \rightarrow a_1 \) is an edge in \( H \) then the edge between \( a_1 \) and \( b \) must be directed towards \( a_1 \). Also if \( b \rightarrow b_0 \) is an edge of \( H \) then the edge between \( v \) and \( b_0 \) is directed toward \( b_0 \) since \( v \) has maximal outdegree. Therefore the circuit

\[
a_0 \rightarrow b \rightarrow a_1 \rightarrow \ldots \rightarrow a \rightarrow v \rightarrow b_0 \rightarrow \ldots \rightarrow a_0
\]

(where the dots indicate we follow the circuit \( H \)) will be another Hamiltonian circuit in \( T \), different from \( H \). This is not possible, so we conclude that the edge \( b \rightarrow a \) cannot be in \( T \).

Thus, \( S \) has at least one Hamiltonian circuit:

\[ C: \ a \rightarrow b \rightarrow b_0 \rightarrow \ldots \rightarrow a \ . \]

Suppose \( S \) has a second Hamiltonian circuit \( C_0 \), different from \( C \).

Let \( a \rightarrow b_1 \) be the edge of \( C_0 \) which is directed away from \( a \) (\( b_1 \) may equal \( b \)).

If, in \( C_0 \), we replace \( a \rightarrow b_1 \) by the edges \( a \rightarrow v \rightarrow b_1 \) then we will have a Hamiltonian circuit in \( T \) which is different from \( H \). This is impossible, so we conclude that we cannot have a second Hamiltonian circuit in \( S \), i.e.,
Thus if \( T \in A_{n+1} \) then \( T \) can be gotten by applying the construction to \( S \in T_n \).

**THEOREM 4:** \( C = T_{n+1} \).

In words, by applying the construction of section 2 to \( T_n \), we obtain \( T_{n+1} \) and each member of \( T_{n+1} \) is constructed in a uniquely determined fashion.

**PROOF:** Follows from (1), Lemma 1, and Theorem 3.

5. **Recurrence.**

From the description of the construction we see that

\[
A_{n+1} = C_1(T_n) \cup C_3(T_n)
\]
\[
B_{n+1} = C_2(T_n) \cup C_4(T_n) \cup C_5(T_n)
\]

where the unions are disjoint, and by Theorem 2

\[
A_{n+1} = |A_{n+1}| = |C_1(T_n)| + |C_3(T_n)|
\]
\[
= |A_n| + |B_n|
\]
\[
= A_n + B_n
\]

and
\[ B_{n+1} = |\mathcal{B}_{n+1}| = |C_2(T_n)| + |C_4(T_n)| + |C_5(T_n)| \]
\[ = |A_n| + |B_n| + |B_n| \]
\[ = A_n + 2B_n \]
\[ = A_{n+1} + B_n \] (3)

where the least equality follows by applying (2).

**THEOREM 5:** The sequences \( \{A_n\} \) and \( \{B_n\} \) both satisfy the recurrence

\[ x_{n+2} = 3x_{n+1} - x_n, \quad (n \geq 4). \]

**PROOF:** From (2) and (4),

\[ A_{n+2} - A_{n+1} = B_{n+1} \]
\[ = A_{n+1} + B_n \]
\[ = A_{n+1} + (A_{n+1} - A_n) \]
\[ = 2A_{n+1} - A_n \]

so that

\[ A_{n+2} = 3A_{n+1} - A_n. \]

Also from (3) and (4)

\[ B_{n+2} = A_{n+1} + 2B_{n+1} \]
\[ = (B_{n+1} - B_n) + 2B_{n+1} \]
\[ = 3B_{n+1} - B_n. \]
THEOREM 6: The sequence \( \{T_n\} \) which counts the number of nonisomorphic tournaments on \( n \) vertices, which have a unique Hamiltonian circuit, satisfies

\[
T_{n+2} = 3T_{n+1} - T_n \quad (n \geq 4).
\]

PROOF: Follows from Theorem 5 since \( T_n = A_n + B_n \) for every \( n \).
Bibliography


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