ASYMPTOTIC DISTRIBUTIONS FOR HOTELLING'S T_0^2 AND PILLAI'S STATISTIC FOR TESTING INDEPENDENCE - A DIFFERENT APPROACH

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Asymptotic Distributions for Hotelling's T_0^2 and Pillai's Statistic for Testing Independence - A Different Approach

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1. **Introduction.** Let $A(p \times p)$ be distributed Wishart with covariance matrix Σ and n degrees of freedom. Partition A and Σ as

$$A = \begin{pmatrix} A_{11} \ (q \times q) & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \Sigma_{11} \ (q \times q) & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}.$$

Let $P = \Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$ define the generalized population correlation matrix and $R = A_{11}^{-1}A_{12}A_{22}^{-1}A_{21}$ define the generalized sample correlation matrix. Fujikoshi [2] derived the asymptotic distributions under the alternative of the two test statistics

(i) $n \operatorname{tr} R$ - Pillai's-statistic,

(ii) $m \operatorname{tr} R(I - R)^{-1}$ - Hotelling's T_0^2, $m = n - p + q$

for testing the hypotheses H_0: $P = 0$ against the alternative H_1: $P \neq 0$. We shall derive the same two noncentral distributions in this paper but will use a different approach. The approach will be to consider A_{22} as fixed. The asymptotic characteristic functions of the two test statistics conditional on A_{22} is known from Fujikoshi [2] and by averaging out over A_{22} the unconditional characteristic functions will be derived. The asymptotic distributions will be considered under the assumption that $P = \frac{1}{n}\theta$ as pointed out by Sugiura [3]. The following preliminary results will be

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necessary:

2. Preliminary Results:

Lemma 2.1. Let A and Σ be partitioned as in Section 1 with A distributed $W(\Sigma,n)$, then $A_{11.2} = A_{11} - A_{12}A_{22}^{-1}A_{21}$ is distributed $W(\Sigma_{11.2}, n - p + q)$ independently of $G = A_{12}A_{22}^{-1}A_{21}$ and G conditional on A_{22} is distributed $W(\Sigma_{11.2}, p - q, \Omega)$, i.e. a noncentral Wishart with noncentrality parameter $\Omega = \Sigma_{11.2}^{-\frac{1}{2}}\beta A_{22}\beta'\Sigma_{11.2}^{-\frac{1}{2}}$, $\beta = \Sigma_{12}\Sigma_{22}^{-1}$.

Proof: The result is well known and can be proved from Anderson [1], Theorem 4.3.2.

Lemma 2.2. If $A_{11.2}$ is distributed $W(\Sigma_{11.2}, n - p + q)$ and G conditional on A_{22} is distributed $W(\Sigma_{11.2}, p - q, \Omega)$, then conditional on A_{22} the characteristic function of $n \operatorname{tr} R = n \operatorname{tr}(A_{11.2} + G)^{-1}G$ is given by (Fujikoshi [2], Equation 5.17).

$$
\begin{aligned}
(1)\quad g_1(t|A_{22}) = {} & (1 - 2it)^{-\frac{1}{2}q(p-q)} \operatorname{etr}\left\{\frac{it}{1-2it}\Omega\right\} \Big[1 - \frac{1}{4n}\Big\{q(p - q)(p + 1) \\
& \left(1 - 2(1 - 2it)^{-1}\right) + \left(q(p - q)(p + 1) - 4(p + 1)\operatorname{tr}\tfrac{1}{2}\Omega - 4\operatorname{tr}(\tfrac{1}{2}\Omega)^2\right) \\
& (1 - 2it)^{-2} + 4(p + 1)(1 - 2it)^{-3}\operatorname{tr}\tfrac{1}{2}\Omega + 4(1 - 2it)^{-4}\operatorname{tr}(\tfrac{1}{2}\Omega)^2\Big\} + \\
& \frac{1}{96n^2}\Big\{q(p - q)\left(h_0 - h_1(1 - 2it)^{-1}\right) + \textstyle\sum_{\alpha=2}^{8} A_\alpha(\tfrac{1}{2}\Omega)(1 - 2it)^{-\alpha}\Big\} + \\
& O(n^{-3}) \Big]
\end{aligned}
$$

where

$$A_2(\tfrac{1}{2}\Omega) = q(p - q)h_2 - 2h_1 tr\left(\frac{\Omega}{2}\right) - 24q(p - q)(p + 1)tr\left(\frac{\Omega}{2}\right)^2$$

$$A_3(\tfrac{1}{2}\Omega) = -q(p - q)h_3 + 4h_2 tr\left(\frac{\Omega}{2}\right) + 48\{(p - q)q^2 + \left((p - q)^2 + (p - q) + 4\right)q + 4(p - q + 1)\}tr\left(\frac{\Omega}{2}\right)^2 + 128 tr\left(\frac{\Omega}{2}\right)^3$$

$$A_4(\tfrac{1}{2}\Omega) = q(p - q)h_4 - 6h_3 tr\left(\frac{\Omega}{2}\right) + 48\{q^2 + 2(p - q + 1)q + (p - q)^2 + 2(p - q) - 1 \left(tr\frac{\Omega}{2}\right)^2 - 96(p + 2)tr\left(\frac{\Omega}{2}\right)^2 + 96(p + 1)tr\left(\frac{\Omega}{2}\right)tr\left(\frac{\Omega}{2}\right)^2 + 48\left(tr\left(\frac{\Omega}{2}\right)^2\right)^2$$

$$A_5(\tfrac{1}{2}\Omega) = 8h_4 tr\left(\frac{\Omega}{2}\right) - 96\{q^2 + 2(p - q + 1)q + (p - q)^2 + 2(p - q) + 3\}\left(tr\frac{\Omega}{2}\right)^2 - 48\{(p - q)q^2 + \left((p - q)^2 + (p - q) + 12\right)q + 4\left(3(p - q) + 4\right)\}tr\left(\frac{\Omega}{2}\right)^2 - 96(p + 1)(tr\ \Omega)tr\left(\frac{\Omega}{2}\right)^2 - 384 tr\left(\frac{\Omega}{2}\right)^3$$

$$A_6(\tfrac{1}{2}\Omega) = 48\{q^2 + 2(p - q + 1)q + (p - q)^2 + 2(p - q) + 7\}\left(tr\frac{\Omega}{2}\right)^2 + 24\{(p - q)q^2 + \left((p - q)^2 + (p - q) + 20\right)q + 4\left(5(p - q) + 8\right)\}tr\left(\frac{\Omega}{2}\right)^2 - 96(p + 1)(tr\ \Omega)\, tr\left(\frac{\Omega}{2}\right)^2 - 128 tr\left(\frac{\Omega}{2}\right)^3 - 96\left(tr\left(\frac{\Omega}{2}\right)^2\right)^2$$

$$A_7(\tfrac{1}{2}\Omega) = 96(p + 1)\left(tr\frac{\Omega}{2}\right)tr\left(\frac{\Omega}{2}\right)^2 + 384 tr\left(\frac{\Omega}{2}\right)^3$$

$$A_8(\tfrac{1}{2}\Omega) = 48\left(tr\left(\frac{\Omega}{2}\right)^2\right)^2$$

and

$$h_0 = 3(p - q)q^3 + 2\left(3(p - q)^2 + 3(p - q) - 4\right)q^2 + 3(p - q + 1)\left((p - q)^2 + (p - q) - 4\right)q - 4\left(2(p - q)^2 + 3(p - q) - 1\right)$$

$$h_1 = 12q(p - q)(p + 1)^2$$

$$h_2 = 6\{3(p - q)q^3 + 2\left(3(p - q)^2 + 3(p - q) + 4\right)q^2 + (p - q + 1)\left(3(p - q)^2 + 3(p - q) + 16\right)q + 8(p - q + 1)^2\}$$

$$h_3 = 4\{3(p - q)q^3 + 2\big(3(p - q)^2 + 3(p - q) + 8\big)q^2 + 3(p - q + 1)\big((p - q)^2 + (p - q) + 12\big)q + 4\big(4(p - q)^2 + 9(p - q) + 7\big)\}$$

$$h_4 = 3\{(p - q)q^3 + 2\big((p - q)^2 + (p - q) + 4\big)q^2 + (p - q + 1)\big((p - q)^2 + (p - q) + 20\big)q + 4\big(2(p - q)^2 + 5(p - q) + 5\big)\}$$

$$\Omega = \beta'\Sigma_{11.2}^{-1}\beta A_{22}\ , \quad \beta = \Sigma_{12}\Sigma_{22}^{-1}$$

Lemma 2.3. If $A_{11.2}$ is distributed $W(\Sigma_{11.2}, n - p + q)$ and G conditional on A_{22} is distributed $W(\Sigma_{11.2}, p - q, \Omega)$, $\Omega = \beta'\Sigma_{11.2}^{-1}\beta A_{22}$, $\beta = \Sigma_{12}\Sigma_{22}^{-1}$, then conditional on A_{22} the characteristic function of $m \operatorname{tr} R(I - R)^{-1} = m \operatorname{tr} A_{11.2}^{-1}G$, $m = n - p + q$ is given by (Fujikoshi [2], Equation 6.16).

$$(2) \quad g_2(t|A_{22}) = (1 - 2it)^{-q(p-q)} \operatorname{etr}\left\{\frac{it}{1-2it}\Omega\right\} \Big[1 + \frac{1}{4m}\{q(p - q)(p - 2q - 1) - 2(p - q)\big(q(p - q) - \operatorname{tr}(\Omega)\big)(1 - 2it)^{-1} + \big(q(p - q)(p + 1) - 2(-q + 2p + 1)\operatorname{tr}(\Omega) + \operatorname{tr}(\Omega)^2\big)(1 - 2it)^{-2} + 2\big((p + 1)\operatorname{tr}(\Omega) - \operatorname{tr}(\Omega)^2\big)(1 - 2it)^{-3} + \operatorname{tr}(\Omega)^2(1 - 2it)^{-4}\} + \frac{1}{96m^2}\sum_{\alpha=0}^{8} B_\alpha(\tfrac{1}{2}\Omega)(1 - 2it)^{-\alpha} + O(m^{-3}) \Big]$$

where $B_0(\tfrac{1}{2}\Omega) = q(p - q)\ell_0$

$$B_1(\tfrac{1}{2}\Omega) = -\ell_1\big(q(p - q) - \operatorname{tr}(\Omega)\big)$$

$$B_2(\tfrac{1}{2}\Omega) = q(p - q)\ell_2 - (\ell_1 + 2\ell_2)\operatorname{tr}(\Omega) + 12(p - q)^2\operatorname{tr}^2(\Omega) - 6(p - q)\big(q^2 - q(p - q - 1) - 4\big)\operatorname{tr}(\Omega)^2$$

$$B_3(\tfrac{1}{2}\Omega) = -q(p - q)\ell_3 + (2\ell_2 + 3\ell_3)\operatorname{tr}(\Omega) - 24\big(q(p - q) + 2(p - q)^2 + p - q + 2\big)\operatorname{tr}^2(\Omega) + 12\Big\{q^2(p - q) - \big(2(p - q)^2 - p + q + 4\big)q - 8\big(2(p - q) + 1\big)\Big\}\operatorname{tr}(\Omega)^2 + 12(p - q)\operatorname{tr}(\Omega)\operatorname{tr}(\Omega)^2 + 16\operatorname{tr}(\Omega)^3$$

$$B_4(\tfrac{1}{2}\Omega) = q(p - q)\ell_4 - (3\ell_3 + 4\ell_4)tr(\Omega) + 12\Big(q^2 + 2\big(3(p - q) + 1\big)q + 3\big(2(p - q)^2 + 2(p - q) + 5\big)\Big)tr^2(\Omega) + 12\Big(3\big((p - q)^2 + 6\big)q + 4\big(9(p - q) + 8\big)\Big)tr(\Omega)^2 - 12\big(q + 4(p - q) + 1\big)tr(\Omega)tr(\Omega)^2 - 96tr(\Omega)^3 + 3tr^2(\Omega)^2$$

$$B_5(\tfrac{1}{2}\Omega) = 4\ell_4 tr(\Omega) - 24\Big(q^2 + q\big(3(p - q) + 2\big) + 2(p - q)^2 + 3(p - q) + 9\Big)tr^2(\Omega) - 12\Big(q^2(p - q) + \big(2(p - q)^2 + p - q + 24\big)q + 8\big(4(p - q) + 5\big)\Big)tr(\Omega)^2 + 36\big(q + 2(p - q) + 1\big)tr(\Omega)tr(\Omega)^2 + 192tr(\Omega)^3 - 12tr^2(\Omega)^2$$

$$B_6(\tfrac{1}{2}\Omega) = 12\big(q^2 + 2(p - q + 1)q + (p - q)^2 + 2(p - q) + 7\big)tr^2(\Omega) + 6\Big(q^2(p - q) + q(p + q)^2 + q(p - q) + 20q + 4\big(5(p - q) + 8\big)\Big)tr(\Omega)^2 - 12\big(3q + 4(p - q) + 3\big)tr(\Omega)tr(\Omega)^2 - 160tr(\Omega)^3 + 18tr^2(\Omega)^2$$

$$B_7(\tfrac{1}{2}\Omega) = 12(p + 1)tr(\Omega)tr(\Omega)^2 + 48tr(\Omega)^3 - 12tr^2(\Omega)^2$$

$$B_8(\tfrac{1}{2}\Omega) = 6tr^2(\Omega)^2$$

$$\ell_0 = 3(p - q)q^3 - 2\big(3(p - q)^2 - 3(p - q) + 4\big)q^2 + 3q(p - q - 1)\big((p - q)^2 - p + q + 4\big) - 4\big(2(p - q)^2 - 3(p - q) - 1\big)$$

$$\ell_1 = -12q(p - q)^2(q + 1)$$

$$\ell_2 = -6(p - q)\Big(q^3 + 2q^2 - 3q\big((p - q)^2 + 1\big) - 4\big(2(p - q) + 1\big)\Big)$$

$$\ell_3 = 4\{\big(3(p - q)^2 + 4\big)q^2 + 3q\big((p - q)^3 + (p - q)^2 + 8(p - q) + 4\big) + 8\big(2(p - q)^2 + 3(p - q) + 2\big)\}$$

$$\ell_4 = 3\{q^3(p - q) + 2\big((p - q)^2 + (p - q) + 4\big)q^2 + (p - q + 1)\big((p - q)^2 + (p - q) + 20\big)q + 4\big(2(p - q)^2 + 5(p - q) + 5\big)\}$$

Lemma 2.4. Let $A_{22}(p - q \times p - q)$ be distributed $W(\Sigma_{22}, n)$, then for $\beta = \Sigma_{12}\Sigma_{22}^{-1}$, $\Sigma_{11.2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$ and $P = \Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} = \frac{1}{n}\theta$

$$(3)\quad E\ \mathrm{etr}\left\{\frac{it}{1-2it}\beta'\Sigma_{11.2}^{-1}\beta A_{22}\right\} = \mathrm{etr}\left\{\frac{it\theta}{1-2it}\right\}\Big[1 + \frac{1}{n}[\tfrac{1}{2}\mathrm{tr}\theta^2\big((1 - 2it)^{-1} - 1\big) + \tfrac{1}{4}\mathrm{tr}\theta^2\big((1 - 2it)^{-2} - 2(1 - 2it)^{-1} + 1\big)] + \frac{1}{n^2}[\tfrac{1}{2}\mathrm{tr}\theta^3\big((1 - 2it)^{-1} - 1\big) + \tfrac{1}{2}\mathrm{tr}\theta^3\big((1 - 2it)^{-2} - 2(1 - 2it)^{-1} + 1\big) + \tfrac{2}{3}\mathrm{tr}\theta^3\big((1 - 2it)^{-3} - 3(1 - 2it)^{-2} + 3(1 - 2it)^{-1} - 1\big) + \big\{ \tfrac{1}{8}\big((1 - 2it)^{-2} - 2(1 - 2it)^{-1} + 1\big)\mathrm{tr}^2\theta^2 + \tfrac{1}{8}\mathrm{tr}^2\theta^2\big((1 - 2it)^{-3} - 3(1 - 2it)^{-2} + 3(1 - 2it)^{-1} - 1\big) + \tfrac{1}{32}\mathrm{tr}^2\theta^2\big((1 - 2it)^{-4} - 4(1 - 2it)^{-3} + 6(1 - 2it)^{-2} - 4(1 - 2it)^{-1} + 1\big)] + 0(n^{-3})\Big]$$

$$(4)\quad E\ \mathrm{etr}\left\{\frac{it}{1-2it}\beta'\Sigma_{11.2}^{-1}\beta A_{22}\right\}\mathrm{tr}(\tfrac{1}{2}\beta'\Sigma_{11.2}^{-1}\beta A_{22}) = \mathrm{etr}\left\{\frac{it\theta}{1-2it}\right\}[\tfrac{1}{2}\mathrm{tr}\theta + \frac{1}{2n}\{\mathrm{tr}\theta^2 + \mathrm{tr}\theta^2\big((1 - 2it)^{-1} - 1\big) + \tfrac{1}{2}\mathrm{tr}\theta\mathrm{tr}\theta^2\big((1 - 2it)^{-1} - 1\big) + \tfrac{1}{4}\mathrm{tr}\theta\mathrm{tr}\theta^2\big((1 - 2it)^{-2} - 2(1 - 2it)^{-1} + 1\big) + 0(n^{-2})\Big]$$

(5) $E\,\mathrm{etr}\left\{\frac{it}{1-2it}\beta'\Sigma_{11.2}^{-1}\beta A_{22}\right\}\mathrm{tr}(\tfrac{1}{2}\beta'\Sigma_{11.2}^{-1}\beta A_{22})^2 =$

$$= \mathrm{etr}\left\{\frac{it\theta}{1-2it}\right\}[\tfrac{1}{4}\mathrm{tr}\theta^2 + \frac{1}{4n}\{\tfrac{1}{2}\mathrm{tr}^2\theta^2\big((1-2it)^{-1}-1\big) + \tfrac{1}{4}\mathrm{tr}^2\theta^2\big((1-2it)^{-2}-2(1-2it)^{-1}+1\big) + \mathrm{tr}\theta^2 + 2\mathrm{tr}\theta^3 + 2\mathrm{tr}\theta^3\big((1-2it)^{-1}-1\big) + \mathrm{tr}^2\theta\} + 0(n^{-2})]$$

(6) $E\,\mathrm{etr}\left\{\frac{it}{1-2it}\beta'\Sigma_{11.2}^{-1}\beta A_{22}\right\}\mathrm{tr}(\tfrac{1}{2}\beta'\Sigma_{11.2}^{-1}\beta A_{22})^3 = \mathrm{etr}\left\{\frac{it\theta}{1-2it}\right\}[\tfrac{1}{8}\mathrm{tr}\theta^3 + 0(n^{-1})]$

(7) $E\,\mathrm{etr}\left\{\frac{it}{1-2it}\beta'\Sigma_{11.2}^{-1}\beta A_{22}\right\}\mathrm{tr}^2(\tfrac{1}{2}\beta'\Sigma_{11.2}^{-1}\beta A_{22}) = \mathrm{etr}\left\{\frac{it\theta}{1-2it}\right\}[\tfrac{1}{4}\mathrm{tr}^2\theta + 0(n^{-1})]$

(8) $E\,\mathrm{etr}\left\{\frac{it}{1-2it}\beta'\Sigma_{11.2}^{-1}\beta A_{22}\right\}\mathrm{tr}(\tfrac{1}{2}\beta'\Sigma_{11.2}^{-1}\beta A_{22})\mathrm{tr}(\tfrac{1}{2}\beta'\Sigma_{11.2}^{-1}\beta A_{22})^2 =$

$$= \mathrm{etr}\left\{\frac{it\theta}{1-2it}\right\}[\tfrac{1}{8}\mathrm{tr}\theta\,\mathrm{tr}\theta^2 + 0(n^{-1})]$$

(9) $E\,\mathrm{etr}\left\{\frac{it}{1-2it}\beta'\Sigma_{11.2}^{-1}\beta A_{22}\right\}\mathrm{tr}^2(\tfrac{1}{2}\beta'\Sigma_{11.2}^{-1}\beta A_{22})^2 = \mathrm{etr}\left\{\frac{it\theta}{1-2it}\right\}[\tfrac{1}{16}\mathrm{tr}^2\theta^2 + 0(n^{-1})]$

Proof: Since the zonal polynomial $C_K(\tfrac{1}{2}\beta'\Sigma_{11.2}^{-1}\beta A_{22})$ can be expressed in terms of sums of powers of the characteristic roots of the matrix and since the inverse is also true, the key integral will be

(10) $$T_K = \frac{1}{\Gamma_{p-q}(\tfrac{1}{2}n)}|2\Sigma_{22}|^{-\frac{1}{2}n}\int_{A_{22}>0}\mathrm{etr}(-\tfrac{1}{2}\Sigma_{22}^{-1}A_{22})|A_{22}|^{\frac{1}{2}(n-p+q-1)}\,\mathrm{etr}\left\{\frac{it}{1-2it}\beta'\Sigma_{11.2}^{-1}\beta A_{22}\right\}C_K(\tfrac{1}{2}\beta'\Sigma_{11.2}^{-1}\beta A_{22})dA_{22}$$

$$= (\tfrac{1}{2}n)_K\left|I - \frac{2it}{1-2it}\Sigma_{22}\beta'\Sigma_{11.2}^{-1}\beta\right|^{-\frac{1}{2}n}C_K\big(\beta'\Sigma_{11.2}^{-1}\beta(\Sigma_{22}^{-1} - \frac{2it}{1-2it}\beta'\Sigma_{11.2}^{-1}\beta)^{-1}\big)$$

But

(11) $$\Sigma_{11.2}^{-1} = \Sigma_{11}^{-\frac{1}{2}}(I - \frac{1}{n}\theta)\Sigma_{11}^{-\frac{1}{2}}$$

$$= \Sigma_{11}^{-\frac{1}{2}}\left(I + \frac{1}{n}\theta + \frac{1}{n^2}\theta^2 + 0(n^{-3})\right)\Sigma_{11}^{-\frac{1}{2}}$$

Let $\frac{1}{\sqrt{n}}\theta_1 = \Sigma_{22}^{\frac{1}{2}}\beta'\Sigma_{11}^{-\frac{1}{2}}$ such that $\theta_1'\theta_1 = nP = \theta$, then it follows that

(12) $$T_K = \operatorname{etr}\left\{\frac{it\theta}{1-2it}\right\}\Bigg[1 + \frac{1}{n}\left\{\alpha_1(K) + \left(\frac{it}{1-2it}\right)\operatorname{tr}\theta^2 + \left(\frac{it}{1-2it}\right)^2\operatorname{tr}\theta^2\right\} + \frac{1}{n^2}\Bigg\{\left(\frac{it}{1-2it}\right)\operatorname{tr}\theta^3 +$$

$$2\left(\frac{it}{1-2it}\right)^2\operatorname{tr}\theta^3 + \frac{4}{3}\left(\frac{it}{1-2it}\right)^3\operatorname{tr}\theta^3 + \frac{1}{2}\left(\frac{it}{1-2it}\right)^2\operatorname{tr}^2\theta^2 + \left(\frac{it}{1-2it}\right)^3\operatorname{tr}^2\theta^2 +$$

$$\frac{1}{2}\left(\frac{it}{1-2it}\right)^4\operatorname{tr}^2\theta^2 + \frac{1}{6}\left(k - \alpha_2(K) + 3\alpha_1^2(K)\right) + \alpha_1(K)\left\{\frac{it}{1-2it} + \left(\frac{it}{1-2it}\right)^2\right\}\operatorname{tr}\theta^2\Bigg\} +$$

$$0(n^{-2})\Bigg]\, C_K\left(\tfrac{1}{2}\theta + \frac{1}{2n}(\theta^2 + \frac{2it}{1-2it}\theta^2) + 0(n^{-2})\right)$$

where $\alpha_1(K) = \sum_{i=1}^q k_i(k_i - i)$

$\alpha_2(K) = \sum_{i=1}^q k_i(4k_i^2 - 6k_i + 3)$. Use was made of the binomial expansion into a series and

$$(\tfrac{1}{2}n)_K = (\tfrac{1}{2}n)^k\left[1 + \frac{1}{n}\alpha_1(K) + \frac{1}{6n^2}\left(k - \alpha_2(K) + 3\alpha_1^2(K)\right) + 0(n^{-3})\right]$$

3. Asymptotic Distributions.

Theorem 3.1. The asymptotic distribution of Pillai's statistic $n \operatorname{tr} R$ if $P = \frac{1}{n}\theta$, is given by

(13) $$P[n\ \mathrm{tr}\ R < z] = P[\chi_f^2(\delta^2) < z] - \frac{1}{4n}\left\{\sum_{\alpha=0}^{4} a_\alpha P\left[\ \chi^2_{f+2\alpha}\ (\delta^2) < z\ \right]\right\} + \frac{1}{96n^2}\left\{\sum_{\alpha=0}^{8} b_\alpha\ P\left[\ \chi^2_{f+2\alpha}\ (\delta^2) < z\ \right]\right\} + 0(n^{-3})$$

where a_α , $\alpha = 0, \ldots ,4$ and b_α , $\alpha = 0, \ldots ,8$ are given in (15) and (16) and $\delta^2 = \mathrm{tr}\frac{1}{2}\theta$, the noncentrality parameter of the noncentral $\chi^2_{f+2\alpha}$ distribution. $f = q(p - q)$.

<u>Proof</u>: Applying Lemma 2.4 on Lemma 2.2, the unconditional characteristic function of $n\ \mathrm{tr}\ R$ is given by

(14) $$\begin{aligned} q_1(t) &= E_{A_{22}}\ g_1(t|A_{22}) \\ &= \mathrm{etr}\left\{\frac{it\theta}{1-2it}\right\}(1 - 2it)^{-\frac{1}{2}q(p-q)} \left[1 - \frac{1}{4n}\left\{\sum_{\alpha=0}^{4} a_\alpha(1 - 2it)^{-\alpha}\right\} + \frac{1}{96n^2}\left\{\sum_{\alpha=0}^{8} b_\alpha(1 - 2it)^{-\alpha}\right\} + 0(n^{-3}) \right] \end{aligned}$$

where

(15) $$\begin{aligned} a_0 &= q(p - q)(p + 1) + 4\mathrm{tr}(\tfrac{1}{2}\theta)^2 \\ a_1 &= -2q(p - q)(p + 1) \\ a_2 &= q(p - q)(p + 1) - 4(p + 1)\mathrm{tr}\tfrac{1}{2}\theta - 8\mathrm{tr}(\tfrac{1}{2}\theta)^2 \\ a_3 &= 4(p + 1)\mathrm{tr}\tfrac{1}{2}\theta \\ a_4 &= 4\mathrm{tr}(\tfrac{1}{2}\theta)^2 \end{aligned}$$

and

(16) $$\begin{aligned} b_0 &= q(p - q)h_0 + 24q(p - q)(p + 1)\mathrm{tr}(\tfrac{1}{2}\theta)^2 - 128\mathrm{tr}(\tfrac{1}{2}\theta)^3 + 48\mathrm{tr}^2(\tfrac{1}{2}\theta)^2 \\ b_1 &= -q(p - q)h_1 - 48q(p - q)(p + 1)\mathrm{tr}(\tfrac{1}{2}\theta)^2 \\ b_2 &= q(p - q)h_2 - 2h_1\mathrm{tr}(\tfrac{1}{2}\ ) + 96\mathrm{tr}^2(\tfrac{1}{2}\theta) - 24\left(q(p - q)(p + 1) - 4\right)\mathrm{tr}(\tfrac{1}{2}\theta)^2 - \\ &\quad 96(p + 1)\mathrm{tr}\tfrac{1}{2}\theta\mathrm{tr}(\tfrac{1}{2}\theta)^2 - 192\mathrm{tr}^2(\tfrac{1}{2}\theta)^2 \end{aligned}$$

$$b_3 = -q(p - q)h_3 + 4h_2 tr\tfrac{1}{2}\theta + 96[(p - q)q^2 + q(p - q)^2 + q(p - q) + 4q + 4(p - q + 1)]tr(\tfrac{1}{2}\theta)^2 + 96(p + 1)tr\tfrac{1}{2}\theta tr(\tfrac{1}{2}\theta)^2 + 640tr(\tfrac{1}{2}\theta)^3$$

$$b_4 = q(p - q)h_4 - 6h_3 tr(\tfrac{1}{2}\theta) + 48\left(q^2 + 2(p - q + 1)q + (p - q)^2 + 2(p - q) - 3\right) tr^2(\tfrac{1}{2}\theta) - 24\left[(p - q)q^2 + \left((p - q)^2 + (p - q) + 12\right)q + 4\left(3(p - q) + 5\right)\right] tr(\tfrac{1}{2}\theta)^2 + 192(p + 1)tr(\tfrac{1}{2}\theta)tr(\tfrac{1}{2}\theta)^2 + 288tr^2(\tfrac{1}{2}\theta)^2$$

$$b_5 = 8h_4 tr\tfrac{1}{2}\theta - 96\left(q^2 + 2(p - q + 1)q + (p - q)^2 + 2(p - q) + 3\right)tr^2(\tfrac{1}{2}\theta) - 48\left((p - q)q^2 + q(p - q)^2 + q(p - q) + 12q + 12(p - q) + 16\right)tr(\tfrac{1}{2}\theta)^2 - 192(p + 1)tr(\tfrac{1}{2}\theta)tr(\tfrac{1}{2}\theta)^2 - 768tr(\tfrac{1}{2}\theta)^3$$

$$b_6 = 48\left(q^2 + 2(p - q + 1)q + (p - q)^2 + 2(p - q) + 7\right)tr^2(\tfrac{1}{2}\theta) + 24\left(q^2(p - q) + q(p - q)^2 + q(p - q) + 20q + 20(p - q) + 32\right)tr(\tfrac{1}{2}\theta)^2 - 96(p + 1)tr(\tfrac{1}{2}\theta) tr(\tfrac{1}{2}\theta)^2 - 128tr(\tfrac{1}{2}\theta)^3 - 192tr^2(\tfrac{1}{2}\theta)^2$$

$$b_7 = 96(p + 1)tr(\tfrac{1}{2}\theta)tr(\tfrac{1}{2}\theta)^2 + 384tr(\tfrac{1}{2}\theta)^3$$

$$b_8 = 48tr^2(\tfrac{1}{2}\theta)^2$$

and h_i , $i = 1, \ldots ,4$ given in Lemma 2.2.

<u>Theorem 3.2</u>. The asymptotic distribution of Hotelling's-T_0^2 statistic $m\ tr\ R(I - R)^{-1}$, $m = n - p + q$, if $P = \frac{1}{n}\theta$, is given by

$$(17)\quad P[m\ tr\ R(I - R)^{-1} < z] = P[\chi_f^2(\delta^2) < z] + \frac{1}{4m}\left\{\sum_{\alpha=0}^{4} c_\alpha\ P[\chi_{f+2\alpha}^2(\delta^2) < z]\right\} + \frac{1}{96m^2}\left\{\sum_{\alpha=0}^{8} d_\alpha\ P[\chi_{f+2\alpha}^2(\delta^2) < z]\right\} + 0(m^{-3})$$

where $\delta^2 = \tfrac{1}{2}tr\theta$, $f = q(p - q)$ and c_α , $\alpha = 0, \ldots ,4$ and d_α , $\alpha = 0, \ldots ,8$ given in (19) and (20).

Proof: Applying Lemma 2.4 on Lemma 2.3, the unconditional characteristic function of $m \operatorname{tr} R(I - R)^{-1}$ is given by

$$(18)\quad q_2(t) = E_{A_{22}} g_2(t|A_{22})$$
$$= \operatorname{etr}\left\{\frac{it\theta}{1-2it}\right\}(1 - 2it)^{-\frac{1}{2}q(p-q)} \left[1 + \frac{1}{4m}\left\{\sum_{\alpha=0}^{4} c_\alpha(1 - 2it)^{-\alpha}\right\} + \frac{1}{96m^2}\left\{\sum_{\alpha=0}^{8} d_\alpha(1 - 2it)^{-\alpha}\right\} + 0(m^{-3}) \right]$$

where

$$(19)\quad \begin{aligned} c_0 &= q(p - q)(p - 2q - 1) - 4\operatorname{tr}(\tfrac{1}{2}\theta)^2 \\ c_1 &= -2(p - q)\big(q(p - q) - 2\operatorname{tr}(\tfrac{1}{2}\theta)\big) \\ c_2 &= q(p - q)(p + 1) - 4(2p - q + 1)\operatorname{tr}(\tfrac{1}{2}\theta) + 8\operatorname{tr}(\tfrac{1}{2}\theta)^2 \\ c_3 &= 4\big((p + 1)\operatorname{tr}\tfrac{1}{2}\theta - 2\operatorname{tr}(\tfrac{1}{2}\theta)^2\big) \\ c_4 &= 4\operatorname{tr}(\tfrac{1}{2}\theta)^2 \end{aligned}$$

and

$$(20)\quad d_0 = q(p - q)\ell_0 + 24(p - q)\{q^2 - q(p - q - 1) + 4\}\operatorname{tr}(\tfrac{1}{2}\theta)^2 - 128\operatorname{tr}(\tfrac{1}{2}\theta)^3 + 48\operatorname{tr}^2(\tfrac{1}{2}\theta)^2$$

$$d_1 = -\ell_1\big(q(p - q) - 2\operatorname{tr}(\tfrac{1}{2}\theta)\big) + 48q(p - q)^2\operatorname{tr}(\tfrac{1}{2}\theta)^2 - 96(p - q)\operatorname{tr}(\tfrac{1}{2}\theta)\operatorname{tr}(\tfrac{1}{2}\theta)^2$$

$$d_2 = q(p - q)\ell_2 - 2(\ell_1 + 2\ell_2)\operatorname{tr}(\tfrac{1}{2}\theta) + 48\big((p - q)^2 + 2\big)\operatorname{tr}^2(\tfrac{1}{2}\theta) - 24\{3(p - q)q^2 - (p - q)(p - q - 3)q - 4\big(3(p - q) + 1\big)\}\operatorname{tr}(\tfrac{1}{2}\theta)^2 + 96(2p - q + 1)\operatorname{tr}(\tfrac{1}{2}\theta)\operatorname{tr}(\tfrac{1}{2}\theta)^2 - 192\operatorname{tr}^2(\tfrac{1}{2}\theta)^2$$

$$d_3 = -q(p - q)\ell_3 + 2(2\ell_2 + 3\ell_3)tr(\tfrac12\theta) - 96\big((p - q)q + 2(p - q)^2 + p - q + 4) tr^2(\tfrac12\theta) + 48\{(p - q)q^2 - \big(3(p - q)^2 - p + q + 8\big)q - 8\big(3(p - q) + 2\big)\} tr(\tfrac12\theta)^2 - 96(2q-p+1)tr(\tfrac12\theta)tr(\tfrac12\theta)^2 + 640tr(\tfrac12\theta)^3 + 192tr^2(\tfrac12\theta)^2$$

$$d_4 = q(p - q)\ell_4 - 2(3\ell_3 + 4\ell_4)tr(\ \theta) + 48\{q^2 + 2q(3p - 3q + 1) + 6(p - q)^2 + 6(p - q) + 17\}tr^2(\tfrac12\theta) + 24\{q^2(p - q) + \big(7(p - q)^2 + (p - q) + 44\big)q + 4\big(20(p - q) + 19\big)tr(\tfrac12\theta)^2 - 192\big(q + 3(p - q) + 1\big)tr(\tfrac12\theta)tr(\tfrac12\theta)^2 - 1536tr(\tfrac12\theta)^3 + 96tr^2(\tfrac12\theta)^2$$

$$d_5 = 8\ell_4 tr(\tfrac12\theta) - 96\{q^2 + q(3p - 3q + 2) + 2(p - q)^2 + 3(p - q) + 9\}tr^2(\tfrac12\theta) - 48\{q^2(p - q) + q\big(2(p - q)^2 + p - q + 24\big) + 8(4p - 4q + 5)tr(\tfrac12\theta)^2 + 96(4q + 7p - 7q + 4)tr(\tfrac12\ )tr(\tfrac12\theta)^2 + 1920tr(\tfrac12\theta)^3 - 384tr^2(\tfrac12\theta)^2$$

$$d_6 = 48\{q^2 + 2q(p - q + 1) + (p - q)^2 + 2(p - q) + 7\}tr^2(\tfrac12\theta) + 24\{q^2(p - q) + q\big((p - q)^2 + p - q + 20\big) + 4(5p - 5q + 8)\}tr(\tfrac12\theta)^2 - 96(3q + 4p - 4q + 3) tr(\tfrac12\theta)tr(\tfrac12\theta)^2 - 1280tr(\tfrac12\theta)^3 + 384tr^2(\tfrac12\theta)^2$$

$$d_7 = 96\{(p + 1)tr(\tfrac12\theta)tr(\tfrac12\theta)^2 + 4tr(\tfrac12\theta)^3 - 2tr^2(\tfrac12\theta)^2\}$$

$$d_8 = 48tr^2(\tfrac12\theta)^2$$

ℓ_1, ℓ_2, ℓ_3 and ℓ_4 are given in Lemma 2.3.

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