Homogenization models for 2-D grid structures

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1 Introduction

In the past several years we have pursued efforts related to the development of accurate models for the dynamics of flexible structures made of composite materials. Most of these efforts have focused on structures with very simple geometry such as beams with tip masses, and not on the more complex structures typically used in engineering applications. One example of these more complex structures is grids (essentially thin plates with holes). A characteristic feature of grids is that they are composed of members that are quite thin and have a highly periodic structure. Unfortunately these characteristics, coupled with the difficulties related to the unknown material properties such as stiffness and internal damping, make application of traditional computational methods to find approximate solutions to the vibrational problem challenging.

In this paper, rather than viewing periodicity and sparseness as obstacles to be overcome, we exploit them to our advantage. We consider a variational problem on a domain that has large, periodically distributed holes. Using homogenization techniques we show that the solution to this problem is in some topology 'close' to the solution of a similar problem that holds on a much simpler domain. In Section 2 we study the behavior of the solution of the variational problem as the holes increase in number, but decrease in size in such a way that the total amount of material remains constant. The result will be an equation that is in general more complex, but with a domain that is simply connected rather than perforated. In Section 3 we study the limit of the solution as the amount of material goes to zero. This second limit will, in most cases, retrieve much of the simplicity that was lost in the first limit without sacrificing the simplicity of the domain. Finally in Section 4 we show that these results can be applied to the case of a vibrating Love-Kirchhoff plate with Kelvin-Voigt damping.

We rely heavily on earlier results of [Du], [CS] for the static, undamped Love-Kirchhoff equation. Our efforts here result in a modification of those results to include both time dependence and Kelvin-Voigt damping.

2 The Homogenization Process

2.1 Problem Statement and Notation

Let Ω be a bounded, open subset of $\mathbb{R}^2$. Let $\Omega_{\epsilon\mu}$ denote that part of Ω that is covered by material (see the left half of Figure 1). A natural subset of $\Omega_{\epsilon\mu}$ is enclosed in the dashed box. We shall call this subset a "cell" of $\Omega_{\epsilon\mu}$. As depicted in the figure the size of each cell is ϵl_1 by ϵl_2, where ϵ is some "small" (small in comparison to the other lengths in the problem) parameter. Each cell contains a hole of dimension $\epsilon(l_1 - \mu) \times \epsilon(l_2 - \mu)$. Our goal is to study the behavior of the solution of a variational problem first as $\epsilon \to 0$ and then as $\mu \to 0$. In order to avoid difficulties with shifting domains we introduce the fixed (relative to ϵ) cell Y_μ where Y_μ has length l_1 by l_2. By applying the transformation $y = \frac{x}{\epsilon}$ we can go from a cell in $\Omega_{\epsilon\mu}$ to the fixed cell Y_μ. We assume that the holes of $\Omega_{\epsilon\mu}$ do not meet the boundary; this assumption restricts both the geometry of Ω and the permissible values of ϵ (e.g., $\epsilon \in \{2^{-n}\}$)

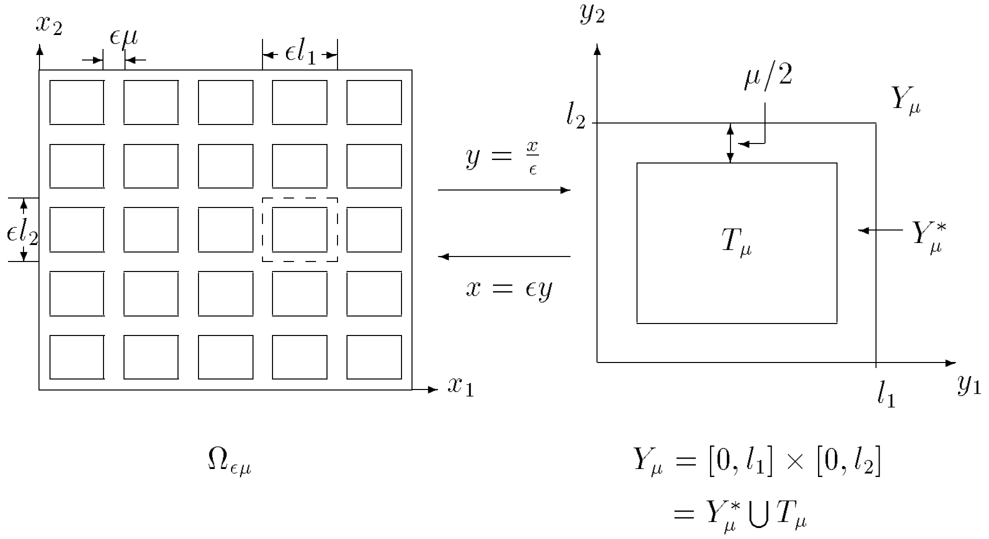

Figure 1: Structure of Ω, $\Omega_{\epsilon\mu}$, Y_μ, and Y_μ^*.

Define $H_{\epsilon\mu} = L^2(\Omega_{\epsilon\mu})$ and $V_{\epsilon\mu} = H_b^2(\Omega_{\epsilon\mu}) = \{\psi : \psi \in H^2(\Omega_{\epsilon\mu})\ ,\ \psi = \frac{\partial \psi}{\partial x_2} = 0 \text{ on } x_2 = 0\}$. None of what follows depends on this particular choice of boundary conditions which are those for a grid clamped along the boundary

$x_2 = 0$. We assume it only for the sake of specificity. The arguments below are valid for any boundary conditions in which ψ and the normal derivative of ψ vanish on an open interval of the outer boundary. Using this notation we consider the following problem

$$< \gamma u_{tt}^{\epsilon\mu}, \psi > + \sigma^{\epsilon,\mu}(u^{\epsilon\mu}, \psi) = < f, \psi > \qquad \forall \psi \in V_{\epsilon\mu} \tag{1}$$

$$u^{\epsilon\mu}(0) = u_0^{\epsilon\mu} \in V_{\epsilon\mu} \qquad u_t^{\epsilon\mu}(0) = v_0^{\epsilon\mu} \in H_{\epsilon\mu} \tag{2}$$

where γ is a positive constant and

$$\sigma^{\epsilon\mu}(\varphi, \psi) = \int_{\Omega_{\epsilon\mu}} a_{ijkh}^{\epsilon\mu}(x) \frac{\partial^2 \varphi}{\partial x_k \partial x_h} \frac{\partial^2 \psi}{\partial x_i \partial x_j}\, dx$$

and where in writing the above equations we have adopted the summation convention (i.e., we sum on repeated indices). We assume that each of the coefficients $a_{ijkh}^{\epsilon\mu}$ is Y-periodic and an element of $L^\infty(\mathbb{R}^2)$ and that the following ellipticity condition holds

$$a_{ijkh}^{\epsilon\mu} \xi_{ij} \xi_{kh} \geq A \xi_{ij} \xi_{ij}, \qquad A > 0, \quad \text{for all } \xi = (\xi_{ij}), \text{ with } \xi_{ij} = \xi_{ji}. \tag{3}$$

If the forcing function $f = f^{\epsilon\mu}$ is in $L^2(0, T; L^2(\Omega_{\epsilon\mu}))$ then there is a unique weak solution to (1). (See e.g., [LM], or [W], p. 434–442).

We adopt the following notation. For any measurable set E, 1_E denotes the characteristic function of E and $|E|$ denotes the measure of E. If $g \in L^2(E)$, we will denote the mean value of g on E by $\mathcal{M}_E(g)$:

$$\mathcal{M}_E(g) = \frac{1}{|E|} \int_E g(y)\, dy$$

For any function $g \in L^2(\Omega_{\epsilon\mu})$ we will denote by $\tilde{g}$ the extension by zero to all of Ω. We will denote many different constants which are independent of ϵ by the one symbol C. The set of test functions on Ω (i.e., the set of all bounded, infinitely differentiable functions on Ω with compact support) will be denoted by $\mathcal{D}(\Omega)$. The symbol $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_j \geq 0$ will denote a multi-index, with $|\alpha| = \Sigma_{j=1}^n \alpha_j$. Similarly, $D_j^k = \partial^k / \partial x_j^k$ for k scalar and for $1 \leq j \leq n$, and so

$$D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \ldots D_n^{\alpha_n}$$

denotes a differential operator of order $|\alpha|$. Lastly, to simplify notation while we study the limit of $u^{\epsilon\mu}$ as $\epsilon \to 0$, we will temporarily suppress the index μ (i.e., $u^\epsilon \equiv u^{\epsilon\mu}$, $\Omega_\epsilon \equiv \Omega_{\epsilon\mu}$, $V_\epsilon = V_{\epsilon\mu}$, etc.).

2.2 Preliminary Lemmas

We will have occasion to use the following lemma frequently.

Lemma 2.1 *Suppose that $g \in L^2(Y)$ is Y-periodic and is extended periodically to all of $\mathbb{R}^2$. If we define $g_\epsilon(x) = g\left(\frac{x}{\epsilon}\right)$, $x \in \Omega$, then as $\epsilon \to 0$,*

$$g_\epsilon \to \frac{1}{|Y|}\int_Y g(y)\,dy = \mathcal{M}_Y(g) \qquad \textit{in } L^2(\Omega) \textit{ weakly.}$$

Moreover, if $g \in L^\infty(Y)$ then as $\epsilon \to 0$, $g_\epsilon \to \mathcal{M}_Y(g)$ in $L^\infty(\Omega)$ weak-.*

Proof: See Chapter 5, Lemma 4.1 of [SP].

Hence if we let $1_{\cup Y^*}$ denote the extension by periodicity of the characteristic function of Y^* to all of $\mathbb{R}^2$, then $1_{\Omega_{\epsilon\mu}}(x) = 1_{\cup Y^*}\left(\frac{x}{\epsilon}\right)$ and so by the previous lemma

$$1_{\Omega_{\epsilon\mu}} \to \mathcal{M}_Y\left(1_{Y^*}\right) = \frac{|Y^*|}{|Y|} \qquad \text{in } L^2(\Omega) \text{ weakly.}$$

The quantity $\mathcal{M}_Y\left(1_{Y^*}\right)$ will appear frequently in what follows, and therefore we will simply denote it by θ. It follows from the above lemma that if $g \in L^2(\Omega)$, then $1_{\Omega_{\epsilon\mu}} g \to \theta g$ weakly in $L^2(\Omega)$. In order to study the limit u^ϵ as $\epsilon \to 0$, we need to extend it from the perforated domain Ω_ϵ to the domain Ω. Since $u^\epsilon \in V_\epsilon \subset H^2(\Omega_\epsilon)$ we cannot simply extend u^ϵ by zero into the holes and preserve the smoothness of u^ϵ. A more sophisticated extension is required, the existence of which is guaranteed by the following two lemmas.

Lemma 2.2 *Assume Y^* has the uniform cone property, then there exists an extension operator $Q \in \mathcal{L}(H^2(Y^*), H^2(Y))$ such that*

$$\sum_{|\alpha|=2} |D^\alpha Q\varphi|_{L^2(Y)} \le C \sum_{|\alpha|=2} |D^\alpha \varphi|_{L^2(Y^*)},$$

for any $\varphi \in H^2(Y^*)$, where C is a constant independent of φ.

Proof: See Lemma 1 of [Du].

Lemma 2.3 *Assume Ω_ϵ has the uniform cone property, then there exists an extension operator $P^\epsilon \in \mathcal{L}(L^\infty(0,T;H^2(\Omega_\epsilon)), L^\infty(0,T;H^2(\Omega)))$ such that $P^\epsilon\varphi_t = (P^\epsilon\varphi)_t$ in $(0,T)\times\Omega$, and*

$$\sum_{|\alpha|=2} |D^\alpha P^\epsilon \varphi|_{L^\infty(0,T;L^2(\Omega))} \leq C \sum_{|\alpha|=2} |D^\alpha \varphi|_{L^\infty(0,T;L^2(\Omega_\epsilon))},$$

for any $\varphi \in L^\infty(0,T;H^2(\Omega_\epsilon))$, where C is a constant independent of φ and ϵ.

Proof: Lemma 2 of [Du] yields the existence of an extension operator $Q^\epsilon \in \mathcal{L}(H^2(\Omega_\epsilon), H^2(\Omega))$ with the property

$$\sum_{|\alpha|=2} |D^\alpha Q^\epsilon \psi|_{L^2(\Omega)} \leq C \sum_{|\alpha|=2} |D^\alpha \psi|_{L^2(\Omega_\epsilon)} \qquad \text{for all } \psi \in H^2(\Omega_\epsilon) \tag{4}$$

where C is a constant independent of ψ and ϵ. We define the operator P^ϵ by

$$P^\epsilon\varphi(t,x) = [Q^\epsilon\varphi(t,\cdot)](x) \qquad \text{for } \varphi \in L^\infty(0,T;H^2(\Omega_\epsilon)). \tag{5}$$

Then from the construction of Q^ϵ in [Du] it follows that

$$\frac{\partial}{\partial t}\left[P^\epsilon\varphi(t,x)\right] = \frac{\partial}{\partial t}\left\{Q^\epsilon\varphi(t,\cdot)\right\}(x) = \left[Q^\epsilon\varphi_t(t,\cdot)\right](x) = P^\epsilon\left(\varphi_t\right)(t,x).$$

Now from (4) we have for each $t \in (0,T)$,

$$\sum_{|\alpha|=2} |D^\alpha Q^\epsilon \varphi(t)|_{L^2(\Omega)} \leq C \sum_{|\alpha|=2} |D^\alpha \varphi(t)|_{L^2(\Omega_\epsilon)}.$$

So taking the essential supremum over $(0,T)$ of the right side of the inequality above, and then of the lefthand side we obtain the desired result.

We remark that the construction of the extension operator Q^ϵ and hence that of P^ϵ is such that the boundary conditions of $H^2_b(\Omega_\epsilon)$ are preserved; i.e., $P^\epsilon \in \mathcal{L}(L^\infty(0,T;H^2_b(\Omega_\epsilon)), L^\infty(0,T;H^2_b(\Omega)))$.

Lemma 2.4 *Suppose $\{\phi_\epsilon\} \in V_\epsilon$, $|\phi_\epsilon|_{V_\epsilon} \leq C$, C independent of ϵ; then there exists a subsequence ϕ_{ϵ_n}, such that $\phi_{\epsilon_n} \to \phi$ in $L^2(\Omega)$ weakly as $\epsilon_n \to 0$. Moreover, the previous lemma ensures that ϕ is actually in $H^2(\Omega)$.*

Before turning to *a priori* estimates needed in the subsequent theoretical considerations, we discuss briefly how the approximation results developed here may be used in practice. Of particular interest is how one treats forcing functions and initial data $(f, u_0^{\epsilon\mu}, v_0^{\epsilon\mu}$ in (1)-(2)). In the typical application, one will have a given fixed structure corresponding to a domain $\Omega_{\bar{\epsilon}\bar{\mu}}$ for fixed values of ϵ and μ. For this structure initial data $u_0 = u_0^{\bar{\epsilon}\bar{\mu}} \in H_b^2(\Omega_{\bar{\epsilon}\bar{\mu}}) = V_{\bar{\epsilon}\bar{\mu}}$, $v_0 = v_0^{\bar{\epsilon}\bar{\mu}} \in L^2(\Omega_{\bar{\epsilon}\bar{\mu}}) = H_{\bar{\epsilon}\bar{\mu}}$ as well as a forcing function $f^{\bar{\epsilon}\bar{\mu}} \in L^2(\Omega_{\bar{\epsilon}\bar{\mu}})$ will be available. One wishes to use a homogenized model or limiting model of (1)-(2) as $\epsilon, \mu \to 0$ as an approximation to the fixed grid with domain $\Omega_{\bar{\epsilon}\bar{\mu}}$. This model will be defined on Ω (this is the whole point of our efforts) and one must somehow construct data u_0, v_0 and forcing function f (each with domain Ω) to be used in this approximation system. One can use the data $u_0^{\bar{\epsilon}\bar{\mu}}, v_0^{\bar{\epsilon}\bar{\mu}}, f^{\bar{\epsilon}\bar{\mu}}$ for the actual structure to construct data for the approximation system.

For the forcing function it is reasonable to use the "zero outside $\Omega_{\bar{\epsilon}\bar{\mu}}$" extension, i.e. $f = f^{\bar{\epsilon}\bar{\mu}} \in L^2(\Omega)$. For the initial data u_0 the $\tilde{}$ extension is not reasonable (nor will it provide needed smoothness for u_0). However, Lemma 2.3 allows one to extend functions in $H_b^2(\Omega_{\epsilon\mu})$ to $H_b^2(\Omega)$, thus maintaining smoothness and exterior boundary conditions. The extension operators $P^{\epsilon\mu}$ guaranteed by this lemma reduce to $Q^{\epsilon\mu}$ in $\mathcal{L}(H_b^2(\Omega_{\epsilon\mu}), H_b^2(\Omega))$ on elements that are constant functions in t. One can then define initial data $\hat{u}_0 \equiv Q^{\bar{\epsilon}\bar{\mu}} u_0^{\bar{\epsilon}\bar{\mu}}$ and $\hat{v}_0 \equiv \tilde{v}_0^{\bar{\epsilon}\bar{\mu}}$ in $H_b^2(\Omega)$ and $L^2(\Omega)$, respectively, that are natural to use in the limit equation on Ω. In the sequence of approximate systems (1)-(2) as $\epsilon, \mu \to 0$, one might then use the forcing function f and the restrictions (to $\Omega_{\epsilon\mu}$) of the initial data i.e., $u_0^{\epsilon\mu} \equiv Q^{\bar{\epsilon}\bar{\mu}} u_0^{\bar{\epsilon}\bar{\mu}}|_{\Omega\epsilon\mu}$, $v_0^{\epsilon\mu} \equiv \tilde{v}_0^{\bar{\epsilon}\bar{\mu}}|_{\Omega\epsilon\mu}$ for each ϵ, μ with $\epsilon < \bar{\epsilon}$, $\mu < \bar{\mu}$. Note that since $\hat{u}_0$, $\hat{v}_0 \in L^2(\Omega)$ we have $\tilde{u}_0^{\epsilon\mu} = 1_{\Omega_{\epsilon\mu}} \hat{u}_0 \to \theta \hat{u}_0$ in $L^2(\Omega)$ weakly with $\hat{u}_0 \in H_b^2(\Omega)$ and $\tilde{v}_0^{\epsilon\mu} = 1_{\Omega_{\epsilon\mu}} \hat{v}_0 \to \theta \hat{v}_0$ weakly in $L^2(\Omega)$. Also observe that $|u_0^{\epsilon\mu}|_{V_{\epsilon\mu}}, |v_0^{\epsilon\mu}|_{H_{\epsilon\mu}}$ are uniformly bounded as ϵ, $\mu \to 0$.

2.3 A Priori Estimates

We let $\mu > 0$ be fixed and consider solutions $u^{\epsilon} = u^{\epsilon\mu}$ of (1)-(2) corresponding to $f \in L^2(0, T; L^2(\Omega))$ and initial data $u_0^{\epsilon\mu}$, $v_0^{\epsilon\mu}$ such that $|u_0^{\epsilon\mu}|_{V_{\epsilon\mu}}$ and $|v_0^{\epsilon\mu}|_{H_{\epsilon\mu}}$ are uniformly bounded (in ϵ, μ). We remind the reader that in our notation below we will only temporarily suppress the dependence on the fixed value of μ). Then we have

Theorem 2.1 *There exists an extension operator P^ϵ*

$$P^\epsilon \in \mathcal{L}(L^\infty(0,T;V_\epsilon), L^\infty(0,T;H_b^2(\Omega)))$$

and a function $u = u^\mu$ such that for some sequence $\epsilon_n \to 0$ we have

$$P^{\epsilon_n} u^{\epsilon_n} \to u \textit{ in } L^\infty(0,T;H_b^2(\Omega)) \textit{weak-*},$$

$$(P^{\epsilon_n} u^{\epsilon_n})_t = P^{\epsilon_n} u_t^{\epsilon_n} \to u_t \textit{ in } L^\infty(0,T;L^2(\Omega)) \textit{ weak-*}.$$

Proof: The arguments underlying the fundamental existence and uniqueness results (see [W], p. 439-442 and also [BIW]) for (1)-(2) can be used to show that solutions also satisfy

$$|u_t^\epsilon(t)|_{H_\epsilon}^2 + |u^\epsilon(t)|_{V_\epsilon} \le C\left(|u_0^\epsilon|_{V_\epsilon}^2 + |v_0^\epsilon|_{H_\epsilon}^2 + \int_0^T |f(t)|_{H_\epsilon}^2\, dt\right)$$

for almost every t in $(0,T)$, where the constant C is independent of ϵ. From the uniform boundedness assumption on the initial data, we thus conclude that u^ϵ and u_t^ϵ are uniformly bounded in $L^\infty(0,T;V_\epsilon)$ and $L^\infty(0,T;H_\epsilon)$, respectively. We can thus use the extension operator P^ϵ constructed in Lemma 2.3 to conclude that $P^\epsilon u^\epsilon$ and $P^\epsilon u_t^\epsilon$ are bounded in $L^\infty(0,T;H_b^2(\Omega))$ and $L^\infty(0,T;L^2(\Omega))$, respectively. Hence there are subsequences $P^{\epsilon_n}u^{\epsilon_n}$, $P^{\epsilon_n}u_t^{\epsilon_n}$, such that as $\epsilon_n \to 0$ we have $P^{\epsilon_n}u^{\epsilon_n} \to u$ in $L^\infty(0,T;H_b^2(\Omega))$ weak-*, and $P^{\epsilon_n}u_t^{\epsilon_n} \to u_t$ in $L^\infty(0,T;L^2(\Omega))$ weak-*.

2.4 Limit of $P^\epsilon u^\epsilon$ as $\epsilon \to 0$

Now that we have established the existence of a u that is the subsequential limit of $P^\epsilon u^\epsilon$, we need to determine the equation that u satisfies. Define

$$\zeta_{ij}^\epsilon = a_{ijkh}^\epsilon(x)\frac{\partial^2 u^\epsilon}{\partial x_k \partial x_h} = a_{ijkh}\left(\frac{x}{\epsilon}\right)\frac{\partial^2 u^\epsilon}{\partial x_k \partial x_h}\,.$$

We then extend the weak formulation of the plate equation (1) to all of Ω by writing

$$\int_0^T\int_\Omega \tilde{\zeta}_{ij}^\epsilon \frac{\partial^2\psi}{\partial x_i \partial x_j} v\, dx\, dt = \int_0^T\int_\Omega 1_{\Omega_\epsilon} f\psi v\, dx\, dt - \int_0^T\int_\Omega \gamma P^\epsilon u_{tt}^\epsilon 1_{\Omega_\epsilon}\psi v\, dx\, dt. \tag{6}$$

for all $\psi \in H^2_b(\Omega)$, $v \in \mathcal{D}(0,T)$ with $v(0) = v(T) = v'(T) = 0$. Since $|u^\epsilon(t)|_{V_\epsilon}$ can be bounded by a constant C independent of ϵ and t, and the a^ϵ_{ijkh} are in $L^\infty(\Omega)$ with $|a^\epsilon_{ijkh}|_{L^\infty(\Omega)} \le C$ independent of ϵ, we have

$$|\tilde{\zeta}^\epsilon_{ij}|_{L^\infty(0,T;L^2(\Omega))} \le C$$

which implies the existence of a $\tilde{\zeta}^*_{ij} \in L^\infty(0,T;L^2(\Omega))$ and a sequence $\epsilon_n \to 0$ such that

$$\tilde{\zeta}^{\epsilon_n}_{ij} \to \tilde{\zeta}^*_{ij} \text{ in } L^\infty(0,T;L^2(\Omega)) \text{ weak } *. \tag{7}$$

Hence as $\epsilon_n \to 0$ we have using Lemma 2.1 and Theorem 2.1,

$$\int_0^T \int_\Omega \tilde{\zeta}^*_{ij} \frac{\partial^2 \psi}{\partial x_i \partial x_j} v \, dx \, dt = \int_0^T \int_\Omega \theta f \psi v \, dx \, dt - \int_0^T \int_\Omega \theta \gamma u_{tt} \psi v \, dx \, dt. \tag{8}$$

We can obtain the claimed initial conditions by performing two integrations by parts with respect to t and applying the conditions on v and v' at 0 and T given above. The remaining task then, is to somehow relate the limit $\tilde{\zeta}^*_{ij}$ to u. We introduce the function $w_{lm}(y) \in H^2(Y^*)$ which is the solution of

$$\int_{Y^*} a_{ijkh}(y) \frac{\partial^2 w_{lm}(y)}{\partial y_i \partial y_j} \frac{\partial^2 \psi}{\partial y_k \partial y_h} \, dy = 0 \qquad \forall \psi \in H^2(Y^*) \tag{9}$$

$$W_{lm}(y) \equiv w_{lm}(y) - \mathcal{P}_{lm}(y) \text{ Y-periodic} \tag{10}$$

where $\mathcal{P}_{lm}(y) = \frac{1}{2} y_l y_m$. Set

$$W^\epsilon_{lm}(x) = \epsilon^2 (Q W_{lm}) \left(\frac{x}{\epsilon} \right) \qquad \text{for } x \in \epsilon Y$$

and extend periodically to all of $\mathbb{R}^2$, where Q is the extension operator of Lemma 2.2. Then define $w^\epsilon_{lm}(x) \equiv W^\epsilon_{lm}(x) + \frac{1}{2} x_l x_m$. It follows from (9) that W^ϵ_{lm}, and hence w^ϵ_{lm}, can be bounded, independent of ϵ, in $H^2(\Omega)$ so that for some sequence $\epsilon_n \to 0$,

$$w^{\epsilon_n}_{lm} \to w^*_{lm} \text{ in } H^2(\Omega) \text{ weakly,} \tag{11}$$

and for any α, $|\alpha| = 2$,

$$D^\alpha w^{\epsilon_n}_{lm} \to D^\alpha \mathcal{P}_{lm} \text{ in } L^2(\Omega) \text{ weakly.} \tag{12}$$

We introduce $\eta^\epsilon_{kh} = \eta^\epsilon_{khlm}$ by

$$\begin{aligned}\eta^\epsilon_{kh} &= a^\epsilon_{ijkh}(x)\frac{\partial^2 w^\epsilon_{lm}(x)}{\partial x_i \partial x_j}\\ &= a_{ijkh}\left(\frac{x}{\epsilon}\right)\frac{\partial^2 w_{lm}(\frac{x}{\epsilon})}{\partial y_i \partial y_j}\end{aligned}$$

By construction its extension $\tilde{\eta}^\epsilon_{kh}$ to Ω, satisfies

$$\int_\Omega \tilde{\eta}^\epsilon_{kh}\frac{\partial^2 \psi}{\partial x_k \partial x_h}\,dx = 0. \tag{13}$$

Furthermore, using Lemma 2.1 as $\epsilon \to 0$

$$\tilde{\eta}^\epsilon_{kh} \to \frac{1}{|Y|}\int_{Y^*} a_{ijkh}(y)\frac{\partial^2 w_{lm}(y)}{\partial y_i \partial y_j}\,dy \equiv \beta_{lmkh} \qquad \text{in } L^2(\Omega) \text{ weakly.} \tag{14}$$

We are now ready to prove

Theorem 2.2 *Let u^ϵ be the solution of (1)-(2) corresponding to $f \in L^2(0,T;L^2(\Omega))$ and initial data $u^\epsilon_0, v^\epsilon_0$ uniformly bounded in V_ϵ, H_ϵ satisfying*

$$\tilde{u}^\epsilon_0 \to u_0 \textit{ in } L^2(\Omega) \textit{ weakly, with } u_0 \in H^2_b(\Omega)$$

$$\tilde{u}^\epsilon_1 \to v_0 \textit{ in } L^2(\Omega) \textit{ weakly.}$$

Then there exists an extension operator $P^\epsilon \in \mathcal{L}(L^\infty(0,T;H^2_b(\Omega_\epsilon)), L^\infty(0,T;H^2_b(\Omega)))$ such that for some sequence $\epsilon_n \to 0$,

$$P^{\epsilon_n} u^{\epsilon_n} \to u \textit{ in } L^\infty(0,T;H^2_b(\Omega)) \textit{ weak-*}$$

$$P^{\epsilon_n} u^{\epsilon_n}_t \to u_t \textit{ in } L^\infty(0,T;L^2(\Omega)) \textit{ weak-*}$$

where u is the solution to the homogenized equation:

$$< \theta\gamma u_{tt}(t), \psi > + \sigma(u(t),\psi) = < \theta f, \psi > \textit{ on } (0,T), \qquad \forall \psi \in H^2_b(\Omega) \tag{15}$$

$$u(0) = u_0/\theta \qquad u_t(0) = v_0/\theta.$$

The homogenized sesquilinear form

$$\sigma(\varphi,\psi) = \int_\Omega q_{ijkh}\frac{\partial^2 \varphi}{\partial x_k \partial x_h}\frac{\partial^2 \psi}{\partial x_i \partial x_j}\,dx$$

is defined by the constant coefficients

$$q_{ijkh} = \frac{1}{|Y|} \int_{Y^*} \left[a_{ijkh}(y) - a_{lmkh}(y) \frac{\partial^2 \chi^{ij}(y)}{\partial y_l \partial y_m} \right] dy \tag{16}$$

where the functions $\chi^{ij}(y)$ are the Y-periodic solutions of

$$\int_{Y^*} a_{lmkh}(y) \frac{\partial^2 (\chi^{ij}(y) - \mathcal{P}^{ij}(y))}{\partial y_l \partial y_m} \frac{\partial^2 \psi}{\partial y_k \partial y_h}\, dy = 0 \qquad \forall \psi \in H^2(Y^*) \tag{17}$$

$$\psi \textit{ periodic in } Y^*$$

and where $\mathcal{P}^{ij}(y) = \frac{1}{2} y_i y_j$.

Proof: In (13) choose $\psi = \phi P^\epsilon u^\epsilon$, where $\phi \in \mathcal{D}(\Omega)$ and then multiply by $v \in \mathcal{D}(0,T)$ and integrate from 0 to T. Now in (6) choose $\psi = \phi w^\epsilon_{lm}$, where $\phi \in \mathcal{D}(\Omega)$ and subtract the two equations to obtain

$$\begin{aligned}
&\int_0^T \int_\Omega \tilde{\zeta}^\epsilon_{ij} \frac{\partial^2(\phi w^\epsilon_{lm})}{\partial x_i \partial x_j} v\, dx\, dt - \int_0^T \int_\Omega \tilde{\eta}^\epsilon_{ij} \frac{\partial^2(\phi P^\epsilon u^\epsilon)}{\partial x_i \partial x_j} v\, dx\, dt \\
&= \int_0^T \int_\Omega 1_{\Omega_\epsilon} f w^\epsilon_{lm} \phi v\, dx\, dt - \int_0^T \int_\Omega \gamma P^\epsilon u^\epsilon_{tt} 1_{\Omega_\epsilon} w^\epsilon_{lm} \phi v\, dx\, dt.
\end{aligned}$$

If we apply the product rule for the differentiations we have

$$\begin{aligned}
&\int_0^T \int_\Omega \tilde{\zeta}^\epsilon_{ij} \frac{\partial^2 \phi}{\partial x_i \partial x_j} w^\epsilon_{lm} v\, dx\, dt + \int_0^T \int_\Omega \tilde{\zeta}^\epsilon_{ij} \frac{\partial \phi}{\partial x_i} \frac{\partial w^\epsilon_{lm}}{\partial x_j} v\, dx\, dt \\
&+ \int_0^T \int_\Omega \tilde{\zeta}^\epsilon_{ij} \frac{\partial \phi}{\partial x_j} \frac{\partial w^\epsilon_{lm}}{\partial x_i} v\, dx\, dt + \int_0^T \int_\Omega \tilde{\zeta}^\epsilon_{ij} \phi \frac{\partial^2 w^\epsilon_{lm}}{\partial x_i \partial x_j} v\, dx\, dt \\
&- \int_0^T \int_\Omega \tilde{\eta}^\epsilon_{ij} \frac{\partial^2 \phi}{\partial x_i \partial x_j} P^\epsilon u^\epsilon v\, dx\, dt - \int_0^T \int_\Omega \tilde{\eta}^\epsilon_{kh} \frac{\partial \phi}{\partial x_k} \frac{\partial (P^\epsilon u^\epsilon)}{\partial x_h} v\, dx\, dt \\
&- \int_0^T \int_\Omega \tilde{\eta}^\epsilon_{kh} \frac{\partial \phi}{\partial x_h} \frac{\partial (P^\epsilon u^\epsilon)}{\partial x_k} v\, dx\, dt - \int_0^T \int_\Omega \tilde{\eta}^\epsilon_{kh} \phi \frac{\partial^2 (P^\epsilon u^\epsilon)}{\partial x_k \partial x_h} v\, dx\, dt \\
&= \int_0^T \int_\Omega 1_{\Omega_\epsilon} f w^\epsilon_{lm} \phi v\, dx\, dt - \int_0^T \int_\Omega \gamma P^\epsilon u^\epsilon_{tt} 1_{\Omega_\epsilon} w^\epsilon_{lm} \phi v\, dx\, dt.
\end{aligned}$$

Examining the above integrals one by one, we discover that all but two of them have limits as $\epsilon \to 0$. The two troublesome terms are

$$\int_0^T \int_\Omega \tilde{\zeta}^\epsilon_{ij} \phi \frac{\partial^2 w^\epsilon_{lm}}{\partial x_i \partial x_j} v \, dx \, dt \quad \text{and} \quad \int_0^T \int_\Omega \tilde{\eta}^\epsilon_{kh} \phi \frac{\partial^2 (P^\epsilon u^\epsilon)}{\partial x_k \partial x_h} v \, dx \, dt.$$

Now since $\tilde{\cdot}$ denotes the extension from Ω_ϵ by 0 to all of Ω we have

$$\int_0^T \int_\Omega \tilde{\zeta}^\epsilon_{ij} \phi \frac{\partial^2 w^\epsilon_{lm}}{\partial x_i \partial x_j} v \, dx \, dt = \int_0^T \int_{\Omega_\epsilon} a^\epsilon_{ijkh} \frac{\partial^2 u^\epsilon}{\partial x_k \partial x_h} \phi \frac{\partial^2 w^\epsilon_{lm}}{\partial x_i \partial x_j} v \, dx \, dt$$

and

$$\int_0^T \int_\Omega \tilde{\eta}^\epsilon_{kh} \phi \frac{\partial^2 (P^\epsilon u^\epsilon)}{\partial x_k \partial x_h} v \, dx \, dt = \int_0^T \int_{\Omega_\epsilon} a^\epsilon_{ijkh} \frac{\partial^2 w^\epsilon_{lm}}{\partial x_i \partial x_j} \phi \frac{\partial^2 u^\epsilon}{\partial x_k \partial x_h} v \, dx \, dt.$$

But here is where the adjoint system contributes to our analysis, yielding that the two terms are in fact equal. So now using the convergences (7), (11), and (14) we have

$$\begin{aligned}
&\int_0^T \int_\Omega \tilde{\zeta}^*_{ij} \frac{\partial^2 \phi}{\partial x_i \partial x_j} w^*_{lm} v \, dx \, dt + \int_0^T \int_\Omega \tilde{\zeta}^*_{ij} \frac{\partial \phi}{\partial x_i} \frac{\partial w^*_{lm}}{\partial x_j} v \, dx \, dt \\
&+ \int_0^T \int_\Omega \tilde{\zeta}_{ij} \frac{\partial \phi}{\partial x_j} \frac{\partial w^*_{lm}}{\partial x_i} v \, dx \, dt - \int_0^T \int_\Omega \beta_{lmkh} \frac{\partial^2 \phi}{\partial x_k \partial x_h} uv \, dx \, dt \\
&- \int_0^T \int_\Omega \beta_{lmkh} \frac{\partial \phi}{\partial x_k} \frac{\partial u}{\partial x_h} v \, dx \, dt - \int_0^T \int_\Omega \beta_{lmkh} \frac{\partial \phi}{\partial x_h} \frac{\partial u}{\partial x_k} v \, dx \, dt \\
&= \int_0^T \int_\Omega \theta f w^*_{lm} \phi v \, dx \, dt - \int_0^T \int_\Omega \gamma \theta u_{tt} w^*_{lm} \phi v \, dx \, dt.
\end{aligned}$$

We can rewrite this as

$$\begin{aligned}
&\int_0^T \int_\Omega \tilde{\zeta}^*_{ij} \frac{\partial^2 (\phi w^*_{lm})}{\partial x_i \partial x_j} v \, dx \, dt - \int_0^T \int_\Omega \tilde{\zeta}^*_{ij} \phi \frac{\partial^2 w^*_{lm}}{\partial x_i \partial x_j} v \, dx \, dt \\
&- \int_0^T \int_\Omega \beta_{lmkh} \frac{\partial^2 (\phi u)}{\partial x_k \partial x_h} v \, dx \, dt + \int_0^T \int_\Omega \beta_{lmkh} \phi \frac{\partial^2 u}{\partial x_k \partial x_h} v \, dx \, dt \\
&= \int_0^T \int_\Omega \theta f \phi w^*_{lm} v \, dx \, dt - \int_0^T \int_\Omega \theta \gamma u_{tt} w^*_{lm} \phi v \, dx \, dt.
\end{aligned}$$

Then using (8) and (13) we have

$$\int_0^T \int_\Omega \tilde{\zeta}^*_{ij} \phi \frac{\partial^2 w^*_{lm}}{\partial x_i \partial x_j} v \, dx \, dt = \int_0^T \int_\Omega \beta_{lmkh} \phi \frac{\partial^2 u}{\partial x_k \partial x_h} v \, dx \, dt$$

and finally using (12), we obtain

$$\tilde{\zeta}^*_{lm} = \beta_{lmkh}\frac{\partial^2 u}{\partial x_k \partial x_h}.$$

Now let $\chi^{lm}(y) = \mathcal{P}^{lm}(y) - w_{lm}(y)$; it is easy to verify that $\beta_{lmkh} = q_{lmkh}$ where the q_{lmkh} are as given in equation (16). Now making this substitution into (8) we have the homogenized equation (15).

Finally we show that the q_{ijkh} satisfy an ellipticity condition. Recall that

$$\begin{aligned} q_{ijkh} &= \frac{1}{|Y|}\int_{Y^*}\left(a_{ijkh} - a_{lmkh}\frac{\partial^2 \chi^{ij}(y)}{\partial y_l \partial y_m}\right) dy \\ &= -\frac{1}{|Y|}\int_{Y^*} a_{lmkh}\frac{\partial^2 \left(\chi^{ij}(y) - \mathcal{P}^{ij}\right)}{\partial y_l \partial y_m}\, dy \\ &= \frac{1}{|Y|}\int_{Y^*} a_{lmno}\frac{\partial^2 \left(\chi^{ij}(y) - \mathcal{P}^{ij}\right)}{\partial y_l \partial y_m}\frac{\partial^2(-\mathcal{P}^{kh})}{\partial y_n \partial y_o}\, dy. \end{aligned}$$

We define the sesquilinear form $\sigma_{Y^*}(\phi,\psi)$ by

$$\sigma_{Y^*}(\phi,\psi) = \int_{Y^*} a_{lmno}\frac{\partial^2 \phi}{\partial y_l \partial y_m}\frac{\partial^2 \psi}{\partial y_n \partial y_o}\, dy.$$

Therefore

$$q_{ijkh} = \frac{1}{|Y|}\sigma_{Y^*}(\chi^{ij} - \mathcal{P}^{ij}, -\mathcal{P}^{kh}). \tag{18}$$

Now observe that χ^{ij} can be characterized as the Y-periodic solution of

$$\sigma_{Y^*}(\chi^{ij} - \mathcal{P}^{ij}, \psi) = 0 \qquad \forall \psi \in H^2(Y^*),\ \psi \text{ Y-periodic}.$$

In particular if we choose $\psi = \chi^{kh}$ we have

$$\sigma_{Y^*}(\chi^{ij} - \mathcal{P}^{ij}, \chi^{kh}) = 0. \tag{19}$$

So now combining (18) and (19)

$$q_{ijkh} = \frac{1}{|Y|}\sigma_{Y^*}(\chi^{ij} - \mathcal{P}^{ij}, \chi^{kh} - \mathcal{P}^{kh}).$$

Since the a^ϵ_{ijkh} satisfy an ellipticity condition it is easy to show that σ_{Y^*} is a coercive sesquilinear form, which in turn makes it easy to demonstrate that the q_{ijkh} satisfy an ellipticity condition.

Remark 2.1 *Note that the homogenized coefficients (see (16)) are the average of the a_{ijkh} plus a corrector term. Furthermore note that the homogenization process takes us from having to solve our problem on a perforated domain to solving a problem on a continuous domain.*

2.5 Limit of $P^\epsilon u^\epsilon$ for the Case with Damping

We now extend our previous results to include a damping term. In proving this extension we will use Laplace transforms and the following idea of homogenization with parameter taken from [BLP].

Suppose we have the coefficient $c_{ijkh}(\lambda, y)$, where λ is a parameter belonging to some topological set Λ. Assume that the $c_{ijkh}(\lambda, y)$ are Y-periodic for any $\lambda \in \Lambda$ and that the $c_{ijkh}(\lambda, y)$ satisfy the following ellipticity condition

$$c_{ijkh}(\lambda, y)\xi_{ij}\xi_{kh} \geq C\xi_{ij}\xi_{ij}, \qquad C > 0, \qquad \forall \xi = (\xi_{ij}) \text{ with } \xi_{ij} = \xi_{ji}. \tag{20}$$

Then the sesquilinear form

$$\sigma_\epsilon^{(\lambda)}(\phi, \psi) = \int_{Y^*} c_{ijkh}(\lambda, y)\frac{\partial^2 \phi}{\partial y_k \partial y_h}\frac{\partial^2 \psi}{\partial y_i \partial y_j}\, dy$$

is homogenized by the formulas derived in the previous section, with λ being carried along as a parameter. This process is called " homogenization with parameter".

We now consider the problem

$$< \gamma u_{tt}^\epsilon, \psi > + \sigma_1^\epsilon(u^\epsilon, \psi) + \sigma_2^\epsilon(u_t^\epsilon, \psi) = < f, \psi > \qquad \forall \psi \in V_\epsilon \tag{21}$$

$$u^\epsilon(0) = u_0^\epsilon \in V_\epsilon \qquad u_t^\epsilon = v_0^\epsilon \in H_\epsilon \equiv L^2(\Omega_{\epsilon\mu}) \tag{22}$$

where

$$\sigma_1^\epsilon(\phi, \psi) = \int_{\Omega_\epsilon} a_{ijkh}^\epsilon(x)\frac{\partial^2 \phi}{\partial x_k \partial x_h}\frac{\partial^2 \psi}{\partial x_i \partial x_j}\, dx\ ,$$

$$\sigma_2^\epsilon(\phi, \psi) = \int_{\Omega_\epsilon} b_{ijkh}^\epsilon(x)\frac{\partial^2 \phi}{\partial x_k \partial x_h}\frac{\partial^2 \psi}{\partial x_i \partial x_j}\, dx\ ,$$

and

$$V_\epsilon = \{\psi\ :\ \psi \in H^2(\Omega_{\epsilon\mu})\ ,\ \psi = \frac{\partial \psi}{x_2} = 0 \text{ on } x_2 = 0\}$$

This is the same problem we have been considering but with an additional term that corresponds to damping. The Y-periodic damping coefficients $b^\epsilon_{ijkh} \in L^\infty(\mathbb{R}^2)$ are assumed to satisfy an ellipticity condition like that of (3). Just as in the previous case we can find an extension of the solution satisfying a priori bounds.

Theorem 2.3 *Let $u^\epsilon = u^{\epsilon\mu}$ be the solution to (21)-(22) with $f \in L(0,T;L^2(\Omega))$ and initial data u^ϵ_0, v^ϵ_0 as in Theorem 2.1. Then there exists an extension operator P^ϵ*

$$P^\epsilon \in \mathcal{L}(L^\infty(0,T;V_\epsilon)), L^\infty(0,T;H^2_b(\Omega)))$$

and $u \in L^\infty(0,T;H^2_b(\Omega))$ such that for some sequence $\epsilon_n \to 0$

$$P^{\epsilon_n} u^{\epsilon_n} \to u \text{ in } L^\infty(0,T;H^2_b(\Omega)) weak\text{-}*$$

and

$$P^{\epsilon_n} u^{\epsilon_n}_t \to u_t \text{ in } L^\infty(0,T;L^2(\Omega)) \text{ weak-}*.$$

> *Proof:* Since the ellipticity assumption yields that σ^ϵ_2 is V_ϵ coercive, we may use an estimate (given in [BIW]) exactly like that in the proof of Theorem 2.1. The arguments are then the same as those for the undamped case.

We now take the Laplace transform of u^ϵ which is given by

$$\widehat{u}^\epsilon(s) = \int_0^\infty e^{-st} u^\epsilon(t)\, dt$$

Taking the Laplace transform of (21) we have

$$< \gamma s^2 \widehat{u}^\epsilon(s), \psi > + \sigma^\epsilon_1(\widehat{u}^\epsilon(s), \psi) + s\sigma^\epsilon_2(\hat{u}^\epsilon(s), \psi) = < \hat{f}(s), \psi > \qquad \forall \psi \in V_\epsilon \tag{23}$$

which holds for all s with positive real part. We can rewrite this as

$$< \gamma s^2 \widehat{u}^\epsilon(s), \psi > + \sigma^\epsilon(s)(\widehat{u}^\epsilon(s), \psi) = < \hat{f}(s), \psi > \qquad \forall \psi \in V_\epsilon \tag{24}$$

where

$$\sigma^\epsilon(s)(\widehat{u}^\epsilon(s), \psi) = \int_\Omega (a^\epsilon_{ijkh}(x) + s b^\epsilon_{ijkh}(x)) \frac{\partial^2 \widehat{u}^\epsilon(s)}{\partial x_k \partial x_h} \frac{\partial^2 \psi}{\partial x_i \partial x_j}\, dx.$$

Now let us consider the homogenized problem for (24). Let $\hat{v}(s)$ denote the solution to

$$< \theta\gamma s^2\widehat{v}(s), \psi > +\sigma(s)(\widehat{v}(s), \psi) =< \theta\hat{f}(s), \psi > \qquad \forall\psi \in H_b^2(\Omega) \tag{25}$$

where

$$\sigma(s)(\widehat{v}(s), \psi) = \int_\Omega \widehat{q}_{ijkh}(s)\frac{\partial^2\widehat{v}(s)}{\partial x_k \partial x_h}\frac{\partial^2\psi}{\partial x_i \partial x_j}\, dy$$

and where the $\widehat{q}_{ijkh}(s)$ are defined by

$$\widehat{q}_{ijkh}(s) = \frac{1}{|Y|}\int_{Y^*}\left(a_{ijkh}(y) + sb_{ijkh}(y) - (a_{lmkh}(y) + sb_{lmkh}(y))\frac{\partial^2\chi^{ij}(s,y)}{\partial y_l \partial y_m}\right) dy$$

and the functions χ^{ij} are the Y-periodic solutions of

$$\int_{Y^*}(a_{lmkh}(y)+sb_{lmkh}(y))\frac{\partial^2(\chi^{ij}(s,y) + \mathcal{P}^{ij}(y))}{\partial y_l \partial y_m}\frac{\partial^2\psi}{\partial y_k \partial y_h}\, dy = 0 \text{ in } Y^* \ \forall\psi \in H^2(Y^*)$$

$$\psi \text{ periodic in } Y^*$$

with $\mathcal{P}^{ij}(y) = \frac{1}{2}y_i y_j$. The arguments in Chapter 6, Section 4 of [SP] can be used without essential modification to show existence and uniqueness of a solution $\hat{v}(s)$ of (25) for s with positive real part. Since by Theorem 2.3

$$P^{\epsilon_n}u^{\epsilon_n} \to u \text{ in } L^\infty(0,T;H_b^2(\Omega)) \text{ weak-*}$$

and

$$P^{\epsilon_n}u_t^{\epsilon_n} \to u_t \text{ in } L^\infty(0,T;L^2(\Omega)) \text{ weak-*}$$

as $\epsilon_n \to 0$, we have, taking Laplace transforms, that for real positive s

$$\widehat{P^{\epsilon_n}u^{\epsilon_n}}(s) \to \widehat{u}(s) \text{ in } H_b^2(\Omega) \text{ weakly}$$

and

$$\widehat{P^{\epsilon_n}u_t^{\epsilon_n}}(s) \to \widehat{u_t}(s) \text{ in } L^2(\Omega) \text{ weakly.}$$

But by homogenization with parameter $\widehat{u}(s)$ also satisfies (25) for real positive s. By the uniqueness of solutions of (25), $\widehat{u}(s) = \widehat{v}(s)$ for real positive s and by analytic continuation for any s with positive real part. Finally by the uniqueness of the inverse Laplace transform we have $u = v$. We have thus shown that

Theorem 2.4 *Let u^ϵ be the solution of (21)-(22), corresponding to $f \in L^2(0,T;L^2(\Omega))$ and initial data u_0^ϵ, v_0^ϵ as in Theorem 2.2 satisfying*

$$\tilde{u}_0^\epsilon \to u_0 \text{ in } L^2(\Omega) \text{ weakly, with } u_0 \in H_b^2(\Omega)$$

$$\tilde{u}_1^\epsilon \to v_0 \text{ in } L^2(\Omega) \text{ weakly.}$$

Then there exists an extension operator $P^\epsilon \in \mathcal{L}(L^\infty(0,T;V_\epsilon), L^\infty(0,T;H_b^2(\Omega)))$ such that for some sequence $\epsilon_n \to 0$,

$$P^{\epsilon_n} u^{\epsilon_n} \to u \text{ in } L^\infty(0,T;H_b^2(\Omega)) \text{ weak-*}$$

$$P^{\epsilon_n} u_t^{\epsilon_n} \to u_t \text{ in } L^\infty(0,T;L^2(\Omega)) \text{ weak-*}$$

where u is the solution to the homogenized equation:

$$< \theta\gamma u_{tt}(t), \psi > + \sigma(t)(u(\cdot),\psi) = < \theta f, \psi > \quad \text{on } (0,T) \quad \forall \psi \in H_b^2(\Omega)$$

$$u(0) = u_0/\theta \qquad \dot{u}(0) = v_0/\theta \, ,$$

corresponding to the homogenized hysteresis sesquilinear form

$$\sigma(t)(u(\cdot),\psi) = \int_\Omega \int_0^t q_{ijkh}(t-\tau) \frac{\partial^2 u(\tau,x)}{\partial x_k \partial x_h} \frac{\partial^2 \psi}{\partial x_i \partial x_j} \, d\tau \, dx.$$

The coefficients $q_{ijkh}(t) = \mathcal{L}^{-1}[\hat{q}_{ijkh}(s)]$ are the inverse Laplace transforms of the functions

$$\widehat{q}_{ijkh}(s) \equiv \frac{1}{|Y|} \int_{Y^*} \left(a_{ijkh}(y) + s b_{ijkh}(y) - (a_{lmkh}(y) + s b_{lmkh}(y)) \frac{\partial^2 \chi^{ij}(s,y)}{\partial y_l \partial y_m} \right) dy$$

and the functions $\chi^{ij}(s,y)$ are the Y-periodic solutions of

$$\int_{Y^*} (a_{lmkh}(y) + s b_{lmkh}(y)) \frac{\partial^2(\chi^{ij}(s,y) - \mathcal{P}^{ij}(y))}{\partial y_l \partial y_m} \frac{\partial^2 \psi}{\partial y_k \partial y_h} \, dy = 0 \text{ in } Y^* \;\; \forall \psi \in H^2(Y^*)$$

$$\psi \text{ periodic in } Y^*$$

where $\mathcal{P}^{ij}(y) = \frac{1}{2} y_i y_j$.

3 The Dependence of the Homogenized Coefficients on μ

3.1 Introduction

In this section we consider the limit behavior of the homogenized coefficients as the thickness of the members $\mu \to 0$. For the sake of simplicity we will assume that the the original coefficients a^{ϵ}_{ijkh} and b^{ϵ}_{ijkh} are constants and that $l_1 = l_2 = 1$. Hence $|Y| = 1$, $|Y^*| = \mu(2-\mu)$ and $\theta = \mu(2-\mu)$. Since we want to study the dependence of u (the solution of the homogenized equation) on the thickness μ of the material as $\mu \to 0$ we now reintroduce the μ superscript (e.g. $q^{\mu}_{ijkh} \equiv q_{ijkh}$, $\chi^{ij}_{\mu} \equiv \chi^{ij}$, $u^{\mu} \equiv u$ etc.).

3.2 A Priori Estimates and the Limiting Equation for the Case without Damping

We again prove some a priori estimates to show the existence of convergent subsequences. We begin by choosing $\psi = \chi^{ij}_{\mu}(y)$ in equation (17) yielding .

$$\int_{Y^*_\mu} a_{lmkh}(y) \frac{\partial^2 \chi^{ij}_{\mu}(y)}{\partial y_l \partial y_m} \frac{\partial^2 \chi^{ij}_{\mu}(y)}{\partial y_k \partial y_h}\, dy = \int_{Y^*_\mu} a_{ijkh}(y) \frac{\partial^2 \chi^{ij}_{\mu}(y)}{\partial y_k \partial y_h}$$

Then from the ellipticity of the a_{ijkh} and an application of Hölder's inequality

$$A \sum_{|\alpha|=2} |D^{\alpha} \chi^{ij}_{\mu}|^2_{L^2(Y^*_\mu)} \leq C \sum_{|\alpha|=2} |D^{\alpha} \chi^{ij}_{\mu}|_{L^2(Y^*_\mu)} |Y^*_\mu|^{1/2}. \tag{26}$$

Thus

$$\sum_{|\alpha|=2} |D^{\alpha} \chi^{ij}_{\mu}|_{L^2(Y^*_\mu)} \leq C |Y^*_\mu|^{1/2} \leq C \mu^{1/2}, \tag{27}$$

where, as explained earlier, we use C to denote a generic constant independent of μ throughout. We can then use this bound on the χ^{ij}_{μ} to bound the q^{μ}_{ijkh}. Recall that

$$q^{\mu}_{ijkh} = \frac{1}{|Y|} \int_{Y^*_\mu} \left(a_{ijkh}(y) - a_{lmkh}(y) \frac{\partial^2 \chi^{ij}_{\mu}(y)}{\partial y_l \partial y_m} \right) dy \tag{28}$$

and using the estimate (27) we obtain

$$\mu^{-1} q^{\mu}_{ijkh} \le C \left[(2-\mu) + \mu^{-1} |Y^*_\mu|^{1/2} \sum_{|\alpha|=2} |D^\alpha \chi^{ij}_\mu|_{L^2(Y^*_\mu)} \right] \le C.$$

Thus there is a sequence μ_n such that as $\mu_n \to 0$,

$$\mu_n^{-1} q^{\mu_n}_{ijkh} \to q^*_{ijkh}. \tag{29}$$

Examining the homogenized equation (15) we see that if the q^*_{ijkh} give rise to a coercive sesquilinear form, then as $\mu \to 0$

$$u^\mu \to u^* \qquad \text{in } L^\infty(0,T; H^2_b(\Omega)) \text{ weak-}^*$$

where $u^* \in H^2_b(\Omega)$ is the weak solution to

$$2\gamma u^*_{tt} + q^*_{ijkh} \frac{\partial^4 u^*}{x_i x_j x_k x_h} = 2f \qquad \text{in } \Omega$$

$$u^*(0) = u_0/2 \qquad \text{and } \dot{u}^*(0) = v_0/2.$$

3.3 Calculation of the q^*_{ijkh}.

We wish to investigate the limit of equation (28) as $\mu \to 0$. But (28) depends on integrals over Y^*_μ so it will be necessary to map Y^*_μ to some fixed region. Moreover note that the integrands over Y^* contain only constants a_{ijkh} and the Y-periodic functions χ^{ij}_μ. Thus we may take integrals over any affine transformation of Y^*_μ and still preserve the values of the integrals and our a priori bounds on the χ^{ij}_μ. In particular we will find it convenient to consider the translated cell $\mathcal{Y} = (-1/2, -1/2) + Y$ depicted in the figure below. We will find it useful to decompose $\mathcal{Y}^*_\mu$ into horizontal, vertical and central parts by

$$H_\mu = \{|y_1| \le 1/2,\ |y_2| \le \mu/2\} \qquad \text{the horizontal part of } \mathcal{Y}^*_\mu$$

$$V_\mu = \{|y_1| \le \mu/2,\ |y_2| \le 1/2\} \qquad \text{the vertical part of } \mathcal{Y}^*_\mu$$

$$K_\mu = \{|y_1| \le \mu/2,\ |y_2| \le \mu/2\} \qquad \text{the central square of } \mathcal{Y}^*_\mu\ .$$

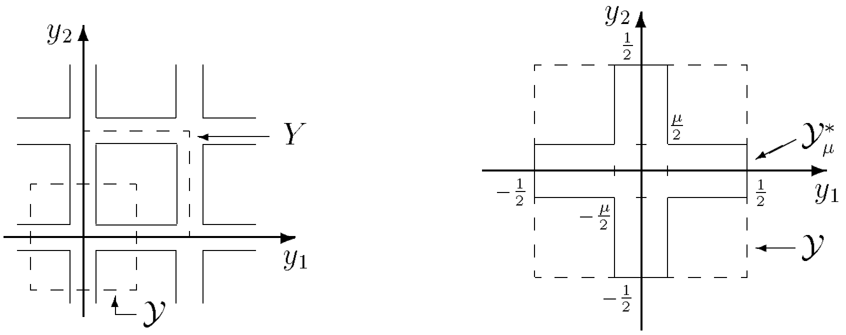

Figure 2: Structure of Y, Y_μ^*, $\mathcal{Y}$, and $\mathcal{Y}_\mu^*$.

Now using the definition of the q_{ijkh}^μ in (28) and the above decomposition of $\mathcal{Y}_\mu^*$ we have

$$\begin{aligned} \mu^{-1} q_{ijkh}^\mu = \mu^{-1} |\mathcal{Y}_\mu^*| a_{ijkh} \quad &- \quad \mu^{-1} \int_{H_\mu} a_{lmkh} \frac{\partial^2 \chi_\mu^{ij}}{\partial y_l \partial y_m}\, dy - \mu^{-1} \int_{V_\mu} a_{lmkh} \frac{\partial^2 \chi_\mu^{ij}}{\partial y_l \partial y_m}\, dy \\ &+ \quad \int_{K_\mu} \mu^{-1} a_{lmkh} \frac{\partial^2 \chi_\mu^{ij}}{\partial y_l \partial y_m}\, dy\ . \end{aligned} \tag{30}$$

As noted above, because the domains in the above integrals depend on μ and since we are interested in the limit of these integrals as $\mu \to 0$ we will find it convenient to map K_μ, H_μ, and V_μ onto the fixed domain $\mathcal{Y}$. For any function $\phi(y_1, y_2)$ defined on H_μ we define

$$\phi_H(y_1, y_2/\mu) = \phi(y_1, y_2).$$

Similarly for V_μ and K_μ respectively

$$\phi_V(y_1/\mu, y_2) = \phi(y_1, y_2)$$

and

$$\phi_K(y_1/\mu, y_2/\mu) = \phi(y_1, y_2).$$

Using the a priori bound (27) we have that

$$\int_{H_\mu} \left[\left(\frac{\partial^2 \chi_\mu^{ij}}{\partial y_1^2} \right)^2 + \left(\frac{\partial^2 \chi_\mu^{ij}}{\partial y_1 \partial y_2} \right)^2 + \left(\frac{\partial^2 \chi_\mu^{ij}}{\partial y_2 \partial y_1} \right)^2 + \left(\frac{\partial^2 \chi_\mu^{ij}}{\partial y_2^2} \right)^2 \right] dy \leq C\mu.$$

If we then introduce the variables $z_1 = y_1$, $z_2 = y_2/\mu$ we see that the above bound can be rewritten as

$$\int_{\mathcal{Y}} \left[\left(\frac{\partial^2 \chi^{ij}_{\mu,H}}{\partial z_1^2} \right)^2 + \mu^{-1} \left(\frac{\partial^2 \chi^{ij}_{\mu,H}}{\partial z_1 \partial z_2} \right)^2 + \mu^{-1} \left(\frac{\partial^2 \chi^{ij}_{\mu,H}}{\partial z_2 \partial z_1} \right)^2 + \mu^{-2} \left(\frac{\partial^2 \chi^{ij}_{\mu,H}}{\partial z_2^2} \right)^2 \right] \mu \, dz \leq C\mu$$

thus

$$\frac{\partial^2 \chi^{ij}_{\mu,H}}{\partial z_1^2} \to h^{ij}_{11} \qquad \text{in } L^2(\mathcal{Y}) \text{ weakly} \tag{31}$$

$$\mu^{-1} \frac{\partial^2 \chi^{ij}_{\mu,H}}{\partial z_2 \partial z_1} \to h^{ij}_{21} \qquad \text{in } L^2(\mathcal{Y}) \text{ weakly} \tag{32}$$

$$\mu^{-1} \frac{\partial^2 \chi^{ij}_{\mu,H}}{\partial z_1 \partial z_2} \to h^{ij}_{12} \qquad \text{in } L^2(\mathcal{Y}) \text{ weakly} \tag{33}$$

$$\mu^{-2} \frac{\partial^2 \chi^{ij}_{\mu,H}}{\partial z_2^2} \to h^{ij}_{22} \qquad \text{in } L^2(\mathcal{Y}) \text{ weakly.} \tag{34}$$

Similarly for V_μ we have $z_1' = y_1/\mu$, $z_2' = y_2$ and

$$\mu^{-2} \frac{\partial^2 \chi^{ij}_{\mu,V}}{\partial z_1'^2} \to v^{ij}_{11} \qquad \text{in } L^2(\mathcal{Y}) \text{ weakly} \tag{35}$$

$$\mu^{-1} \frac{\partial^2 \chi^{ij}_{\mu,V}}{\partial z_1' \partial z_2'} \to v^{ij}_{12} \qquad \text{in } L^2(\mathcal{Y}) \text{ weakly} \tag{36}$$

$$\mu^{-1} \frac{\partial^2 \chi^{ij}_{\mu,V}}{\partial z_1' \partial z_2'} \to v^{ij}_{21} \qquad \text{in } L^2(\mathcal{Y}) \text{ weakly} \tag{37}$$

$$\frac{\partial^2 \chi^{ij}_{\mu,V}}{\partial z_2'^2} \to v^{ij}_{22} \qquad \text{in } L^2(\mathcal{Y}) \text{ weakly} \tag{38}$$

and lastly for K_μ ($z_1'' = y_1/\mu$, $z_2'' = y_2/\mu$) we have

$$\mu^{-3/2} \frac{\partial^2 \chi^{ij}_{\mu,K}}{\partial z_1''^2} \to k^{ij}_{11} \qquad \text{in } L^2(\mathcal{Y}) \text{ weakly} \tag{39}$$

$$\mu^{-3/2} \frac{\partial^2 \chi^{ij}_{\mu,K}}{\partial z_1'' \partial z_2''} \to k^{ij}_{12} \qquad \text{in } L^2(\mathcal{Y}) \text{ weakly} \tag{40}$$

$$\mu^{-3/2}\frac{\partial^2\chi^{ij}_{\mu,K}}{\partial z''_1\partial z''_2} \to k^{ij}_{21} \qquad \text{in } L^2(\mathcal{Y}) \text{ weakly} \tag{41}$$

$$\mu^{-3/2}\frac{\partial^2\chi^{ij}_{\mu,K}}{\partial z''^2_2} \to k^{ij}_{22} \qquad \text{in } L^2(\mathcal{Y}) \text{ weakly.} \tag{42}$$

Now observe that because of the periodicity of χ^{ij}_μ we have

$$\int_{\mathcal{Y}} h^{ij}_{11}\,dy = \int_{\mathcal{Y}} h^{ij}_{12}\,dy = \int_{\mathcal{Y}} h^{ij}_{21}\,dy = 0.$$

Similarly,

$$\int_{\mathcal{Y}} v^{ij}_{12}\,dy = \int_{\mathcal{Y}} v^{ij}_{22}\,dy = \int_{\mathcal{Y}} v^{ij}_{22}\,dy = 0.$$

We are now ready to begin the evaluation of the q^*_{ijkh} using (30). We first note that (27) implies that the last term of (30) goes to 0 as $\mu \to 0$ since

$$|\mu^{-1}\int_{K_\mu} a_{lmkh}\frac{\partial^2\chi^{ij}_\mu}{\partial y_l\partial y_m}\,dy| \le C\mu^{1/2}.$$

So now applying the convergencies (31)-(38) to (29) we have

$$q^*_{ijkh} = 2a_{ijkh} - a_{11kh}\int_{\mathcal{Y}} v^{ij}_{11}\,dy - a_{22kh}\int_{\mathcal{Y}} h^{ij}_{22}\,dy. \tag{43}$$

Our remaining task then is to evaluate the integral terms

$$\int_{\mathcal{Y}} v^{ij}_{11}\,dy \qquad \text{and} \qquad \int_{\mathcal{Y}} h^{ij}_{22}\,dy.$$

To that end let ϕ be a smooth function, periodic in $\mathcal{Y}$ and independent of y_1. Choose $\psi = \phi$ in (17) then

$$\mu^{-1}\int_{\mathcal{Y}^*_\mu} a_{lm22}\frac{\partial^2\chi^{ij}_\mu}{\partial y_l\partial y_m}\frac{\partial^2\phi}{\partial y_2^2}\,dy = \mu^{-1}\int_{\mathcal{Y}^*_\mu} a_{ij22}\frac{\partial^2\phi}{\partial y_2^2}\,dy. \tag{44}$$

As before we decompose the integrals on $\mathcal{Y}^*_\mu$ into integrals on H_μ, V_μ, and K_μ. For the right side observe that the contribution from the V_μ term is 0 since ϕ is periodic and that as $\mu \to 0$ the integrals on K_μ go to zero. Therefore

$$\begin{aligned}
\lim_{\mu\to 0}\mu^{-1}\int_{\mathcal{Y}^*_\mu} a_{ij22}\frac{\partial^2\phi}{\partial y_2^2}\,dy &= \lim_{\mu\to 0}\mu^{-1}\int_{H_\mu} a_{ij22}\frac{\partial^2\phi}{\partial y_2^2}\,dy \\
&= \lim_{\mu\to 0} a_{ij22}\int_{-1/2}^{1/2} dy_1\left[\mu^{-1}\int_{-\mu/2}^{\mu/2}\frac{\partial^2\phi}{\partial y_2^2}\,dy_2\right] \\
&= a_{ij22}\frac{\partial^2\phi}{\partial y_2^2}(0).
\end{aligned} \tag{45}$$

We now deal with the left side terms using the same decomposition of $\mathcal{Y}^*_\mu$ as before

$$
\begin{aligned}
&\mu^{-1}\int_{\mathcal{Y}^*_\mu} a_{lm22}\frac{\partial^2\chi^{ij}_\mu}{\partial y_l\partial y_m}\frac{\partial^2\phi}{\partial y_2^2}\,dy\\
&= \mu^{-1}\int_{H_\mu} a_{lm22}\frac{\partial^2\chi^{ij}_\mu}{\partial y_l\partial y_m}\frac{\partial^2\phi}{\partial y_2^2}\,dy+\mu^{-1}\int_{V_\mu} a_{lm22}\frac{\partial^2\chi^{ij}_\mu}{\partial y_l\partial y_m}\frac{\partial^2\phi}{\partial y_2^2}\,dy\\
&\qquad -\mu^{-1}\int_{K_\mu} a_{lm22}\frac{\partial^2\chi^{ij}_\mu}{\partial y_l\partial y_m}\frac{\partial^2\phi}{\partial y_2^2}\,dy\\
&= \mu^{-1}\int_{\mathcal{Y}} a_{1122}\frac{\partial^2\chi^{ij}_{\mu,H}}{\partial z_1^2}\mu^{-2}\frac{\partial^2\phi_H}{\partial z_2^2}\mu\,dz+\mu^{-1}\int_{\mathcal{Y}} a_{1222}\mu^{-1}\frac{\partial^2\chi^{ij}_{\mu,H}}{\partial z_1\partial z_2}\mu^{-2}\frac{\partial^2\phi_H}{\partial z_2^2}\mu\,dz\\
&\quad +\mu^{-1}\int_{\mathcal{Y}} a_{2122}\mu^{-1}\frac{\partial^2\chi^{ij}_{\mu,H}}{\partial z_2\partial z_1}\mu^{-2}\frac{\partial^2\phi_H}{\partial z_2^2}\mu\,dz+\mu^{-1}\int_{\mathcal{Y}} a_{2222}\mu^{-2}\frac{\partial^2\chi^{ij}_{\mu,H}}{\partial z_2^2}\mu^{-2}\frac{\partial^2\phi_H}{\partial z_2^2}\mu\,dz\\
&\quad +\mu^{-1}\int_{\mathcal{Y}} a_{2222}\frac{\partial^2\chi^{ij}_{\mu,V}}{\partial z_2'^2}\mu^{-2}\frac{\partial^2\phi_V}{\partial z_2'^2}\mu\,dz'+\mu^{-1}\int_{\mathcal{Y}} a_{2122}\mu^{-1}\frac{\partial^2\chi^{ij}_{\mu,V}}{\partial z_2'\partial z_1'}\mu^{-2}\frac{\partial^2\phi_V}{\partial z_2'^2}\mu\,dz'\\
&\quad +\mu^{-1}\int_{\mathcal{Y}} a_{1222}\mu^{-1}\frac{\partial^2\chi^{ij}_{\mu,V}}{\partial z_1'\partial z_2'}\mu^{-2}\frac{\partial^2\phi_V}{\partial z_2'^2}\mu\,dz'+\mu^{-1}\int_{\mathcal{Y}} a_{1122}\mu^{-2}\frac{\partial^2\chi^{ij}_{\mu,V}}{\partial z_1'^2}\mu^{-2}\frac{\partial^2\phi_V}{\partial z_2'^2}\mu\,dz'\\
&\quad -\mu^{-1}\int_{\mathcal{Y}} a_{1122}\mu^{-2}\frac{\partial^2\chi^{ij}_{\mu,K}}{\partial z_1''^2}\mu^{-2}\frac{\partial^2\phi_K}{\partial z_2''^2}\mu^2\,dz''-\mu^{-1}\int_{\mathcal{Y}} a_{1222}\mu^{-2}\frac{\partial^2\chi^{ij}_{\mu,K}}{\partial z_1''\partial z_2''}\mu^{-2}\frac{\partial^2\phi_K}{\partial z_2''^2}\mu^2\,dz''\\
&\quad -\mu^{-1}\int_{\mathcal{Y}} a_{2122}\mu^{-2}\frac{\partial^2\chi^{ij}_{\mu,K}}{\partial z_2''\partial z_1''}\mu^{-2}\frac{\partial^2\phi_K}{\partial z_2''^2}\mu^2\,dz''-\mu^{-1}\int_{\mathcal{Y}} a_{2222}\mu^{-2}\frac{\partial^2\chi^{ij}_{\mu,K}}{\partial z_2''^2}\mu^{-2}\frac{\partial^2\phi_K}{\partial z_2''^2}\mu^2\,dz''
\end{aligned}
$$

Since ϕ is independent of y_1, $\phi_V=\phi$ and $\phi_H=\phi_K$. Furthermore

$$\mu^{-2}\frac{\partial^2\phi_H}{\partial z_2^2}(z_2)=\frac{\partial^2\phi}{\partial y_2^2}(\mu y_2)\overset{\mu\to 0}{\to}\frac{\partial^2\phi}{\partial y_2^2}(0). \tag{46}$$

The integrals over K_μ converge to 0 by the estimate

$$|\mu^{-1}\int_{K_\mu} a_{lmkh}\frac{\partial^2\chi^{ij}_\mu}{\partial y_l\partial y_m}\,dy|\le C\mu^{1/2}.$$

Then observe that

$$\mu^{-1}\int_{\mathcal{Y}} a_{1122}\frac{\partial^2\chi^{ij}_{\mu,H}}{\partial z_1^2}\mu^{-2}\frac{\partial^2\phi_H}{\partial z_2^2}\mu\,dz \;=\; \mu^{-1}\int_{H_\mu} a_{1122}\frac{\partial^2\chi^{ij}_\mu}{\partial y_1^2}\frac{\partial^2\phi}{\partial y_2^2}\,dy$$

$$
\begin{aligned}
&= \mu^{-1} a_{1122} \int_{-\mu/2}^{\mu/2} \frac{\partial^2 \phi}{\partial y_2^2} \left[\int_{-1/2}^{1/2} \frac{\partial^2 \chi_\mu^{ij}}{\partial y_1^2} \, dy_1 \right] dy_2 \\
&= 0
\end{aligned}
$$

by the periodicity of the χ_μ^{ij}. By similar arguments we can show that

$$
\mu^{-1} \int_{\mathcal{Y}} a_{1222} \mu^{-1} \frac{\partial^2 \chi_{\mu,H}^{ij}}{\partial z_1 \partial z_2} \mu^{-2} \frac{\partial^2 \phi_H}{\partial z_2^2} \mu \, dz = 0
$$

and

$$
\mu^{-1} \int_{\mathcal{Y}} a_{2122} \mu^{-1} \frac{\partial^2 \chi_{\mu,H}^{ij}}{\partial z_2 \partial z_1} \mu^{-2} \frac{\partial^2 \phi_H}{\partial z_2^2} \mu \, dz = 0.
$$

Then using the weak convergencies (34), (35)-(38) and the strong convergence (46) we obtain

$$
\mu^{-1} \int_{\mathcal{Y}_\mu^*} a_{lm22} \frac{\partial^2 \chi_\mu^{ij}}{\partial y_l \partial y_m} \frac{\partial^2 \phi}{\partial y_2^2} \, dy = a_{lm22} \int_{\mathcal{Y}} v_{lm}^{ij}(z') \frac{\partial^2 \phi}{\partial z_2'^2} \, dz' + a_{2222} \frac{\partial^2 \phi}{\partial y_2^2}(0) \int_{\mathcal{Y}} h_{22}^{ij}(z) \, dz.
$$

At long last recalling (44), (45) and the equation above we have

$$
a_{lm22} \int_{\mathcal{Y}} v_{lm}^{ij}(z') \frac{\partial^2 \phi}{\partial z_2'^2} \, dz' + \frac{\partial^2 \phi}{\partial y_2^2}(0) \left[a_{2222} \int_{\mathcal{Y}} h_{22}^{ij}(z) \, dz - a_{ij22} \right] = 0. \qquad (47)
$$

We now use the following lemma from [CS].

Lemma 3.1 *Let w be a function in $L^2(-\frac{1}{2}, \frac{1}{2})$, periodic and let a be a real constant. If*

$$
a\Upsilon(0) + \int_{-1/2}^{1/2} w(x)\Upsilon(x) \, dx = 0
$$

holds for any smooth function Υ periodic on $[-\frac{1}{2}\ \frac{1}{2}]$ and such that

$$
\int_{-1/2}^{1/2} \Upsilon(x) \, dx = 0,
$$

then

$$
a = 0 \qquad \text{and} \qquad w = \text{constant}.
$$

Proof: See [CS].

If we now apply this lemma to (47) with

$$a = a_{2222} \int_{\mathcal{Y}} h_{22}^{ij}(z)\, dz - a_{ij22} \qquad \Upsilon = \frac{\partial^2 \phi}{\partial y_2^2}$$

and

$$w(z_2') = a_{lm22} \int_{-1/2}^{1/2} v_{lm}^{ij}(z_1', z_2')\, dz_1'$$

it follows that

$$\int_{\mathcal{Y}} h_{22}^{ij}(z)\, dz = \frac{a_{ij22}}{a_{2222}}.$$

Now if we were to repeat all the above calculations with a smooth function ϕ that was independent of y_2 we would see that

$$\int_{\mathcal{Y}} v_{11}^{ij}(z)\, dz = \frac{a_{ij11}}{a_{1111}}.$$

Inserting these values into (43) we have

$$q_{ijkh}^* = 2a_{ijkh} - \sum_{l=1}^{2} \frac{a_{ijll} a_{llkh}}{a_{llll}}.$$

3.4 Extension to the Case with Damping

The extension of the above results to include the case of damping is straightforward. We again use Laplace transforms and homogenization with parameter and instead of doing the above calculations for a_{ijkh} we do them for $a_{ijkh} + sb_{ijkh}$ to obtain the homogenized coefficients

$$q_{ijkh}^*(s) = 2(a_{ijkh} + sb_{ijkh}) - \sum_{l=1}^{2} \frac{(a_{ijll} + sb_{ijll})(a_{llkh} + sb_{llkh})}{a_{llll} + sb_{llll}} \tag{48}$$

Under the usual assumptions on the a_{ijkh} and b_{ijkh} it is not hard to see that the $q_{ijkh}^*(s)$ satisfy an ellipticity condition.

4 Consequences for the Love-Kirchhoff Plate

The above results can be specialized to the case of a Love-Kirchhoff plate with Kelvin-Voigt damping. In (21)-(22) we let $\gamma = \rho h$, where ρ is the volume mass density and h the thickness of plate (i.e., γ is the area mass density) and we set the coefficients $a_{ijkh}^{\epsilon\mu}$, $b_{ijkh}^{\epsilon\mu}$ as follows

$$a_{1111}^{\epsilon\mu} = a_{2222}^{\epsilon\mu} = \frac{EI}{1-\nu^2}$$

$$a_{1122}^{\epsilon\mu} = a_{2211}^{\epsilon\mu} = \nu\frac{EI}{1-\nu^2}$$

$$a_{1212}^{\epsilon\mu} = a_{2121}^{\epsilon\mu} = a_{1221}^{\epsilon\mu} = a_{2112}^{\epsilon\mu} = \frac{EI}{2(1+\nu)}$$

and all other $a_{ijkh}^{\epsilon\mu}$ zero, and

$$b_{1111}^{\epsilon\mu} = b_{2222}^{\epsilon\mu} = \frac{c_D I}{1-\nu^2}$$

$$b_{1122}^{\epsilon\mu} = b_{2211}^{\epsilon\mu} = \nu\frac{c_D I}{1-\nu^2}$$

$$b_{1212}^{\epsilon\mu} = b_{2121}^{\epsilon\mu} = b_{1221}^{\epsilon\mu} = b_{2112}^{\epsilon\mu} = \frac{c_D I}{2(1+\nu)}$$

and all other $b_{ijkh}^{\epsilon\mu}$ zero where ν is Poisson's ratio and EI and $c_D I$ are the usual stiffness and Kelvin-Voigt damping. Then we obtain the weak form of the Love-Kirchhoff plate equation with Kelvin-Voigt damping.To clarify the exposition of the implications of the results of sections 2 and 3 we now put the resulting equation in strong form.

$$\rho h\frac{\partial^2 u^{\epsilon\mu}}{\partial t^2} + \frac{\partial^2 M^1}{\partial x_1^2} + 2\frac{\partial^2 M^{12}}{\partial x_1 \partial x_2} + \frac{\partial^2 M^2}{\partial x_2^2} = f \qquad t > 0$$

where

$$M^1 = \frac{EI}{1-\nu^2}\left\{\frac{\partial^2 u^{\epsilon\mu}}{\partial x_1^2} + \nu\frac{\partial^2 u^{\epsilon\mu}}{\partial x_2^2}\right\} + \frac{c_D I}{1-\nu^2}\left\{\frac{\partial^3 u^{\epsilon\mu}}{\partial x_1^2 \partial t} + \nu\frac{\partial^3 u^{\epsilon\mu}}{\partial x_2^2 \partial t}\right\}$$

$$M^2 = \frac{EI}{1-\nu^2}\left\{\frac{\partial^2 u^{\epsilon\mu}}{\partial x_2^2} + \nu\frac{\partial^2 u^{\epsilon\mu}}{\partial x_1^2}\right\} + \frac{c_D I}{1-\nu^2}\left\{\frac{\partial^3 u^{\epsilon\mu}}{\partial x_2^2 \partial t} + \nu\frac{\partial^3 u^{\epsilon\mu}}{\partial x_1^2 \partial t}\right\}$$

$$M^{12} = \frac{EI}{1+\nu}\frac{\partial^2 u^{\epsilon\mu}}{\partial x_1 \partial x_2} + \frac{c_D I}{1+\nu}\frac{\partial^3 u^{\epsilon\mu}}{\partial x_1 \partial x_2 \partial t}$$

The space $V_{\epsilon\mu}$ specifies the essential boundary condition

$$u = \frac{\partial u^{\epsilon\mu}}{\partial x^2} = 0 \text{ for } x_2 = 0.$$

The conditions on the rest of the boundary are natural ones and correspond to zero moment and zero shear. That is,

$$M^1 = 0, \qquad \frac{\partial M^1}{\partial x_1} + 2\frac{\partial M^{12}}{\partial x_2} = 0 \tag{49}$$

on edges parallel to the x_2 axis, and

$$M^2 = 0, \qquad \frac{\partial M^2}{\partial x_2} + 2\frac{\partial M^{12}}{\partial x_1} = 0 \tag{50}$$

on edges parallel to the x_1 axis. The results of sections 2 and 3 then imply that there exists an extension of the solution to this problem $P^\epsilon u^{\epsilon\mu}$ that converges in $L^\infty(0,T;H_b^2(\Omega))$ as $\epsilon \to 0$ and then $\mu \to 0$ to u^*, the unique solution to

$$\rho h \frac{\partial^2 u^*}{\partial t^2} + \frac{\partial^2 M_*^1}{\partial x_1^2} + 2\frac{\partial^2 M_*^{12}}{\partial x_1 \partial x_2} + \frac{\partial^2 M_*^2}{\partial x_2^2} = f \qquad t > 0$$

$$u^*(0) = u_0/2, \qquad u_t^*(0) = v_0/2$$

where

$$M_*^1 = \frac{EI}{2}\frac{\partial^2 u^*}{\partial x_1^2} + \frac{c_D I}{2}\frac{\partial^3 u^*}{\partial x_1^2 \partial t}$$

$$M_*^2 = \frac{EI}{2}\frac{\partial^2 u^*}{\partial x_2^2} + \frac{c_D I}{2}\frac{\partial^3 u^*}{\partial x_2^2 \partial t}$$

$$M_*^{12} = \frac{EI}{1+\nu}\frac{\partial^2 u^*}{\partial x_1 \partial x_2} + \frac{c_D I}{1+\nu}\frac{\partial^3 u^*}{\partial x_1 \partial x_2 \partial t}\ .$$

Of course u^* must also satisfy the clamped boundary condition on the edge $x_2 = 0$ and the moments M_*^1, M_*^2,M_*^{12} must satisfy the zero moment, zero shear specified in (49) and (49).

Finally we remark that we have made initial numerical investigations into the question of the validity and utility of this approximate model by comparing experimentally observed modal properties of a metal plate cut into the shape of a rectangular grid with the modal properties of the corresponding homogenized equation. A summary of those results is contained in [B].

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