

ESTIMATION OF OPTIMAL BLOCK REPLACEMENT POLICIES

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SUMMARY Stochastic approximation methodology is applied to the estimation of optimal block replacement policies. Here optimal is taken to mean the block replacement policy which minimizes long-run expected cost. The basic procedure suggested is refined to provide a faster rate of convergence for the estimator.

0. NOTATION

In the following the symbol \circ is used to denote the composition of functions. A superscript of the form $^{(p)}$ where p is a nonnegative integer will indicate the p -th derivative of a function. For example, if H and g are differentiable functions, then $H \circ g^{(1)}(t) = \frac{d}{dt}H(g(t))$. Wherever used, K_1, K_2, \dots will represent appropriate constants. E and P will be used for expectation and probability, respectively. Convergence in distribution will be denoted by \xrightarrow{D} . Finally, $I[\]$ will be the indicator function of the event within the brackets.

1. INTRODUCTION

An important topic in reliability theory is maintenance policies for systems (mechanical, electrical, etc.) whose components have random lifetimes. Consider a system consisting of a single component or part. It is assumed that this part can be replaced by a like part at any time and is not repaired following failure. Suppose that all parts used have independent and identically distributed lifetimes with distribution F . Further assume that the cost C_1 of having to unexpectedly replace a failed part is greater than the cost C_2 of making a replacement at a planned time. If F has increasing failure rate, then it may be less expensive to substitute new parts for aging parts near failure than to only make replacements following failure. Two maintenance plans motivated by this consideration are the block replacement policy (BRP) and the age replacement policy (ARP). Both of these policies are treated in detail by Barlow and Proschan (1965, 1975), where some historical development is also given.

Suppose a new part is installed at time 0. In the simplest BRP, the part in use is replaced by a new part at times T_1, T_2, T_3, \dots , and replaced following failure at any other time. Such a plan is termed periodic if $T_1 = T, T_2 = 2T, T_3 = 3T, \dots$ where T is some positive real number. For the

periodic BRP, the expected long-run cost per unit time as a function of T is

$$B(T) = \frac{C_1 H(T) + C_2}{T}.$$

Here H is the renewal function associated with the lifetime distribution F. Thus H(T) is the expected number of failures for each of the intervals (0, T], (T, 2T], etc. (It is assumed for both the BRP and the ARP that all replacements are made instantaneously.)

Under an ARP, if a part has not failed and been replaced prior to reaching some fixed age T, it is preventively replaced at age T. The expected long-run cost per unit time can be shown to be

$$A(T) = \frac{C_1 F(T) + C_2 S(T)}{\int_0^T S(t) dt},$$

where $S(t) = 1 - F(t)$.

For both the age and block replacement policies an important practical question is what value of T, if any, minimizes expected cost per unit time. Barlow and Proschan (1965) give sufficient conditions for the existence of an optimal ARP T_0 such that T_0 minimizes A(T). For a given distribution F, finding such a T_0 is a deterministic problem. For instance, Glasser (1967) has computed T_0 for selected gamma, Weibull, and truncated normal distributions. Frees (1983) summarizes some approaches that have been taken to the problem of estimating T_0 for the ARP. Consistent estimators were defined by Arunkumar (1972) and Bergman (1979) using an i.i.d. sample from the distribution F. Bather (1977) proposed a sequential scheme that uses estimates of T_0 to censor observations (make planned replacements), essentially a "random" ARP. Bather's procedure has the desirable feature that asymptotically the incurred cost per unit time is minimized, as though the optimal ARP were used all along. Frees (1983) successfully applies stochastic approximation methodology to the problem of finding the optimal ARP when the distribution F is unknown.

Now suppose T_0 minimizes the BRP cost B(T) in T. Again, if F is known the problem of finding T_0 is not statistical. At this time we have not been able to locate any procedure in the literature for estimating T_0 if no parametric form for F is assumed.

This report investigates using stochastic approximation to estimate the optimal BRP in a nonparametric setting. To motivate the application of stochastic approximation to this problem, consider the following ad hoc procedure. Suppose a periodic BRP is being carried out as described above with scheduled replacements at times T, 2T, 3T, . . . for some $T > 0$. If T is less than the optimal value T_0 , then an unnecessary number of planned replacements will be made. This would

suggest that to decrease costs one should increase the value of T . If T is greater than T_0 , then there are apt to be too many part failures which could be remedied by adjusting the value of T downwards. The stochastic approximation algorithm introduced in Section 2 incorporates this idea of making successive adjustments to T to arrive at the value which minimizes cost.

In Section 2 sufficient conditions are given for the a.s. convergence and asymptotic normality of the basic estimator of T_0 . A refinement to speed the rate of convergence of the procedure is explored in Section 3. The work presented here follows fairly closely that of Frees(1983) on estimation of optimal age replacement policies using stochastic approximation.

2. THE BASIC ALGORITHM

Let $\{a_n\}$ be a sequence of positive constants decreasing to zero. Suppose M is a differentiable regression function on the real line which has a minimum at Z_* . Take Z_1 to be an initial estimate of Z_* . The usual stochastic approximation algorithm defines estimates of Z_* recursively by

$$(2.1) \quad Z_{n+1} = Z_n - a_n M_n^{(1)}(Z_n)$$

where $M_n^{(1)}(Z_n)$ is an estimate of $M^{(1)}(Z_n)$. Such a procedure is now considered for finding an optimal BRP.

Recall that with previously defined notation the cost function for the BRP is

$$B(t) = \frac{C_1 H(t) + C_2}{t}$$

where H is the renewal function associated with F and $C_1 > C_2$. The objective is to minimize B in t . Assume that F is absolutely continuous so that the renewal density $h = H^{(1)}$ exists. Define

$$D(t) = tC_1 h(t) - C_1 H(t) - C_2,$$

so that $B^{(1)}(t) = D(t)/t^2$. Suppose that $B(t)$ has a unique minimum at $t = \phi^*$ with $D(\phi^*) = 0$.

Given an initial estimate ϕ_1^* of ϕ^* , (2.1) suggests defining successive estimates of ϕ^* by the formula

$$\phi_{n+1}^* = \phi_n^* - a_n D_n(\phi_n^*)$$

where $D_n(\phi_n^*)$ is an estimate of $D(\phi_n^*)$. This procedure would have the disadvantage of possibly generating negative estimates when necessarily ϕ^* is greater than zero. If it is given that ϕ^* is greater than some known $\epsilon > 0$, then one might use the algorithm

$$\phi_{n+1}^* = \max (\epsilon, \phi_n^* - a_n D_n(\phi_n^*)).$$

Rather than constraining the estimates in this way, the following approach is taken.

Assume there exist known nonnegative constants T_1 and T_2 such that $T_1 < \phi^* < T_2$. We allow the possibility that $T_1 = 0$ and $T_2 = \infty$. Let $g: \mathbb{R} \rightarrow (T_1, T_2)$ be a known strictly increasing function having $s+1$ bounded derivatives. Suppose ϕ minimizes $B \circ g(t)$ in t , so that $\phi^* = g(\phi)$. Define

$$Dg(t) = C_1 g(t) H \circ g^{(1)}(t) - g^{(1)}(t) [C_1 H \circ g(t) + C_2].$$

Since $B \circ g^{(1)}(t) = \frac{Dg(t)}{[g(t)]^2}$ and $Dg(\phi) = 0$, we now consider using a stochastic approximation procedure

to estimate the zero of Dg . Given an initial estimate ϕ_1 , define successive estimates of ϕ by

$$(2.2) \quad \phi_{n+1} = \phi_n - a_n Dg_n(\phi_n),$$

where $Dg_n(\phi_n)$ is an estimate of $Dg(\phi_n)$. Given ϕ_n , take $g(\phi_n)$ as an estimate of ϕ^* .

The next step is to define an estimator $Dg_n(t)$ of $Dg(t)$. Let $\{ X_{ij} \}_{i,j=1}^{\infty}$ be i. i. d. random variables with distribution function F defined on a probability space (Ω, \mathcal{F}, P) . Take $N_n = \{ N_n(t), t \geq 0 \}$ to be the renewal process associated with the sequence $\{ X_{nj} \}_{j=1}^{\infty}$ with renewals occurring at $X_{n1}, X_{n1}+X_{n2}, X_{n1}+X_{n2}+X_{n3}, \dots$ for $n = 1, 2, \dots$. Let $\{ c_n \}$ be a sequence of positive constants decreasing to zero. Define

$$hg_n(t) = \frac{N_n(g(t+c_n)) - N_n(g(t-c_n))}{2c_n},$$

an estimate of $H \circ g^{(1)}(t)$. Finally, let

$$(2.3) \quad Dg_n(t) = C_1 g(t) hg_n(t) - g^{(1)}(t) [C_1 N_n(g(t)) + C_2],$$

with estimates of ϕ defined recursively by (2.2).

We will make use of the following assumptions.

A1: The distribution function F has increasing failure rate. F is absolutely continuous with bounded density f , $F(0)=0$, and $\mu_2 = \int_0^\infty t^2 dF(t) < \infty$.

A2: There exist known nonnegative constants T_1 and T_2 such that $T_1 < \phi^* < T_2 \leq \infty$. For $t \in (T_1, T_2)$, $D(t)(t-\phi^*) > 0$ for $t \neq \phi^*$.

A2': A2 holds with $T_2 < \infty$.

A3: $g: \mathbf{R} \rightarrow (T_1, T_2)$ is a known, continuous, strictly increasing function with $s+1$ bounded derivatives such that $g^{(2)}$ is continuous.

A4: $\{a_n\}$ and $\{c_n\}$ are sequences of positive constants such that $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = 0$,

$$\sum_{n=1}^{\infty} a_n = \infty, \quad \sum_{n=1}^{\infty} a_n c_n < \infty, \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{a_n^2}{c_n} < \infty.$$

A5:(a) $H \circ g^{(2)}(t)$ exists for each $t \in \mathbf{R}$ and is bounded.

(b) A5 (a) holds and $H \circ g^{(2)}$ is continuous at ϕ .

(c) $H \circ g^{(2)}(t)$ exists for each $t \in \mathbf{R}$ and for some constant $K_1 > 0$

$$|H \circ g^{(2)}(x_1) - H \circ g^{(2)}(x_2)| \leq K_1 |x_1 - x_2| \quad \text{for all } x_1, x_2 \in \mathbf{R}.$$

A5':(a) $H \circ g^{(3)}(t)$ exists for each $t \in \mathbf{R}$ and is bounded.

(b) A5' (a) holds and $H \circ g^{(3)}$ is continuous at ϕ .

A6: For constants $A, C > 0$, let $\Gamma = A \frac{d}{dt} Dg(t)|_{t=\phi}$, $a_n = An^{-1}$, and $c_n = Cn^{-\gamma}$, where $\gamma \in (0, 1)$

is such that $1-\gamma < 2\Gamma$.

Before proceeding with the main results of this section, some simple but important consequences of the preceding assumptions are noted. For a fixed $x > 0$, let $Y_n = N_n(g(x+c_n)) - N_n(g(x-c_n))$. By a Taylor expansion and A1,

$$\begin{aligned} EY_n &= H(g(x+c_n)) - H(g(x-c_n)) \\ &= H \circ g^{(1)}(x)2c_n + o(c_n). \end{aligned}$$

For $k \geq 2$,

$$\begin{aligned} P \{ Y_n = k \} &\leq \sum_{j=1}^{\infty} [F^{(j)}(g(x+c_n)) - F^{(j)}(g(x-c_n))] [F(g(x+c_n) - g(x-c_n))]^{k-1} \\ &= [H(g(x+c_n)) - H(g(x-c_n))] [F(g(x+c_n) - g(x-c_n))]^{k-1} \\ &\leq K_1 c_n [F(g(x+c_n) - g(x-c_n))]^{k-1} \end{aligned}$$

Since $c_n \downarrow 0$, ultimately $F(g(x+c_n) - g(x-c_n)) < 1$, and thus for any integer $p > 0$,

$$(2.4) \quad E \{ Y_n^p I [Y_n \geq 2] \} = K_1 c_n \sum_{k=2}^{\infty} k^p [F(g(x+c_n) - g(x-c_n))]^{k-1} = o(c_n)$$

The above implies that

$$P \{ Y_n = 1 \} = H \circ g^{(1)}(x)2c_n + o(c_n)$$

and that for any real number $p > 0$

$$(2.5) \quad E Y_n^p = H \circ g^{(1)}(x)2c_n + o(c_n).$$

Theorems 2.1 and 2.2 give sufficient conditions for a.s. convergence and asymptotic normality of the estimates generated by the procedure.

Theorem 2.1: Assume A1-A4 hold. Further suppose that any one of the assumptions A5(a), A5(c), or A5'(a) holds. Then $\phi_n \rightarrow \phi$ a. s.

Theorem 2.2: Assume A1, A2', A3, and A6 hold. Let $\Sigma = C_1^2 [g(\phi)]^2 \text{Hog}^{(1)}(\phi)/2$. With Γ defined by A6, if either

(a) A5(b) or A5(c) holds with $\frac{1}{3} \leq \gamma < 1$, or

(b) A5'(a) holds with $\frac{1}{5} < \gamma < 1$, then

$$n^{(1-\gamma)/2} (\phi_n - \phi) \xrightarrow{D} N\left(0, \frac{A^2 C^{-1} \Sigma}{2\Gamma - (1-\gamma)}\right).$$

(c) If A5'(b) holds with $\gamma = \frac{1}{5}$, then

$$n^{2/5} (\phi_n - \phi) \xrightarrow{D} N\left(-\frac{AC^2 C_1 g(\phi) \text{Hog}^{(3)}(\phi)}{6[\Gamma - \frac{2}{5}]}, \frac{A^2 C^{-1} \Sigma}{2\Gamma - \frac{4}{5}}\right).$$

The remainder of this section will be devoted to proofs of Theorems 2.1 and 2.2. The following theorem due to Robbins and Siegmund(1971) will be used in the proof of Theorem 1.

Theorem(Robbins-Siegmund)

Let \mathcal{F}_n be a nondecreasing sequence of sub sigma-fields of \mathcal{F} . Suppose that X_n, β_n, η_n , and γ_n are nonnegative \mathcal{F}_n -measurable random variables such that

$$E_{\mathcal{F}_n} X_{n+1} \leq X_n (1 + \beta_n) + \eta_n - \gamma_n \quad \text{for } n = 1, 2, \dots$$

Then, $\lim_{n \rightarrow \infty} X_n$ exists and $\sum_{n=1}^{\infty} \gamma_n < \infty$ on the set such that $\sum_{n=1}^{\infty} \beta_n < \infty$ and $\sum_{n=1}^{\infty} \eta_n < \infty$.

Define the σ -fields $\mathcal{F}_n = \sigma(X_i, j, i=1, 2, \dots, n-1, j=1, 2, \dots)$. Use $E_{\mathcal{F}_n}$ to denote expectation with respect to the σ -field \mathcal{F}_n . Since N_n is \mathcal{F}_n -measurable but is independent of \mathcal{F}_{n-1} , by (2.2) and (2.3) ϕ_n is \mathcal{F}_n -measurable while $Dg_n(\phi_n)$ is independent of \mathcal{F}_{n-1} . Define

$$\Delta_n = E_{\mathcal{F}_n} [Dg_n(\phi_n) - Dg(\phi_n)],$$

and

$$V_n = c_n^{1/2} [Dg_n(\phi_n) - Dg(\phi_n) - \Delta_n].$$

In Lemmas 2.1 and 2.2 bounds are derived for Δ_n and $E_{\mathcal{F}_n} [Dg_n(\phi_n)]^2$, respectively, to be used in

the application of the Robbins-Siegmund result. V_n plays an important role in the proof of Theorem 2.2. The asymptotic variance of V_n will be $\Sigma = C_1^2 [g(\phi)]^2 \text{Hog}^{(1)}(\phi)/2$.

Several properties of increasing failure rate distributions are used in the proofs of Theorems 2.1 and 2.2. These properties are quoted here and referred to as necessary. See Barlow and Proschan (1965) for proofs of these results. Let F be an increasing failure rate distribution with mean μ and associated renewal function H . Suppose $N = \{ N(t), t \geq 0 \}$ is a renewal process with underlying distribution F . Then

$$(2.6) \quad \text{Var}[N(t)] \leq H(t) = E N(t) \leq \frac{t}{\mu}, \text{ for all } t \geq 0,$$

and

$$(2.7) \quad E [N(t)]^k \leq e^{-t/\mu} \sum_{j=0}^{\infty} \frac{j^k (t/\mu)^j}{j!}.$$

Lemma 2.1: Assume A1, A2, A3 and A4 hold.

(a) If A5(a) holds, then $|\Delta_n| \leq [K_1 + K_2|\phi_n - \phi|]c_n$.

(b) If A5(c) or A5'(a) holds, then $|\Delta_n| \leq [K_1 + K_2|\phi_n - \phi|]c_n^2$.

Proof: By definition we have that

$$\Delta_n = C_1 g(\phi_n) \left[\frac{\text{Hog}(\phi_n + c_n) - \text{Hog}(\phi_n - c_n)}{2c_n} - \text{Hog}^{(1)}(\phi_n) \right].$$

If A5(a) holds, then $\text{Hog}(\phi_n + c_n) - \text{Hog}(\phi_n - c_n) = 2c_n \text{Hog}^{(1)}(\phi_n) + \frac{1}{2}c_n^2 [\text{Hog}^{(2)}(\eta_1) - \text{Hog}^{(2)}(\eta_2)]$,

with $|\eta_i - \phi_n| \leq c_n$, $i = 1, 2$. Thus

$$|\Delta_n| \leq \frac{1}{4}C_1 g(\phi_n) |\text{Hog}^{(2)}(\eta_1) - \text{Hog}^{(2)}(\eta_2)| c_n \leq K_1 g(\phi_n) c_n.$$

For some η such that $|\eta - \phi| \leq |\phi_n - \phi|$,

$$g(\phi_n) = g(\phi) + g^{(1)}(\eta)(\phi_n - \phi),$$

which together with the preceding implies (a) since $g^{(1)}$ is bounded.

If A5(c) holds, then

$$|\Delta_n| \leq \frac{1}{4} C_1 g(\phi_n) |H \circ g^{(2)}(\eta_1) - H \circ g^{(2)}(\eta_2)| c_n$$

$$\leq K_2 g(\phi_n) c_n^2 .$$

$$\leq [K_3 + K_4 |\phi_n - \phi|] c_n^2 ,$$

by the Taylor expansion for g .

If A5'(a) holds, then

$$H \circ g(\phi_n + c_n) - H \circ g(\phi_n - c_n) = 2c_n H \circ g^{(1)}(\phi_n) + \frac{1}{6} c_n^3 [H \circ g^{(3)}(\eta_1) + H \circ g^{(3)}(\eta_2)],$$

where again $|\eta_i - \phi_n| \leq c_n$, $i = 1, 2$. In this case

$$|\Delta_n| \leq \frac{1}{12} C_1 g(\phi_n) |H \circ g^{(3)}(\eta_1) + H \circ g^{(3)}(\eta_2)| c_n^2$$

$$\leq K_5 g(\phi_n) c_n^2$$

$$\leq [K_6 + K_7 |\phi_n - \phi|] c_n^2 ,$$

completing the proof of part (b) of the lemma.

Lemma 2.2: Assume A1-A4 hold. Then $E_{\mathcal{F}_n} [Dg_n(\phi_n)]^2 \leq [K_1 + K_2(\phi_n - \phi)^2] / c_n$.

Proof: Using the definition of Dg_n and the fact that $(a+b)^2 \leq 4(a^2+b^2)$, we have that

$$Dg_n^2(\phi_n) \leq K_1 [g(\phi_n)]^2 \left[\frac{N_n(g(\phi_n + c_n)) - N_n(g(\phi_n - c_n))}{2c_n} \right]^2 + K_2 [g^{(1)}(\phi_n)]^2 [C_1 N_n(g(\phi_n)) + C_2]^2 .$$

By (2.5) and A3 we have that

$$\begin{aligned}
 E_{\mathcal{F}_n} [g(\phi_n)]^2 \left[\frac{N_n(g(\phi_n+c_n)) - N_n(g(\phi_n-c_n))}{2c_n} \right]^2 &= [g(\phi_n)]^2 \frac{1}{4c_n^2} [H \circ g^{(1)}(\phi_n) 2c_n + o(c_n)] \\
 &\leq [g(\phi_n)]^2 \frac{K_3}{c_n} \\
 &\leq [K_4 + K_5 |\phi_n - \phi|]^2 \frac{K_3}{c_n} \\
 &\leq [K_6 + K_7 (\phi_n - \phi)^2] / c_n.
 \end{aligned}$$

Since F has increasing failure rate, it follows from (2.6) and A3 that

$$\begin{aligned}
 E_{\mathcal{F}_n} [g^{(1)}(\phi_n)]^2 [C_1 N_n(g(\phi_n)) + C_2]^2 &\leq K_8 + K_9 E_{\mathcal{F}_n} N_n(g(\phi_n)) + K_{10} E_{\mathcal{F}_n} [N_n(g(\phi_n))]^2 \\
 &\leq K_8 + K_9 \left[\frac{g(\phi_n)}{\mu} \right] + 2K_{10} \left[\frac{g(\phi_n)}{\mu} \right]^2 \\
 &\leq K_{11} + K_{12} [g(\phi_n)]^2 \\
 &\leq K_{13} + K_{14} (\phi_n - \phi)^2
 \end{aligned}$$

Noting that $c_n = o(1)$, the lemma follows from the inequality for Dg_n^2 at the beginning of the proof.

Proof of Theorem 2.1: Letting $U_n = \phi_n - \phi$, we have by (2.2) that

$$U_{n+1}^2 = U_n^2 + a_n^2 Dg_n^2(\phi_n) - 2a_n U_n Dg_n(\phi_n).$$

Since ϕ_n is \mathcal{F}_n -measurable,

$$E_{\mathcal{F}_n} U_{n+1}^2 = U_n^2 + a_n^2 E_{\mathcal{F}_n} [Dg_n^2(\phi_n)] - 2a_n U_n E_{\mathcal{F}_n} [Dg_n(\phi_n)].$$

Noting that

$$E_{\mathcal{F}_n} Dg_n(\phi_n) = C_1 g(\phi_n) \left[\frac{H \circ g(\phi_n+c_n) - H \circ g(\phi_n-c_n)}{2c_n} \right] - g^{(1)}(\phi_n) [C_1 H \circ g(\phi_n) + C_2]$$

$$= Dg(\phi_n) + \Delta_n,$$

it follows that

$$E_{\mathcal{F}_n} U_{n+1}^2 \leq U_n^2 + a_n^2 E_{\mathcal{F}_n} [Dg_n^2(\phi_n)] + 2a_n |U_n| |\Delta_n| - 2a_n (\phi_n - \phi) Dg(\phi_n).$$

If A5(a) holds, then by Lemma 2.1

$$|U_n \Delta_n| \leq [K_1 |U_n| + K_2 U_n^2] c_n \leq [K_3 + K_4 U_n^2] c_n$$

Similarly, under A5(c) or A5'(a),

$$|U_n \Delta_n| \leq [K_1 |U_n| + K_2 U_n^2] c_n^2 \leq [K_3 + K_4 U_n^2] c_n,$$

since $c_n = o(1)$. Thus by Lemmas 2.1 and 2.2,

$$\begin{aligned} E_{\mathcal{F}_n} U_{n+1}^2 &\leq U_n^2 + a_n^2 E_{\mathcal{F}_n} [Dg_n^2(\phi_n)] + 2a_n |U_n| |\Delta_n| - 2a_n (\phi_n - \phi) Dg(\phi_n) \\ &\leq U_n^2 + a_n^2 [K_1 + K_2 U_n^2] / c_n + 2a_n [K_3 + K_4 U_n^2] c_n - 2a_n (\phi_n - \phi) Dg(\phi_n) \\ &= U_n^2 [1 + K_2 \frac{a_n^2}{c_n} + 2K_4 a_n c_n] + [K_1 a_n^2 + 2K_3 a_n c_n] - 2a_n (\phi_n - \phi) Dg(\phi_n). \end{aligned}$$

By A2 and A4 the assumptions of the Robbins-Siegmund result are met. Thus for some finite random

variable ξ we have that $\phi_n \rightarrow \xi$ a. s. and $\sum_{n=1}^{\infty} a_n (\phi_n - \phi) Dg(\phi_n) < \infty$. Since $(\phi_n - \phi) Dg(\phi_n) > 0$

for $\phi_n \neq \phi$ by A2 and $\sum_{n=1}^{\infty} a_n = \infty$ by A4, $\phi_n \rightarrow \phi$ a. s.

Before beginning the proof of Theorem 2.2, we quote a theorem of Fabian(1968) on asymptotic normality which is applicable to many stochastic approximation procedures. The univariate version of this theorem given here is taken from Frees(1983).

Theorem (Fabian)

Suppose \mathcal{F}_n is a nondecreasing sequence of sub σ -fields of \mathcal{F} . Suppose U_n, V_n, T_n, Γ_n , and Φ_n are

random variables such that $\Gamma_n, \Phi_{n-1},$ and V_{n-1} are \mathcal{F}_n -measurable. Let $\alpha, \beta, T, \Sigma, \Gamma,$ and Φ be real constants with $\Gamma > 0$ such that

$$\Gamma_n \rightarrow \Gamma, \quad \Phi_n \rightarrow \Phi, \quad T_n \rightarrow T \text{ or } E |T_n - T| \rightarrow 0, \quad E_{\mathcal{F}_n} V_n = 0,$$

and there exists a constant C such that $C > |E_{\mathcal{F}_n} V_n^2 - \Sigma| \rightarrow 0.$

Suppose, with $\sigma_{j,r}^2 = E I[V_j^2 \geq rj^\alpha] V_j^2,$ that

$$\lim_{j \rightarrow \infty} \sigma_{j,r}^2 = 0 \text{ for all } r$$

or

$$\alpha = 1 \text{ and } \lim n^{-1} \sum_{j=1}^n \sigma_{j,r}^2 = 0 \text{ for all } r.$$

Let $\beta_+ = \beta I[\alpha = 1]$ where $0 < \alpha \leq 1, \quad 0 \leq \beta, \quad \beta_+ < 2\Gamma,$ and

$$U_{n+1} = U_n [1 - n^{-\alpha} \Gamma_n] + n^{-(\alpha+\beta)/2} \Phi_n V_n + n^{-\alpha-\beta/2} T_n.$$

Then,
$$n^{\beta/2} U_n \xrightarrow{D} N\left(\frac{T}{(\Gamma - \beta_+/2)}, \frac{\Sigma \Phi^2}{(2\Gamma - \beta_+)} \right).$$

Lemma 2.3: Assume A1, A2', A3, and A4 hold.

- (a) If A5(b) holds, then $\Delta_n = o(c_n).$
- (b) If A5(c) or A5'(a) holds, then $\Delta_n = O(c_n^2).$
- (c) If A5'(b) holds, then

$$\lim_{n \rightarrow \infty} \frac{\Delta_n}{c_n^2} = C_1 g(\phi) \text{Hog}^{(3)}(\phi) / 6.$$

Proof: If A5(b) holds, then from the proof of Lemma 2.1 and since A2' holds,

$$\begin{aligned} |\Delta_n| &\leq \frac{1}{4} C_1 g(\phi_n) |\text{Hog}^{(2)}(\eta_1) - \text{Hog}^{(2)}(\eta_2)| c_n \\ &\leq K_1 |\text{Hog}^{(2)}(\eta_1) - \text{Hog}^{(2)}(\eta_2)| c_n \end{aligned}$$

with $|\eta_i - \phi_n| \leq c_n, i = 1, 2.$ Since $|\phi_n - \phi| \rightarrow 0$ a. s. by Theorem 2.1 and $\text{Hog}^{(2)}$ is continuous in a

neighborhood of $\phi,$ part (a) holds.

It was seen while proving Lemma 2.1 that if A5(c) holds, then

$$|\Delta_n| \leq K_2 g(\phi_n) c_n^2 = O(c_n^2)$$

since $g(\phi_n)$ is bounded by A2'.

Also from the proof of Lemma 2.1, when A5'(a) holds

$$\Delta_n = \frac{1}{i^2} C_1 g(\phi_n) [H \circ g^{(3)}(\eta_1) + H \circ g^{(3)}(\eta_2)] c_n^2.$$

With $g(\phi_n)$ and $H \circ g^{(3)}$ bounded, part (b) follows. Since $\phi_n \rightarrow \phi$ a. s. , when $H \circ g^{(3)}$ is continuous in a neighborhood of ϕ , we also have (c).

Lemma 2.4: Assume A1, A2', and A3 . Then for $t > 0$

$$E_{\mathcal{F}_n} |V_n|^t \leq c_n^{t/2} [K_1 + K_2 E_{\mathcal{F}_n} |hg_n(\phi_n)|^t + K_3 | E_{\mathcal{F}_n} [hg(\phi_n)] |^t].$$

Proof: By definition

$$V_n = c_n^{1/2} \{ C_1 g(\phi_n) [hg_n(\phi_n) - E_{\mathcal{F}_n} hg_n(\phi_n)] - C_1 g^{(1)}(\phi_n) [N_n(g(\phi_n)) + H(g(\phi_n))] \}.$$

Since for any real numbers a, b, c, and d one can write $|a+b+c+d|^t \leq 4^t [|a|^t + |b|^t + |c|^t + |d|^t]$,

$$|V_n|^t \leq c_n^{t/2} 4^t \{ |g(\phi_n) hg_n(\phi_n)|^t + |g(\phi_n) E_{\mathcal{F}_n} hg_n(\phi_n)|^t + |g^{(1)}(\phi_n) N_n(g(\phi_n))|^t + |g^{(1)}(\phi_n) H(g(\phi_n))|^t \}.$$

Since F has increasing failure rate, by (2.7)

$$\begin{aligned} E_{\mathcal{F}_n} |N_n(g(\phi_n))|^t &\leq E_{\mathcal{F}_n} |N_n(g(\phi_n))|^{[t+1]} \\ &\leq e^{-[t+1]} \sum_{j=0}^{\infty} \frac{j^{[t+1]}}{j!} \left[\frac{g(\phi_n)}{\mu} \right]^j \end{aligned}$$

$$\leq e^{-[t+1]} \sum_{j=0}^{\infty} \frac{j^{[t+1]}}{j!} [T_2/\mu]^j < \infty,$$

where $[t+1]$ is the greatest integer less than or equal to $t+1$. Thus using A3 and the assumption

that $T_2 < \infty$,

$$E_{\mathcal{F}_n} |V_n|^t \leq c_n^{t/2} [K_2 E_{\mathcal{F}_n} |hg_n(\phi_n)|^t + K_3 | E_{\mathcal{F}_n} [hg(\phi_n)] |^t + K_4 + K_5],$$

which establishes the result.

Lemma 2.5: Assume A1, A2', A3, and A4 hold. Further suppose that one of the assumptions A5(b), A5(c), A5'(a), or A5'(b) holds. Then

(a) $E_{\mathcal{F}_n} V_n = 0$ and

$$(b) \lim_{n \rightarrow \infty} E_{\mathcal{F}_n} V_n^2 = \Sigma = \frac{C_1^2 [g(\phi)]^2 H \circ g^{(1)}(\phi)}{2}.$$

Proof: For part (a) we note that $E_{\mathcal{F}_n} V_n = 0$ by definition.

Next, let

$$X_n = g(\phi_n) \left[\frac{N_n(g(\phi_n + c_n)) - N_n(g(\phi_n - c_n))}{2c_n} \right],$$

$$Y_n = g(\phi_n) \left[\frac{H \circ g(\phi_n + c_n) - H \circ g(\phi_n - c_n)}{2c_n} \right], \text{ and}$$

$$Z_n = g^{(1)}(\phi_n) [N_n(g(\phi_n)) - H \circ g(\phi_n)],$$

so that $V_n^2 = c_n C_1^2 [X_n - Y_n - Z_n]^2$. To show (b), we consider separately the conditional

expectations $E_{\mathcal{F}_n} X_n^2$, $E_{\mathcal{F}_n} X_n Y_n$, etc.

(i) By (2.5), Theorem 2.1, and the continuity of $H \circ g^{(1)}$ in a neighborhood of ϕ ,

$$\begin{aligned}
 c_n E_{\mathcal{F}_n} X_n^2 &= c_n [g(\phi_n)]^2 \frac{1}{4c_n^2} [H \circ g^{(1)}(x) 2c_n + o(c_n)] \\
 &= [g(\phi_n)]^2 \frac{1}{4c_n} [H \circ g^{(1)}(x) 2c_n + o(c_n)] \\
 &\rightarrow \frac{C_1^2 [g(\phi)]^2 H \circ g^{(1)}(\phi)}{2} \text{ as } n \rightarrow \infty.
 \end{aligned}$$

(ii) Using (2.5) and the fact that $T_2 < \infty$,

$$\begin{aligned}
 c_n E_{\mathcal{F}_n} X_n Y_n &= c_n [g(\phi_n)]^2 \left[\frac{H \circ g(\phi_n + c_n) - H \circ g(\phi_n - c_n)}{2c_n} \right] E_{\mathcal{F}_n} \left[\frac{N_n(g(\phi_n + c_n)) - N_n(g(\phi_n - c_n))}{2c_n} \right] \\
 &\leq c_n K_1 \left[\frac{2H \circ g^{(1)}(\phi_n) c_n + o(c_n)}{2c_n} \right]^2 \\
 &\leq c_n K_1 [K_2 + o(1)] \rightarrow 0 \text{ as } n \rightarrow \infty.
 \end{aligned}$$

(iii) By the conditional version of the Cauchy-Schwartz inequality,

$$\begin{aligned}
 c_n E_{\mathcal{F}_n} |X_n Z_n| &= g(\phi_n) g^{(1)}(\phi_n) \frac{1}{2} E_{\mathcal{F}_n} [N_n(g(\phi_n + c_n)) - N_n(g(\phi_n - c_n))] |N_n(g(\phi_n)) - H \circ g(\phi_n)| \\
 &\leq K_1 \{ E_{\mathcal{F}_n} [N_n(g(\phi_n + c_n)) - N_n(g(\phi_n - c_n))]^2 \}^{1/2} \{ E_{\mathcal{F}_n} [N_n(g(\phi_n)) - H \circ g(\phi_n)]^2 \}^{1/2}.
 \end{aligned}$$

Thus by (2.5), (2.6), and the fact that $T_2 < \infty$,

$$\begin{aligned}
 c_n E_{\mathcal{F}_n} |X_n Z_n| &\leq K_1 \{ 2H \circ g^{(1)}(\phi_n) c_n + o(c_n) \}^{1/2} \left[\frac{g(\phi_n)}{\mu} \right]^{1/2} \\
 &\leq K_2 o(1) \text{ as } n \rightarrow \infty.
 \end{aligned}$$

Thus $c_n E_{\mathcal{F}_n} |X_n Z_n| \rightarrow 0$ as $n \rightarrow \infty$.

(iv) Since $T_2 < \infty$,

$$\begin{aligned}
 c_n E_{\mathcal{F}_n} Y_n^2 &= [g(\phi_n)]^2 \frac{1}{4c_n} [H \circ g(\phi_n + c_n) - H \circ g(\phi_n - c_n)]^2 \\
 &\leq K_1 \frac{1}{c_n} [2H \circ g^{(1)}(\phi_n) c_n + o(c_n)]^2 \rightarrow 0 \text{ as } n \rightarrow \infty.
 \end{aligned}$$

(v) For all n , $E_{\mathcal{F}_n} Y_n Z_n = 0$.

(vi) It follows from the fact that $T_2 < \infty$ and (2.6) that

$$\begin{aligned} E_{\mathcal{F}_n} Z_n^2 &= [g^{(1)}(\phi_n)]^2 E_{\mathcal{F}_n} [N_n(g(\phi_n)) - H \circ g(\phi_n)]^2 \\ &\leq K_1 \left[\frac{g(\phi_n)}{\mu} \right] \\ &\leq K_2. \end{aligned}$$

Thus $\lim_{n \rightarrow \infty} c_n E_{\mathcal{F}_n} Z_n^2 = 0$.

The lemma follows immediately from (i)-(vi) since by definition

$$E_{\mathcal{F}_n} V_n^2 = c_n C_1^2 E_{\mathcal{F}_n} [X_n - Y_n - Z_n]^2.$$

Lemma 2.6: Assume A1, A2', and A3 hold. Then

(a) $E_{\mathcal{F}_n} V_n^2$ is bounded for each n , and

(b) for any $t > 0$, $\lim_{n \rightarrow \infty} E_{\mathcal{F}_n} (c_n^{1/2} V_n)^t = 0$.

Proof: Since the conditions of Lemma 2.4 hold,

$$E_{\mathcal{F}_n} V_n^2 \leq c_n [K_1 + K_2 E_{\mathcal{F}_n} |hg_n(\phi_n)|^2 + K_3 |E_{\mathcal{F}_n} [hg(\phi_n)]|^2].$$

By A1 and (2.5)

$$c_n E_{\mathcal{F}_n} |hg_n(\phi_n)|^2 = \frac{1}{4c_n} [2H \circ g^{(1)}(\phi_n) c_n + o(c_n)] \leq K_1.$$

Again by A1 and (2.5),

$$c_n |E_{\mathcal{F}_n} [hg_n(\phi_n)]|^2 = \frac{1}{4c_n} [2H \circ g^{(1)}(\phi_n) c_n + o(c_n)]^2 \leq K_2.$$

This establishes (a).

By the definition of hg_n and (2.5) we have that

$$\begin{aligned} c_n^t E_{\mathcal{F}_n} |hg_n(\phi_n)|^t &\leq \frac{1}{2^t} E_{\mathcal{F}_n} [N_n(g(\phi_n+c_n)) - N_n(g(\phi_n-c_n))]^{[t+1]} \\ &= \frac{1}{2^t} [2H \circ g^{(1)}(\phi_n)c_n + o(c_n)] . \end{aligned}$$

Thus $\lim_{n \rightarrow \infty} c_n^t E_{\mathcal{F}_n} |hg_n(\phi_n)|^t = 0$.

Also by (2.5),

$$c_n^t | E_{\mathcal{F}_n} [hg(\phi_n)] |^t = \frac{1}{2^t} [2H \circ g^{(1)}(\phi_n)c_n + o(c_n)]^t \rightarrow 0 .$$

Thus using Lemma 2.4,

$$E_{\mathcal{F}_n} |c_n^{t/2} V_n|^t \leq c_n^t [K_2 E_{\mathcal{F}_n} |hg_n(\phi_n)|^t + K_3 | E_{\mathcal{F}_n} [hg(\phi_n)] |^t + K_4 + K_5] \rightarrow 0 ,$$

which establishes (b).

Lemma 2.7: Let $\sigma_{n,r}^2 = E\{ V_n^2 I[V_n^2 \geq rn] \}$ for $r = 1, 2, \dots$ and $n = 1, 2, \dots$. Assume Lemma 2.6

and A6 hold. Then for each r , $\lim_{n \rightarrow \infty} \sigma_{n,r}^2 = 0$.

Proof: For the $\gamma \in (0, 1)$ of assumption A6, take $q > 1$ such that $\gamma \leq \frac{1}{q}$. Then take p

such that $\frac{2}{p} + \frac{1}{q} = 1$. The proof is then identical to that of Lemma 2.7 of Frees(1983), using the fact

that by Lemma 2.6, $E(c_n^{1/2} V_n)^p = o(1)$.

Proof of Theorem 2.2: We apply the result due to Fabian(1971) in the form quoted earlier.

By the definition of V_n

$$c_n^{-1/2} V_n = Dg_n(\phi_n) - Dg(\phi_n) - \Delta_n .$$

Expanding $Dg(\phi_n)$ about $Dg(\phi)=0$ we have that

$$Dg(\phi_n) = (\phi_n - \phi) \frac{d}{dt} Dg(t)|_{t=\eta_n},$$

where $|\eta_n - \phi| \leq |\phi_n - \phi|$. As before let $U_n = \phi_n - \phi$. Then

$$\begin{aligned} U_{n+1} &= U_n - a_n [c_n^{-1/2} V_n + Dg(\phi_n) + \Delta_n] \\ &= U_n [1 - a_n \frac{d}{dt} Dg(t)|_{t=\eta_n}] - A n^{-1} C^{-1/2} n^{\gamma/2} V_n - a_n \Delta_n . \end{aligned}$$

If we set $\Gamma_n = A \frac{d}{dt} Dg(t)|_{t=\eta_n}$, $\Phi_n = \Phi = -AC^{-1/2}$, $T_n = -An^{(1-\gamma)/2} \Delta_n$, $\alpha=1$, and

$\beta=1-\gamma$, then

$$\begin{aligned} U_{n+1} &= U_n [1 - n^{-1} \Gamma_n] + n^{(\frac{\gamma}{2}-1)} \Phi_n V_n + n^{-1} n^{-(1-\gamma)/2} T_n \\ &= U_n [1 - n^{-\alpha} \Gamma_n] + n^{-(\alpha+\beta)/2} \Phi_n V_n + n^{-\alpha-(\beta/2)} T_n . \end{aligned}$$

Let $\Gamma = A \frac{d}{dt} Dg(t)|_{t=\phi}$. Since $\frac{d}{dt} Dg(t)$ is continuous at ϕ and $\phi_n \rightarrow \phi$ a. s. by Theorem 2.1, we have

that $\Gamma_n \rightarrow \Gamma$ as $n \rightarrow \infty$. With $\Sigma = \frac{C_1^2 [g(\phi)]^2 H \circ g^{(1)}(\phi)}{2}$, by Lemmas 2.5 and 2.6 there exists a

constant K such that $|E_{\mathcal{F}_n} V_n^2 - \Sigma| < K$ for all n and $E_{\mathcal{F}_n} V_n^2 \rightarrow \Sigma$ as $n \rightarrow \infty$. Using Lemma 2.7 we

also have that $\lim_{j \rightarrow \infty} \sigma_{j,r}^2 = \lim_{j \rightarrow \infty} E\{ V_j^2 I[V_j^2 \geq rj^\alpha] \} = 0$ for all r . By assuming A6 we have required

that $1-\gamma < 2\Gamma$.

It remains to establish the asymptotic behaviour of T_n . If A5(b) holds and $\gamma \geq \frac{1}{3}$, then by Lemma 2.3

$$|T_n| \leq K_1 n^{(1-\gamma)/2} o(n^{-\gamma}) = K_1 n^{(1-3\gamma)/2} o(1) \rightarrow 0 .$$

Under A5(c) or A5'(a), with $\gamma > \frac{1}{5}$,

$$|T_n| \leq K_1 n^{(1-\gamma)/2} O(n^{-2\gamma}) = K_2 n^{(1-5\gamma)/2} \rightarrow 0 .$$

If A5'(b) holds and $\gamma = \frac{1}{5}$, then

$$T_n = - \frac{A c_n^2 n^{(1-\gamma)/2} \Delta_n}{c_n^2} = - \frac{A C^2 \Delta_n}{c_n^2} \rightarrow - \frac{A C^2 C_1 g(\phi) H o g^{(3)}(\phi)}{6}$$

by Lemma 2.3 . The limiting normal distributions indicated then follow from the Fabian result.

3. IMPROVING THE RATE OF CONVERGENCE

This section is principally concerned with achieving a faster rate of convergence for the procedure (2.2) by using a kernel estimator for $H o g^{(1)}$. Kernel estimation of $H o g^{(p+1)}$ for integer p greater than zero is also considered here for future applications. The basic approach is that taken by Singh (1977) for kernel estimation of derivatives of a distribution function. Frees (1983) used estimators like those suggested by Singh to speed the rate of convergence of a stochastic approximation procedure. The development undertaken here is like that of Frees except that interest now lies in estimating derivatives of $H o g$.

Let $0 \leq p < r$ where p and r are integers. We will consider the problem of estimating $H o g^{(p+1)}(x)$ assuming that $H o g^{(r+1)}(x)$ exists and is bounded. Let $h g_n$ continue to denote an estimator of $H o g^{(1)}$. For $p \geq 0$, $h g_n^{(p)}$ will denote an estimator of $H o g^{(p+1)}$. It is desirable to have estimators $h g_n$ and $h g_n^{(p)}(x)$ which satisfies the following conditions:

$$A7:(a) \sup_x | E h g_n^{(p)}(x) - H o g^{(p+1)}(x) | = O(n^{-(r-p)/(2r+1)}).$$

$$(b) \text{ For } t > -1, \sup_x E | h g_n(x) |^{t+1} = O(n^{+t/(2r+1)}).$$

A8: For any integer $t \geq 0$ and sequence $\{x_n\}$ such that $\lim_{n \rightarrow \infty} x_n = x_0$, there exist constants Φ_1 and $\Phi_{2,t}$ such

that

$$(a) \lim_{n \rightarrow \infty} n^{(r-p)/(2r+1)} [E h g_n^{(p)}(x_n) - H o g^{(p+1)}(x_n)] = \Phi_1 \text{ and}$$

$$(b) \lim_{n \rightarrow \infty} n^{-t/(2r+1)} E[h_{g_n}(x_n)]^{t+1} = \Phi_{2,t}.$$

It will be seen that if an estimator $h_{g_n}(x)$ of $H \circ g^{(1)}(x)$ such that A7 and A8 hold is used in the SA algorithm considered in Section 2, then ϕ_n will converge to ϕ at a faster rate. As a first step a kernel estimator of $H \circ g^{(p+1)}(x)$ is defined which satisfies A7 and A8.

Denote by B the class of all Borel-measurable bounded functions K that vanish outside the interval $(-1, 1)$. Take

$$\mathcal{K}_p = \left\{ K \in B \text{ such that } \frac{1}{j!} \int_{-1}^{+1} u^j K(u) du = \begin{array}{l} 1, \text{ if } j=p, \\ 0, \text{ if } j \neq p, j=0, 1, \dots, r-1 \end{array} \right\}.$$

As in Section 2, let N_n , $n = 1, 2, \dots$, be independent renewal processes, each having underlying distribution F with renewals at times $S_{ni} = X_{n1} + X_{n2} + \dots + X_{ni}$, $i=1, 2, \dots$. For $K \in \mathcal{K}_p$ define

$$(3.1) \quad \begin{aligned} h_{g_n}^{(p)}(x) &= \frac{1}{c_n^{p+1}} \int_0^\infty K\left(\frac{g^{-1}(u) - x}{c_n}\right) dN_n(u) . \\ &= \frac{1}{c_n^{p+1}} \sum_{i=1}^\infty K\left(\frac{g^{-1}(S_{ni}) - x}{c_n}\right) \end{aligned}$$

where $\{ c_n \}$ is a sequence satisfying A4 or A6.

The next several results are analogous to those of Singh(1977) and Frees(1983) concerning estimation of the derivatives of a distribution function. The lemmas will provide sufficient conditions for A7 and A8 to hold for the estimator $h_{g_n}^{(p)}(x)$. Part (a) of Lemma 3.1 corresponds to Theorem 3.1 of Singh, while part (b) is similar in spirit to Lemma 3.1 (b) of Frees.

Lemma 3.1: Let $K \in \mathcal{K}_p$ and $h_{g_n}^{(p)}(x)$ be defined by (3.1). Suppose $H \circ g^{(r+1)}$ is bounded. Then

$$(a) \sup_x | E h_{g_n}^{(p)}(x) - H \circ g^{(p+1)}(x) | = O(c_n^{r-p}).$$

(b) Also suppose that $\{x_n\}$ is a sequence such that $\lim_{n \rightarrow \infty} x_n = x_0$, where $\text{Hog}^{(r+1)}(x)$ is continuous in a neighborhood of x_0 . Then

$$\lim_{n \rightarrow \infty} c_n^{-(r-p)} [E h g_n^{(p)}(x_n) - \text{Hog}^{(p+1)}(x_n)] = \text{Hog}^{(r+1)}(x_0) \frac{1}{r!} \int_{-1}^{+1} u^r K(u) du.$$

Proof: By the substitution $t = \frac{g^{-1}(u) - x}{c_n}$ and an application of bounded convergence we have that

$$\begin{aligned} E h g_n^{(p)}(x) &= \frac{1}{c_n^{p+1}} E \left\{ \int_0^\infty K \left(\frac{g^{-1}(u) - x}{c_n} \right) dN_n(u) \right\} \\ &= \frac{1}{c_n^{p+1}} \int_0^\infty K \left(\frac{g^{-1}(u) - x}{c_n} \right) h(u) du \\ &= \frac{1}{c_n^p} \int_{-1}^{+1} K(t) \text{hog}(x+c_nt) g^{(1)}(x+c_nt) dt \\ &= \frac{1}{c_n^p} \int_{-1}^{+1} K(t) \text{Hog}^{(1)}(x+c_nt) dt. \end{aligned}$$

Expanding $\text{Hog}^{(1)}(x+c_nt)$ in a Taylor series about x ,

$$\text{Hog}^{(1)}(x+c_nt) = \sum_{j=0}^{r-1} \frac{\text{Hog}^{(j+1)}(x)}{j!} (c_nt)^j + R_n(t)$$

where $R_n(t) = \frac{\text{Hog}^{(r+1)}(\eta(t))}{r!} (c_nt)^r$ with $|\eta(t)-x| \leq |c_nt|$. Thus

$$\begin{aligned} E h g_n^{(p)}(x) &= \frac{1}{c_n^p} \sum_{j=0}^{r-1} \frac{\text{Hog}^{(j+1)}(x)}{j!} c_n^j \int_{-1}^{+1} K(t) t^j dt \\ &\quad + \frac{1}{c_n^p} \int_{-1}^{+1} K(t) \frac{\text{Hog}^{(r+1)}(\eta(t))}{r!} (c_nt)^r dt \\ &= \text{Hog}^{(p+1)}(x) + O(c_n^{r-p}) \end{aligned}$$

since $K \in \mathbf{K}_p$ and $Hog^{(r+1)}$ is bounded. This establishes (a).

By part (a) we have that

$$\begin{aligned} & \lim_{n \rightarrow \infty} c_n^{-(r-p)} [E hg_n^{(p)}(x_n) - Hog^{(p+1)}(x_n)] \\ &= \lim_{n \rightarrow \infty} \int_{-1}^{+1} K(t) \frac{Hog^{(r+1)}(\eta(t))}{r!} t^r dt \\ &= Hog^{(r+1)}(x_0) \frac{1}{r!} \int_{-1}^{+1} K(t) t^r dt \text{ as } n \rightarrow \infty, \end{aligned}$$

by the bounded convergence theorem and the continuity of $Hog^{(r+1)}$ at x_0 , completing the proof of the lemma.

We note that under the assumptions of Lemma 3.1, if $c_n = Cn^{-\gamma}$, where $\gamma \geq \frac{1}{2r+1}$, then A7 (a) holds. If $\gamma = \frac{1}{2r+1}$, then by Lemma 3.1 (b) we have that

$$\begin{aligned} & \lim_{n \rightarrow \infty} n^{(r-p)/(2r+1)} (E[hg_n^{(p)}(x_n)] - Hog^{(p+1)}(x_n)) \\ &= \lim_{n \rightarrow \infty} C^{(r-p)} c_n^{-(r-p)} (E[hg_n^{(p)}(x_n)] - Hog^{(p+1)}(x_n)) \\ &= C^{(r-p)} Hog^{(r+1)}(x_0) \frac{1}{r!} \int_{-1}^{+1} K(t) t^r dt \end{aligned}$$

so that A8 (a) holds with $\Phi_1 = C^{(r-p)} Hog^{(r+1)}(x_0) \frac{1}{r!} \int_{-1}^{+1} K(t) t^r dt$.

The following lemma establishes results for $hg_n^{(p)}(x)$ similar to those contained in Lemma 3.2 of Frees(1983).

Lemma 3.2: Let $K \in \mathbf{K}_0$ and $hg_n(x)$ be defined by (3.1) with $p = 0$. Assume $Hog^{(1)}$ is a bounded function that is continuous at x_0 and $\{ x_n \}$ is a sequence such that $\lim_{n \rightarrow \infty} x_n = x_0$. Further assume that $c_n = o(1)$.

(a) If $t > -1$, then $\sup_x E |hg_n(x)|^{t+1} = O(c_n^{-t})$.

(b) If $t \geq 0$ is an integer, then $\lim_{n \rightarrow \infty} c_n^t E [hg_n(x_n)]^{t+1} = H \circ g^{(1)}(x_0) \int_{-1}^1 [K(u)]^{t+1} du$.

Proof: By the definition of hg_n and the boundedness of K ,

$$\begin{aligned} c_n^t E |hg_n(x)|^{t+1} &= \frac{1}{c_n} E \left| \int_0^\infty K \left(\frac{g^{-1}(u) - x}{c_n} \right) dN_n(u) \right|^{t+1} \\ &\leq \|K\|_\infty^{t+1} \frac{1}{c_n} E [N_n(g(x+c_n)) - N_n(g(x-c_n))]^{t+1} \\ &\leq K_1 \frac{1}{c_n} E [N_n(g(x+c_n)) - N_n(g(x-c_n))]^{t+1} \\ &\leq K_2 \end{aligned}$$

by (2.5) and the boundedness of h . This establishes (a).

To determine a limit for $c_n^t E [hg_n(x_n)]^{t+1}$ we condition on the number of renewals occurring in the interval $[g(x_n-c_n), g(x_n+c_n)]$. Let $Y_n = N_n(g(x_n+c_n)) - N_n(g(x_n-c_n))$. By (2.4) and the boundedness of K ,

$$\begin{aligned} &\sum_{j=2}^\infty E \{ c_n^t [hg_n(x_n)]^{t+1} \mid Y_n = j \} P\{Y_n = j\} \\ &= \sum_{j=2}^\infty E \left\{ \frac{1}{c_n} \left[\int_0^\infty K \left(\frac{g^{-1}(u) - x_n}{c_n} \right) dN_n(u) \right]^{t+1} \mid Y_n = j \right\} P\{Y_n = j\} \\ &\leq \frac{1}{c_n} \sum_{j=2}^\infty \|K\|_\infty^{t+1} j^{t+1} P\{Y_n = j\} \\ &\rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

The following result will be used to find the conditional expectation when $Y_n = 1$. Let X_1, X_2, \dots be i.i.d. random variables with distribution function F which define a renewal process with renewals occurring at $S_n = X_1 + \dots + X_n$, $n = 1, 2, \dots$. Define the excess random variable at time t to be

$$\gamma(t) = S_{N(t)+1} - t,$$

so that $\gamma(t)$ is the remaining life of the part in use at time t . This random variable has distribution function

$$P\{ \gamma(t) \leq x \} = \int_0^t [F(t-u+x) - F(t-u)] dH(u) + [F(t+x) - F(t)] .$$

(See Barlow and Proschan(1965).) Thus the distribution function and the density of the first renewal to occur after time $g(x_n - c_n)$ are given by

$$P\{ S_{N(g(x_n - c_n))+1} \leq v \} = \int_0^{g(x_n - c_n)} [F(v-u) - F(g(x_n - c_n)-u)] dH(u) + [F(v) - F(g(x_n - c_n))]$$

and

$$a(v) = h(v) - \int_{g(x_n - c_n)}^v f(v-u) dH(u) ,$$

respectively, for $v \geq g(x_n - c_n)$. Thus we have that

$$P\{ Y_n = 1 \} = \int_{g(x_n - c_n)}^{g(x_n + c_n)} a(v) [1 - F(g(x_n + c_n) - v)] dv$$

and given that $Y_n = 1$, the conditional density of the lone renewal in $[g(x_n - c_n), g(x_n + c_n)]$ is

$$b(v) = \frac{a(v) [1 - F(g(x_n + c_n) - v)]}{P\{ Y_n = 1 \}} .$$

Hence, by a change of variables,

$$\begin{aligned} & E\{ c_n^t [h g_n(x_n)]^{t+1} \mid Y_n = 1 \} P\{ Y_n = 1 \} \\ &= E\left\{ \frac{1}{c_n} \int_{g(x_n - c_n)}^{g(x_n + c_n)} \left[K\left(\frac{g^{-1}(u) - x_n}{c_n} \right) \right]^{t+1} dN_n(u) \mid Y_n = 1 \right\} P\{ Y_n = 1 \} \\ &= \frac{1}{c_n} \int_{g(x_n - c_n)}^{g(x_n + c_n)} \left[K\left(\frac{g^{-1}(u) - x_n}{c_n} \right) \right]^{t+1} b(u) du P\{ Y_n = 1 \} \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{c_n} \int_{g(x_n - c_n)}^{g(x_n + c_n)} \left[K \left(\frac{g^{-1}(u) - x_n}{c_n} \right) \right]^{t+1} a(u) [1 - F(g(x_n + c_n) - u)] du \\
 &= \int_{-1}^{+1} [K(v)]^{t+1} a(g(x_n + c_n v)) [1 - F(g(x_n + c_n) - g(x_n + c_n v))] g^{(1)}(x_n + c_n v) dv
 \end{aligned}$$

where

$$a(g(x_n + c_n v)) = h \circ g(x_n + c_n v) - \int_{g(x_n - c_n)}^{g(x_n + c_n v)} f(g(x_n + c_n v) - u) dH(u) .$$

Since K and f are bounded,

$$\int_{-1}^{+1} [K(v)]^{t+1} \int_{g(x_n - c_n)}^{g(x_n + c_n v)} f(g(x_n + c_n v) - u) dH(u) [1 - F(g(x_n + c_n) - g(x_n + c_n v))] g^{(1)}(x_n + c_n v) dv$$

$$\leq 2 \|K\|_{\infty}^{t+1} \|f\|_{\infty} \|g^{(1)}\|_{\infty} \int_{g(x_n - c_n)}^{g(x_n + c_n)} dH(u) \rightarrow 0$$

by the continuity of F . Similarly,

$$\lim_{n \rightarrow \infty} \int_{-1}^{+1} [K(v)]^{t+1} h \circ g(x_n + c_n v) F(g(x_n + c_n) - g(x_n + c_n v)) g^{(1)}(x_n + c_n v) dv = 0 .$$

Thus,

$$\begin{aligned}
 &\lim_{n \rightarrow \infty} E\{ c_n^t [h g_n(x_n)]^{t+1} \mid Y_n = 1 \} P\{ Y_n = 1 \} \\
 &= \lim_{n \rightarrow \infty} \int_{-1}^{+1} [K(v)]^{t+1} h \circ g(x_n + c_n v) g^{(1)}(x_n + c_n v) dv
 \end{aligned}$$

$$= H \circ g^{(1)}(x_0) \int_{-1}^{+1} [K(v)]^{t+1} dv ,$$

which completes the proof of the lemma.

Suppose $c_n = Cn^{-\gamma}$ where $\gamma = \frac{1}{2r+1}$. Then the conditions of Lemma 3.2 (a) imply that A7 (b) holds for the kernel estimator hg_n . Similarly, hg_n satisfies A8(b) by Lemma 3.2 (b).

Let the sequence $\{\phi_n\}$ of estimates of ϕ be defined recursively by (2.2), where in the definition (2.3) the estimator $hg_n(x)$ is redefined to be any estimator of $H \circ g^{(1)}(x)$. The following theorem establishes some convergence properties for ϕ_n assuming that $hg_n(x)$ satisfies A7 and A8. The subsequent corollary addresses the special case when $hg_n(x)$ is of the form (3.1) with $p = 0$.

Theorem 3.1: Let $\gamma = \frac{1}{2r+1}$ and $\Gamma = A \frac{d}{dt} Dg(t)|_{t=\phi}$.

(a) Assume A1-A4 and A7 hold. Then $\phi_n \rightarrow \phi$ a.s.

(b) Assume A1, A2', A3, A6, A7, and A8 hold. Then $n^{(1-\gamma)/2} (\phi_n - \phi) \xrightarrow{D} N(\mu_0, \sigma_0^2)$ where

$$\mu_0 = - \frac{C_1 A g(\phi) \Phi_1}{\Gamma - (1-\gamma)/2}$$

and

$$\sigma_0^2 = \frac{C_1^2 A^2 [g(\phi)]^2 \Phi_{2,1}}{2\Gamma - (1-\gamma)}.$$

Proof of Theorem 3.1 (a): Again take $U_n = \phi_n - \phi$. By A7 (a) with $p=0$,

$$\sup_x |E[hg_n(x)] - H \circ g^{(1)}(x)| = O(n^{-r/(2r+1)}).$$

This implies that

$$\begin{aligned} |\Delta_n| &= |E_{\mathcal{F}_n}[Dg_n(\phi_n) - Dg(\phi_n)]| \\ &= C_1 g(\phi_n) |E_{\mathcal{F}_n} hg_n(\phi_n) - H \circ g^{(1)}(\phi_n)| \end{aligned}$$

$$\begin{aligned} &\leq K_1 g(\phi_n) n^{-r/(2r+1)} \\ &\leq [K_2 + K_3 |\phi_n - \phi|] n^{-\gamma r}. \end{aligned}$$

By A7 (b) with $t=1$

$$E_{\mathcal{F}_n} |hg_n(\phi_n)|^2 = O(n^{+1/(2r+1)}) = O(n^{+\gamma}),$$

so that by the proof of Lemma 2.2,

$$\begin{aligned} E_{\mathcal{F}_n} [Dg_n(\phi_n)]^2 &\leq K_1 [g(\phi_n)]^2 E_{\mathcal{F}_n} |hg_n(\phi_n)|^2 + [K_2 + K_3 |\phi_n - \phi|^2] \\ &\leq [K_4 + K_5 |\phi_n - \phi|^2] [n^{+\gamma} + 1] \\ &\leq [K_6 + K_7 |\phi_n - \phi|^2] n^{+\gamma}. \end{aligned}$$

Thus from the proof of Theorem 2.1

$$\begin{aligned} E_{\mathcal{F}_n} U_{n+1}^2 &\leq U_n^2 + a_n^2 E_{\mathcal{F}_n} [Dg_n^2(\phi_n)] + 2a_n |U_n| |\Delta_n| - 2a_n (\phi_n - \phi) Dg(\phi_n) \\ &\leq U_n^2 + a_n^2 n^{+\gamma} [K_6 + K_7 |\phi_n - \phi|^2] + 2a_n [K_8 + K_9 |\phi_n - \phi|^2] n^{-\gamma r} - 2a_n (\phi_n - \phi) Dg(\phi_n) \\ &\leq U_n^2 \{1 + K_7 a_n^2 n^{+\gamma} + 2K_9 a_n n^{-\gamma r}\} + \{K_6 a_n^2 n^{+\gamma} + 2K_8 a_n n^{-\gamma r}\} - 2a_n (\phi_n - \phi) Dg(\phi_n). \end{aligned}$$

Since the required summations are finite, $\phi_n \rightarrow \phi$ a. s. by an application of the Robbins-Siegmund result.

The proof of Theorem 3.1 (b) will require the following lemma, similar to Lemma 2.5 (b). As in Section 2, let $V_n = c_n^{1/2} [Dg_n(\phi_n) - Dg(\phi_n) - \Delta_n]$, where $c_n = Cn^{-1/(2r+1)}$ by A6.

Lemma 3.3: Assume A1, A2', A3, A6, A7 and A8 hold. Then

$$\lim_{n \rightarrow \infty} E_{\mathcal{F}_n} V_n^2 = C_1^2 C [g(\phi)]^2 \Phi_{2,1}.$$

Proof: By definition $V_n^2 = c_n C_1^2 [X_n - Y_n - Z_n]^2$ where

$$X_n = g(\phi_n) [hg_n(\phi_n)],$$

$$Y_n = g(\phi_n) E_{\mathcal{F}_n} [hg_n(\phi_n)], \text{ and}$$

$$Z_n = g^{(1)}(\phi_n) [N_n(g(\phi_n)) - H \circ g(\phi_n)].$$

As in Lemma 2.5, we consider separately the conditional expectations $E_{\mathcal{F}_n} X_n^2$, $E_{\mathcal{F}_n} X_n Y_n$, etc. Since the conditions of Theorem 3.1 (a) hold, $\phi_n \rightarrow \phi$ a.s.

(i) Since $\phi_n \rightarrow \phi$ a. s., by A8 (b) with $t=1$

$$\begin{aligned} c_n E_{\mathcal{F}_n} X_n^2 &= C [g(\phi_n)]^2 n^{-1/(2r+1)} E_{\mathcal{F}_n} [hg_n(\phi_n)]^2 \\ &\rightarrow C [g(\phi)]^2 \Phi_{2,1}. \end{aligned}$$

(ii) By A2' and A7 (b) with $t=0$

$$\begin{aligned} c_n E_{\mathcal{F}_n} X_n Y_n &= [g(\phi_n)]^2 c_n [E_{\mathcal{F}_n} hg_n(\phi_n)]^2 \\ &\leq K_1 c_n O(1) \rightarrow 0. \end{aligned}$$

(iii) By (2.6), A2', A7 (b) with $t=1$, and the conditional version of the Cauchy-Schwartz inequality,

$$\begin{aligned} c_n E_{\mathcal{F}_n} |X_n Z_n| &\leq c_n g(\phi_n) g^{(1)}(\phi_n) E_{\mathcal{F}_n} [hg_n(\phi_n)] |N_n(g(\phi_n)) - H \circ g(\phi_n)| \\ &\leq c_n K_1 \{ E_{\mathcal{F}_n} [hg_n(\phi_n)]^2 \}^{1/2} \{ E_{\mathcal{F}_n} [N_n(g(\phi_n)) - H \circ g(\phi_n)]^2 \}^{1/2} \\ &\leq c_n K_2 \{ E_{\mathcal{F}_n} [hg_n(\phi_n)]^2 \}^{1/2} \\ &\leq c_n K_2 \{ O(n^{+1/(2r+1)}) \}^{1/2} \rightarrow 0. \end{aligned}$$

(iv) Also by A2' and A7 (b) (with $t=0$),

$$\begin{aligned} c_n E_{\mathcal{F}_n} Y_n^2 &= c_n [g(\phi_n)]^2 [E_{\mathcal{F}_n} h g_n(\phi_n)]^2 \\ &\leq c_n K_1 \rightarrow 0. \end{aligned}$$

(v) For all n , $E_{\mathcal{F}_n} Y_n Z_n = 0$.

(vi) By Lemma 2.5 (vi), $\lim_{n \rightarrow \infty} E_{\mathcal{F}_n} Z_n^2 = 0$.

The result follows from (i)-(vi).

Proof of Theorem 3.1 (b): Again the theorem on asymptotic normality due to Fabian is used. From the proof of Theorem 2.2 we have that

$$U_{n+1} = U_n [1 - n^{-1} \Gamma_n] + n^{(\frac{\gamma}{2}-1)} \Phi_n V_n + n^{-1} n^{-(1-\gamma)/2} T_n$$

where as before

$$\Gamma_n = A \frac{d}{dt} Dg(t)|_{t=\eta_n} \rightarrow \Gamma = A \frac{d}{dt} Dg(t)|_{t=\phi},$$

$$\Phi_n = \Phi = -AC^{-1/2}, \text{ and}$$

$$T_n = -An^{(1-\gamma)/2} \Delta_n.$$

Since $\phi_n \rightarrow \phi$ a. s., using A8 (a) with $p=0$ gives

$$\begin{aligned} T &= \lim_{n \rightarrow \infty} T_n \\ &= -AC_1 \lim_{n \rightarrow \infty} g(\phi_n) n^{(1-\gamma)/2} \left(E_{\mathcal{F}_n} [h g_n(\phi_n)] - H \circ g^{(1)}(\phi_n) \right) \\ &= -AC_1 \lim_{n \rightarrow \infty} g(\phi_n) n^{r/(2r+1)} \left(E_{\mathcal{F}_n} [h g_n(\phi_n)] - H \circ g^{(1)}(\phi_n) \right) \\ &= -AC_1 g(\phi) \Phi_1. \end{aligned}$$

By definition $E_{\mathcal{F}_n} V_n = 0$. From Lemma 2.4 we have that

$$E_{\mathcal{F}_n} V_n^2 \leq c_n \{ K_1 + K_2 E_{\mathcal{F}_n} [h g_n(\phi_n)]^2 + K_3 [E_{\mathcal{F}_n} h g_n(\phi_n)]^2 \},$$

which is bounded by A7 (b) . By Lemma 3.3,

$$\lim_{n \rightarrow \infty} E_{\mathcal{F}_n} V_n^2 = C_1^2 C [g(\phi)]^2 \Phi_{2,1} .$$

We need to show that $\sigma_{n,r}^2 = E\{ V_n^2 \mid [V_n^2 \geq rn] \} \rightarrow 0$ as $n \rightarrow \infty$ for $r = 1, 2, \dots$. To

accomplish this using Lemma 2.7 it is first shown that for any $p > 2$, $E_{\mathcal{F}_n} | c_n^{1/2} V_n |^p \rightarrow 0$. By

Lemma 2.4,

$$c_n^{p/2} E_{\mathcal{F}_n} |V_n|^p \leq c_n^p \left\{ K_1 + K_2 E_{\mathcal{F}_n} |hg_n(\phi_n)|^p + K_3 | E_{\mathcal{F}_n} [hg_n(\phi_n)] |^p \right\} .$$

By A7 (b),

$$\begin{aligned} c_n^p E_{\mathcal{F}_n} |hg_n(\phi_n)|^p &= c_n^p O(n^{-(p-1)/(2r+1)}) \\ &= O(n^{-1/(2r+1)}) \rightarrow 0 \end{aligned}$$

if $p > 2$. Now taking $t=0$ in A7 (b)

$$c_n^p | E_{\mathcal{F}_n} [hg(\phi_n)] |^p = c_n^p | O(1) | \rightarrow 0 \text{ as } n \rightarrow \infty .$$

For Lemma 2.7, $p, q > 0$ were such that $\frac{2}{p} + \frac{1}{q} = 1$. Thus, taking $p > 2$, Lemma 2.7 holds.

Taking $\alpha=1$ and $\beta=\beta_+ = 1-\gamma$, part (b) now follows from Fabian's theorem.

Now let $K \in \mathcal{K}_0$ and define hg_n by (3.1) with $p = 0$. Lemmas 3.1 and 3.2 apply to hg_n if the following assumption is met.

A9: For some integer $r \geq 1$, $Hog^{(1)}$ and $Hog^{(r+1)}$ exist and are bounded on the entire real line, and both functions are continuous in a neighborhood of ϕ .

Corollary 3.1: Assume A1-A3, A6, and A9 hold. Let $\gamma = \frac{1}{2r+1}$ and $\Gamma = A \frac{d}{dt} Dg(t) |_{t=\phi}$.

Then

(a) $\phi_n \rightarrow \phi$ a. s.

(b) If $A2'$ also holds, then $n^{(1-\gamma)/2} (\phi_n - \phi) \xrightarrow{D} N(\mu_1, \sigma_1^2)$ where

$$\mu_1 = - \frac{C_1 A g(\phi) C^r H \circ g^{(r+1)}(\phi) \beta_1}{\Gamma - (1-\gamma)/2}$$

and

$$\sigma_1^2 = \frac{C_1^2 A^2 C^{-1} [g(\phi)]^2 H \circ g^{(1)}(\phi) \beta_2}{2\Gamma - (1-\gamma)}$$

with $\beta_1 = \frac{1}{r!} \int_{-1}^{+1} u^r K(u) du$ and $\beta_2 = \int_{-1}^{+1} [K(u)]^2 du$.

Proof: By the discussions following Lemmas 3.1 and 3.2, $A9$ implies $A7$ and $A8$ hold for $hg_n(x)$. The corollary is then an immediate consequence of Theorem 3.1 with

$$\begin{aligned} \Phi_1 &= \lim_{n \rightarrow \infty} n^{r/(2r+1)} [E_{\mathcal{F}_n} hg_n(\phi_n) - H \circ g^{(1)}(\phi_n)] \\ &= C^r H \circ g^{(r+1)}(\phi) \frac{1}{r!} \int_{-1}^{+1} u^r K(u) du \end{aligned}$$

and

$$\begin{aligned} \Phi_{2,1} &= \lim_{n \rightarrow \infty} n^{-1/(2r+1)} E_{\mathcal{F}_n} [hg_n(\phi_n)]^2 \\ &= C^{-1} \lim_{n \rightarrow \infty} c_n E_{\mathcal{F}_n} [hg_n(\phi_n)]^2 \\ &= C^{-1} H \circ g^{(1)}(\phi) \int_{-1}^{+1} [K(u)]^2 du . \end{aligned}$$

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