WHITHER DELETE-K JACKKNIFING FOR SMOOTH STATISTICAL FUNCTIONALS?

by

Pranab Kumar Sen

Department of Biostatistics
University of North Carolina at Chapel Hill

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The classical jackknifing based on a resampling scheme with deletion of of one observation at a time serves the dual purpose of bias reduction and variance estimation. Delete-k jackknifing, for some $k \geq 1$, is a variant of this scheme. In the light of second order asymptotic distributional representations, it is shown that for second order (compact) differentiable statistical functionals, for any $k = o(n)$, delete-k jackknifing behaves very much similar to the classical one. This raises the question: To what degree delete-k jackknifing is preferable in practice ?

1. Introduction. Let $X_1,\ldots,X_n$ be n independent and identically distributed random variables (i.i.d.r.v.) with a distribution function (d.f.) F, defined on the real line R. Let F_n be the *sample (empirical) d.f.* For a general statistical functional $\theta = T(F)$, a natural estimator is $T_n = T(F_n)$. In general, T_n may not be unbiased for θ , and its mean square error may not be generally known too. The classical jackknifing serves a dual purpose of reducing the possible bias of T_n and providing an estimator of its mean square error. It consists in identifying the n subsamples of size n-1 each (by eliminating one observation at a time from the basic sample), and incorporating them in the formulation of the *pseudovariables* which in turn provide the jackknifed estimator and its mean square error estimator. This scheme has been extended to the so called *delete-k jackknifing* where $\binom{n}{k}$ subsamples of sizes n-k each are obtained by deleting k observations at a time from the basic sample of size n, and these $\binom{n}{k}$ subsample estimators are incorporated in the formulation of the pseudovariables on which the jackknifed estimator of θ and its estimated mean square error rest.

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Under quite general regularity conditions [viz., Parr(1983,1985) and Sen(1988)], for $T^*_{n,1}$, the delete-1 jackknifed version of T_n, as $n \to \infty$,

(1.1) $$(n-1)\{ T_n - T^*_{n,1} \} \to C(F) \text{ almost surely (a.s.)},$$

where C(F) is a suitable functional of F to be defined more precisely later on. Also, let $V^*_{n,1}$ be the usual (Tukey) jackknifed estimator of σ^2_1, the (asymptotic) mean square error of $n^{\frac{1}{2}}(T_n - \theta)$. Then, under the same regularity conditions,

(1.2) $$V^*_{n,1} \to \sigma^2_1 \text{ a.s., as } n \to \infty .$$

Further, if $T_1(F;x)$ stands for the *influence function* corresponding to T(F) and if we let $\bar{T}_{1n} = n^{-1}\Sigma^n_{i=1} T_1(F;X_i)$ [so that $n^{\frac{1}{2}}\bar{T}_{1n}$ is asymptotically normal with mean 0 and variance σ^2_1], then the following *second order asymptotic distributional representation (SOADR)* result holds [viz., Sen(1988)] :

(1.3) $$n\{ T^*_{n,1} - T(F) - \bar{T}_{1n} \} \xrightarrow{\mathcal{D}} \Sigma_{k\geq 1} \lambda_k \{ Z^2_k - 1 \},$$

where the Z_k are i.i.d.r.v.'s having the standard normal d.f., and the eigenvalues λ_k, $k \geq 1$, depend on the underlying d.f. F and the functional T(.). Note that

(1.4) $$n^{\frac{1}{2}}\{ T^*_{n,1} - \theta \} \underset{p}{\sim} n^{\frac{1}{2}}(T_n - \theta) \xrightarrow{\mathcal{D}} \mathcal{N}(0,\sigma^2_1),$$

even under weaker regularity conditions. Note further that (1.2) and (1.4) are first order properties, while (1.1) and (1.3) are second order ones.

The object of the present study is to focus on general delete-k jackknifed estimators$\{T^*_{n,k}, k \geq 1\}$ and the related versions of the Tukey variance estimators (of σ^2_1), and to examine how far these first and second order properties hold ? In this context, we may note that for $k \geq 2$, one may have either a resampling scheme of [n/k] deletion of distinct sets of k observations from the basic sample or the more natural case of $\binom{n}{k}$ possible subsamples of size n-k from the basic sample of size n. The first scheme has some arbitrariness in the partitioning, and we shall mainly consider the second case. Along with the preliminary notions, the proposed estimators are considered in Section 2. The main results are then presented in Section 3 and their derivations are sketched in Section 4. The last section deals with some useful remarks and general conclusions.

2. Preliminary notions. Note that the sample d.f. F_n is defined by

(2.1) $$F_n(x) = n^{-1}\Sigma_{i=1}^{n} I(X_i \leq x) , x \in R,$$

where I(A) stands for the indicator function of the set A. Thus, we have

(2.2) $$\theta = T(F) \text{ and } T_n = \hat{\theta}_n = T(F_n) ,$$

for a suitable functional T(.). Since we are dealing with statistical functionals, the delete-k jackknifed estimators may all be defined solely in terms of the sub-sample empirical d.f.s.Towards this, we let for every $\underset{\sim}{i} \in I$,

(2.3) $$F_{n-k}^{(\underset{\sim}{i})}(x) = (n-k)^{-1}\Sigma_{j\in S_{n,k}^{(\underset{\sim}{i})}} I(X_j \leq x) ; S_{n,k}^{(\underset{\sim}{i})} = \underset{\sim}{n} \setminus \underset{\sim}{i} ,$$

where

(2.4) $$\underset{\sim}{i} = (i_1,\ldots,i_k) , \underset{\sim}{n} = \{1,\ldots,n\} \text{ and } I = \text{set of all } \binom{n}{k} \underset{\sim}{i} .$$

Let then

(2.5) $$T_{n-k}^{(\underset{\sim}{i})} = T(F_{n-k}^{(\underset{\sim}{i})}) , \text{ for every } \underset{\sim}{i} \in I; \# \text{ of } I = \binom{n}{k}.$$

The pseusovariables generated by the delete-k jackknifing are defined as

(2.6) $$T_{n,\underset{\sim}{i}}^{(k)} = k^{-1}\{ nT_n - (n-k)T_{n-k}^{(\underset{\sim}{i})} \} , \text{ for } \underset{\sim}{i} \in I.$$

Then, the delete-k jackknifed estimator of θ is defined as

(2.7) $$T_{n,k}^{*} = \binom{n}{k}^{-1}\Sigma_{\{\underset{\sim}{i} \in I\}} T_{n,\underset{\sim}{i}}^{(k)} \}, \text{ for } k \geq 1.$$

Side by side, we also introduce the variance functions

(2.8) $$V_{n,k}^{*} = \binom{n}{k}^{-1}\Sigma_{\{\underset{\sim}{i} \in I\}} \{T_{n,\underset{\sim}{i}}^{(k)} - T_{n,k}^{*} \}^2, \text{ for } k \geq 1.$$

Next, we introduce some regularity conditions on T(.). Let A be a topological vector space and K be a class of compact subsets of A , such that every subset consisting of a single point belongs to K. We say that a functional $\tau(.)$ is *Hadamard-continuous* at $F \in A$, if

(2.9) $$|\tau(G) - \tau(F)| \to 0 \text{ with } ||G - F|| \to 0 \text{ [on } G \in K \subseteq K],$$

where $||G - F||$ refers to the usual sup-norm [i.e., $\sup_x | G(x)-F(x)|$]. Next, we say that $\tau(.)$ is *first order compact (Hadamard-) differentiable* at $F \in A$, if

(2.10) $$\tau(G) = \tau(F + G-F) = \tau(F) + \int\tau_1(F;x)d[G(x)-F(x)] + R_1(F;G-F)$$

where

(2.11) $$|R_1(F,G-F)| = o(|| G-F ||), \text{ uniformly in } G \in K,$$

and $\tau_1(F;x)$, the *first order compact derivative* of $\tau(.)$ at F, may be so normalized that

(2.12) $\quad \int \tau_1(F;x)dF(x) = 0$.

Similarly, $\tau(.)$ is *second order compact (Hadamard-) differentiable* at $F \in A$, if

(2.13)
$$\tau(G) = \tau(F) + \int \tau_1(F;x)d[G(x)-F(x)] + \tfrac{1}{2}\iint \tau_2(F;x,y)d[G(x)-F(x)]d[G(y)-F(y)] + R_2(F;G-F) ,$$

where

(2.14) $\quad |R_2(F;G-F)| = o(\,||G - F\,||^2)$, uniformly in $G \in K$,

and $\tau_2(F;.)$, the *second order compact derivative* of $\tau(.)$ at F, may be so normalized that

(2.15) $\quad \int \tau_2(F;x,y)dF(y) = 0 = \int \tau_2(F;x,y)dF(x)$ a.e.

Finally, we let

(2.16) $\quad \tau_2^*(G) = \int \tau_2(G;x,x)dG(x) \quad , \quad G \in K.$

Other notations will be introduced as and when necessary.

3. The main results. For the delete-k jackknifing, we may either set k to be an arbitrary positive number or may even allow k $(=k_n)$ to depend on the sample size n, in such a way that as n increases, k_n is $o(n^{1-\eta})$, for some $\eta > 0$, so that

(3.1) $\quad k_n$ is nondecreasing with $(n-1)/(n-k_n)$ converging to 1, as $n \to \infty$.

First, we consider the following theorem relating to the first order properties.

THEOREM 3.1. *If* $T(F)$ *is first order Hadamard-differentiable at* F *and* $T_1^{**}(G) = \int T_1^2(G,x)dG(x)$ $(<\infty)$ *is Hadamard-continuous at* F *, then for every* k *satisfying* (3.1) *, defining the* $V_{n,k}^*$ *as in* (2.8) *, we have for* $\sigma_1^2 = T_1^{**}(F)$,

(3.2)
$$k(n-1)(n-k)^{-1}V_{n.k}^* = V_{n,1}^* + o(1) \text{ a.s.}$$
$$= \sigma_1^2 + o(1) \text{ a.s., as } n \to \infty .$$

Thus, when adjusted by the scale factor $k(n-1)/(n-k)$ *, the Tukey estimator of the variance in the delete-k jackknifing is strongly consistent for* σ_1^2 .

THEOREM 3.2. *If* $T(F)$ *is second-order Hadamard-differentiable at* F *and* $T_2^*(.)$ *defined by* (2.16) *is Hadamard-continuous at* F *, then*

(3.3) $\quad (n-1)\{\, T_n - T_{n,k}^* \,\} \to (\tfrac{1}{2})T_2^*(F)$ a.s., as $n \to \infty$.

Thus the second order bias property of the classical jackknifing is shared by delete-k jackknifing, for every $k \geq 1$*. Actually,* $(n-1)(\, T_{n,k}^* - T_{n,1}^*) = o(1)$ a.s., *as* $n \to \infty$, *for every* k *satisfying* (3.1) .

THEOREM 3.3. *Assume that*

(3.4) $E_F T_2^2(F;X_1,X_1) = \iint T_2^2(F;x,y)dF(x)dF(y) < \infty$.

Then, under the hypothesis of Theorem 3.2, for every k *satisfying* (3.1) ,

(3.5) $n\{ T^*_{n,k} - T(F) - \bar{T}_{1n} \} \xrightarrow{\mathcal{D}} \Sigma_{r\geq 1} \lambda_r\{ Z_r^2 - 1 \}$,

where the Z_r *are i.i.d.r.v.'s with the standard normal d.f. and the eigenvalues* λ_r *correspond to the orthonormal functions* $\tau_r(.), r \geq 1$,*defined by*

(3.6) $\int T_2(F;x,y)\tau_r(x)dF(x) = \lambda_r \tau_r(y)$ a.e.(F), $r \geq 1$;

(3.7) $\int \tau_r(y)\tau_s(y)dF(y) = 1$ or 0 accordings as $r = s$ or not.

Thus, for any k *satisfying* (3.1), *delete-*k *jackknifing leads to the same SOADR as in the case of* $k = 1$.

4. Proofs of the theorems. Using (2.3), it readily follows that

(4.1) $||F^{(\underset{\sim}{i})}_{n-k} - F_n|| \leq (k/n)$,with probability 1, for every $\underset{\sim}{i} \in I$.

However, this result is not strong enough for our purpose (when we allow k to increase, subject to (3.1)). For this reason, we consider the following Lemma.

Lemma 4.1. Whenever $k = k_n = o(n^{1-\eta})$, for some $\eta > 0$,

(4.2) $\binom{n}{k}^{-1}\Sigma_{\{\underset{\sim}{i}\in I\}} ||F^{(\underset{\sim}{i})}_{n-k} - F_n||^2 = O(k(n-k)^{-2})$ a.s., as $n \to \infty$.

Proof. Parallel to (2.3), we define for every $\underset{\sim}{i} \in I$,

(4.3) $F_{\underset{\sim}{i},k}(x) = k^{-1}\Sigma_{j=1}^k I(X_{i_j} \leq x)$, $x \in R$.

Then, from (2.3) and (4.3), we obtain that

(4.4) $k^{-1}(n-k)|| F^{(\underset{\sim}{i})}_{n-k} - F_n || = ||F_{\underset{\sim}{i},k} - F_n ||$, $\forall \underset{\sim}{i} \in I$.

Thus, the left hand side of (4.2) can be written as

(4.5) $k(n-k)^{-2} \{ \binom{n}{k}^{-1}\Sigma_{\{\underset{\sim}{i}\in I\}} k||F_{\underset{\sim}{i},k} - F_n||^2 \}$

$\leq 2k(n-k)^{-2}\{ \binom{n}{k}^{-1}\Sigma_{\{\underset{\sim}{i}\in I\}} k||F_{\underset{\sim}{i},k} - F||^2 +(k/n)n||F_n - F||^2 \}$.

Now, by the classical results on the Kolmogorov-Smirnov goodness of fit statistic,

(4.6) $n|| F_n - F ||^2 = O((\log\log n)^{\frac{1}{2}})$ a.s., as $n \to \infty$,

while $k/n = o(n^{-\eta})$, so that $k||F_n - F||^2 = o(1)$ a.s., as $n \to \infty$. Further, we may identify $\binom{n}{k}^{-1}\Sigma_{\{\underset{\sim}{i}\in I\}}k||F_{\underset{\sim}{i},k} - F||^2$ as the U-statistic [Hoeffding (1948)] corresponding to the kernel $\phi_k(X_1,\ldots,X_k) = k||F_k - F||^2$. Using the basic results of Kiefer (1961) [on the finite-sample behavior of the Kolmogorov-Smirnov type statistics], it follows that for every finite k (≥ 1),

(4.7) $E_F\{\phi_k(X_1,\ldots,X_k)\} \leq D = \int_0^\infty 4x \exp\{-2x^2\}dx = 1,$

and for every finite integer r (≥ 1),

(4.8) $E_F\{[\phi_k(X_1,\ldots,X_k)]^{2r}\} \leq c_r < \infty$, uniformly in $k \geq 1$.

As such, using the results in Sen and Ghosh (1981) [viz., Sen (1981,p.56)], it follows that for every (fixed) positive integer r , the 2r-th central moment of $\binom{n}{k}^{-1}\Sigma_{\underset{\sim}{i}\varepsilon I}\ k||F_{\underset{\sim}{i},k} - F||^2$ is $O((k/n)^r) = o(n^{-r\eta})$, for every $k = o(n^{1-\eta})$, $\eta > 0$. Combining this with the Markov inequality and the Borel-Cantelli lemma, we immediately obtain by choosing $r : r\eta > 1$, that for $k = o(n^{1-\eta})$, for some $\eta > 0$,

(4.9) $\binom{n}{k}^{-1}\Sigma_{\{\underset{\sim}{i}\varepsilon I\}}\ k||F_{\underset{\sim}{i},k} - F||^2 = O(1)$ a.s., as $n \to \infty$,

Hence, (4.2) follows from (4.5) and the above discussions. Q.E.D.

<u>Lemma 4.2</u>. For any $k : k = o(n^{1-\eta})$, for some $\eta > 0$,

(4.10) $\binom{n}{k}^{-1}\Sigma_{\{\underset{\sim}{i}\ \varepsilon\ I\}}||F^{(\underset{\sim}{i})}_{n-k} - F_n|| = O(k^{\frac{1}{2}}(n-k)^{-1})$ a.s., as $n \to \infty$.

The proof follows directly from (4.5) and the elementary moment inequality, and hence, is omitted.

Let us now start with the proofs of the theorems. First, by an appeal to (2.10)-(2.12), for $G = F^{(\underset{\sim}{i})}_{n-k}$ and $F = F_n$, we immediately obtain that for first order differentiable T(.) ,

(4.11) $T^{(\underset{\sim}{i})}_{n-k} = T(F^{(\underset{\sim}{i})}_{n-k}) = T(F_n + F^{(\underset{\sim}{i})}_{n-k} - F_n)$

$$= T(F_n) + \int T_1(F_n;x)dF^{(\underset{\sim}{i})}_{n-k}(x) + R_1(F_n; F^{(\underset{\sim}{i})}_{n-k} - F_n), \forall \underset{\sim}{i} \varepsilon I ,$$

where

(4.12) $|R_1(F_n; F^{(\underset{\sim}{i})}_{n-k} - F_n)| = o(\ ||F^{(\underset{\sim}{i})}_{n-k} - F_n||\ ), \forall \underset{\sim}{i}\ \varepsilon\ I$.

In view of (2.12), the second term on the right hand side of (4.11) can be expressed as $-(n-k)^{-1}\Sigma_{j=1}^k T_1(F_n;X_{i_j})$. Therefore, by (2.6) and (4.11), we have

(4.13) $T^{(k)}_{n,\underset{\sim}{i}} = T_n + k^{-1}\Sigma_{j=1}^k T_1(F_n;X_{i_j}) + o(k^{-1}(n-k)||F^{(\underset{\sim}{i})}_{n-k} - F_n||), \underset{\sim}{i}\ \varepsilon\ I$.

From (2.7), (4.10) and (4.13), we obtain that

(4.14) $T^*_{n,k} = T_n + 0 + o(k^{-\frac{1}{2}})$ a.s., as $n \to \infty$;

in this context, it may be noted that by writing $\bar{T}^{(n)}_{1,\underset{\sim}{i}} = k^{-1}\Sigma^k_{j=1}T_1(F_n;X_{i_j})$, $\underset{\sim}{i} \in I$,

$$(4.15)\quad \binom{n}{k}^{-1}\Sigma_{\{\underset{\sim}{i}\,\in\, I\}}\ \bar{T}^{(n)}_{1,\underset{\sim}{i}} = n^{-1}\Sigma^n_{i=1}\ T_1(F_n;X_i) = \int T_1(F_n;x)dF_n(x) = 0,$$

[by (2.12)], which accounts for the second term on the right hand side of (4.14).

It follows similarly that

$$(4.16)\quad \begin{aligned} \binom{n}{k}^{-1}\Sigma_{\{\underset{\sim}{i}\,\in I\}}\ (\ \bar{T}^{(n)}_{1,\underset{\sim}{i}}\ )^2 &= \binom{n}{k}^{-1}k^{-2}\Sigma_{\{\underset{\sim}{i}\,\in\, I\}}[\ \Sigma^k_{j=1}T_1(F_n;X_{i_j})\ ]^2 \\ &= \binom{n}{k}^{-1}k^{-2}\Big\{\Sigma_{\{\underset{\sim}{i}\,\in I\}}[\Sigma^k_{j=1}\ T^2_1(F_n;X_{i_j}) + \Sigma^k_{j\neq j'=1}T_1(F_n;X_{i_j})T_1(F_n;X_{i_{j'}})\ ]\Big\} \\ &= (nk)^{-1}\Sigma^n_{i=1}\ T^2_1(F_n;X_i) + [(k-1)/kn(n-1)]\ \Sigma^n_{i\neq j=1}\ T_1(F_n;X_i)T_1(F_n;X_j) \\ &= (kn)^{-1}\Sigma^n_{i=1}\ T^2_1(F_n;X_i) - [(k-1)/kn(n-1)]\Sigma^n_{i=1}T^2_1(F_n;X_i) \qquad [\text{ by (4.15) }] \\ &= [(n-k)/k(n-1)]\ n^{-1}\Sigma^n_{i=1}\ T^2_1(F_n;X_i) \\ &= [(n-k)/k(n-1)]\{\ \int T^2_1(F_n;x)dF_n(x)\} \\ &= [(n-k)/k(n-1)]\ T^{**}_1(F_n), \end{aligned}$$

where $||F_n - F|| \to 0$ a.s., as $n \to \infty$, and hence, by the assumed Hadamard-continuity of $T^{**}_1(.)$, we obtain that $T^{**}_1(F_n) \to T^{**}_1(F) = \sigma^2_1$ a.s., as $n \to \infty$. Thus, from (4.16), it follows that

$$(4.17)\quad \binom{n}{k}^{-1}\Sigma_{\{\underset{\sim}{i}\,\in\, I\}}(\ \bar{T}^{(n)}_{1,\underset{\sim}{i}}\ )^2 = [(n-k)/k(n-1)]\sigma^2_1 + o(k^{-1}) \text{ a.s., as } n \to \infty\ .$$

Next, from (4.13) and (4.14), we obtain that

$$(4.18)\quad \begin{aligned} \binom{n}{k}^{-1}\Sigma_{\{\underset{\sim}{i}\in\, I\}}\{T^{(k)}_{n,\underset{\sim}{i}} - T^*_{n,k} + \bar{T}^{(n)}_{1,\underset{\sim}{i}}\ \}^2 &= o\Big(k^{-1} + \frac{n-k}{k}\binom{n}{k}^{-1}\Sigma_{\underset{\sim}{i}\in I}\ ||F^{(i)}_{n-k} - F_n||^2\Big) \\ &= o(\ k^{-1} + O((n-k)^{-1})\ ) \text{ a.s. [by Lemma 4.1]} \\ &= o(k^{-1}) \text{ a.s., as } n \to \infty\ , \quad \text{as } k = o(n^{1-\eta}), \text{ for some } \eta > 0. \end{aligned}$$

At this stage, we may recall that [viz., Lemma 1 of Sen (1960)] if $\{X_{Ni}\}$ and $\{Y_{Ni}\}$ are two sequences of r.v.'s (not necessarily independent) , such that (i) $N^{-1}\Sigma^N_{i=1}\ X^2_{Ni} \to A\ (\ < \infty)$ a.s. and (ii) $N^{-1}\Sigma^N_{i=1}(X_{Ni} - Y_{Ni})^2 \to 0$ a.s., as $N \to \infty$, then $N^{-1}\Sigma^N_{i=1}\ Y^2_{Ni} \to A$ a.s., as $N \to \infty$. Letting $N = \binom{n}{k}$, we obtain from (2.8), (4.17) and (4.18) that under the hypothesis of Theorem 3.1, as $n \to \infty$,

$$(4.19)\quad k(n-1)(n-k)^{-1}V^*_{n,k} \to \sigma^2_1 \text{ a.s., for every } k\text{: } k = o(n^{1-\eta}), \text{ for some } \eta > 0.$$

This completes the proof of Theorem 3.1.

Next, we consider the proof of (3.3). By using (2.13)-(2.15), we obtain that

$$T^{(i)}_{n\tilde{-}k} = T_n - (n-k)^{-1}\Sigma^k_{j=1}T_1(F_n;X_{i_j}) + [2(n-k)^2]^{-1}\Sigma^k_{j=1}\Sigma^k_{\ell=1}T_2(F_n;X_{i_j},X_{i_\ell})$$

$$+ o(\,||F^{(i)}_{n-k} - F_n\,||^2\,)\,,\ \forall\, \underset{\sim}{i} \in I\,. \tag{4.20}$$

Further, note that by virtue of (2.15), $\Sigma^n_{j=1}\ T_2(F_n;X_i,X_j) = 0$, for every $i(=1,\dots,n)$, and hence, by (2.7), (4.20) and Lemma 4.1, we have

$$\begin{aligned}(4.21)\quad T^*_{n-k} &= T_n - 0 - [2(n-k)]^{-1}\,\{n^{-1}\Sigma^n_{i=1}\ T_2(F_n;X_i,X_i)\} - \\ &\quad (k-1)[2(n-k)]^{-1}\{n(n-1)\}^{-1}\{\Sigma^n_{i\neq j=1}T_2(F_n;X_i,X_j)\ \} + o((n-k)^{-1}) \text{ a.s.}\\ &= T_n - [2(n-k)]^{-1}\{\ 1 - (k-1)/(n-1)\}\{n^{-1}\Sigma^n_{i=1}T_2(F_n;X_i,X_i)\} + o((n-k)^{-1}) \text{ a.s.}\\ &= T_n - [2(n-1)]^{-1} \int T_2(F_n;x,x)dF_n(x) + o((n-k)^{-1}) \text{ a.s., as } n\to\infty\ ,\end{aligned}$$

so that

$$\begin{aligned}(4.21)\quad (n-1)\{\ T_n - T^*_{n,k}\ \} &= \tfrac{1}{2}\int T_2(F_n;x,x)dF_n(x) \ + o(n/(n-k)) \text{ a.s.}\\ &= \tfrac{1}{2}T^*_2(F_n) + o(1) \text{ a.s., as } n\to\infty\ ,\ \forall\ k = o(n^{1-\eta}\,),\eta > 0.\end{aligned}$$

Again $||\ F_n - F\ || \to 0$ a.s., as $n\to\infty$, while by the assumed Hadamard-continuity of $T^*_2(.)$ at F, $T^*_2(F_n) \to T^*_2(F)$ a.s., as $n\to\infty$. This completes the proof of (3.3).

Finally, to prove (3.5), we note that by virtue of (3.3),

$$(4.22)\quad (n-1)\{\ T^*_{n,k} - T^*_{n,1}\ \} \to 0 \text{ a.s., as } n\to\infty\ ,\forall\ k\!: k = o(n^{1-\eta}),\ \eta > 0.$$

As such, it suffices to show that (3.5) holds for the particular case of $k = 1$, and this has already been proved in Theorem 2.2 of Sen (1988). Hence, we omit the details here.

5. Concluding remarks.

From Theorems 3.1 and 3.2 it is quite clear that if instead of the classical jackknifing, one uses the delete-k jackknifing procedure for some $k \geq 1$, as regards the two estimators $T^*_{n,1}$ and $T^*_{n,k}$ are concerned, by virtue of (4.22), they are a.s. equivalent upto the order n^{-1}. Thus, upto the order n^{-1} , there is no change in the bias reduction picture due to higher order delete-k jackknifing. In practice, bias of the order n^{-2} (or higher) are of not much interest. As regards the Tukey form of the variance estimators are concerned, for delete-k jackknifing, one needs the adjustment factor $k(n-1)/(n-k)$, and with that (3.2) ensures their asymptotic (a.s.) equivalence upto the first order. On the other hand, from the computational aspect, delete-k jackknifing involves labor of the order n^k, and hence, with larger values of k, the computational complexities may increase rather abruptly. From all these considerations, it may be concluded that if jackknifing is to be condidered for bias reduction and variance estimation for an estimator of a smooth functional T(F), then there is not much attraction for adaptation of delete-k jackknifing. The picture may, however, change considerably if we deal with so called non-smooth functionals where (2.11) or (2.14) may not hold. Then, of course, a different method of attack may be necessary to study this relative picture.

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