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LIMIT THEOREMS FOR SUMS OF ORDER STATISTICS

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This is a brief summary of recent results on the asymptotic distribution of various ordered portions of sums of independent, identically distributed random variables that have been obtained by a direct probabilistic approach based upon some integral representations of such sums in terms of uniform empirical distribution functions, following a quantile transformation. Almost all the results discussed here have been obtained jointly with Erich Haeusler and David M. Mason. The survey itself forms a guideline for a series of lectures given as a part of the Sixth International Summer School.

1. Introduction

Let $X_1, X_2, \ldots$ be independent, real, non-degenerate random variables with the common distribution function $F(x) = Pr\{X \leq x\}$, $x \in \mathbf{R}$, and introduce the inverse or quantile function Q of F defined as

$$Q(s) = \inf\{x : F(x) \geq s\}, \ 0 < s \leq 1, \ Q(0) = Q(0+).$$

Our motivating point of departure is the simple fact that if $U_1, U_2, \ldots$ are independent random variables on a probability space $(\Omega, \mathcal{F}, P)$ uniformly distributed in $(0,1)$, then for each n the distributional equality

$$(1.1) \qquad \sum_{j=1}^{n} X_j \stackrel{\mathcal{D}}{=} \sum_{j=1}^{n} Q(U_j) = n \int_0^1 Q(s) dG_n(s)$$

holds, where $G_n(s) = n^{-1}\#\{1 \leq j \leq n : U_j \leq s\}$ is the uniform empirical distribution function on $(0,1)$. In fact, if $X_{1,n} \leq \cdots \leq X_{n,n}$ are the order statistics based on the

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sample $X_1, \ldots, X_n$ and $U_{1,n} \leq \cdots \leq U_{n,n}$ denote the order statistics pertaining to $U_1, \ldots, U_n$, then we have

$$\{X_{j,n} : 1 \leq j \leq n,\ n \geq 1\} \stackrel{\mathcal{D}}{=} \{Q(U_{j,n}) : 1 \leq j \leq n,\ n \geq 1\} \tag{1.2}$$

and hence, introducing the natural centering sequence

$$\mu_n(m+1, n-(k+1)) = n \int_{(m+1)/n}^{1-(k+1)/n} Q(u+)du, \tag{1.3}$$

we have, for example,

$$\begin{aligned}\sum_{j=m+1}^{n-k} X_{j,n} - \mu_n(m+1, n-(k+1)) &\stackrel{\mathcal{D}}{=} \sum_{j=m+1}^{n-k} Q(U_{j,n}) - \mu_n(m+1, n-(k+1)) \\ &= \left\{ Q(U_{m+1,n}) + n \int_{(m+1)/n}^{U_{m+1,n}} \left(G_n(s) - \frac{m+1}{n} \right) dQ(s) \right\} \\ &\quad + n \int_{(m+1)/n}^{1-(k+1)/n} (s - G_n(s)) dQ(s) \\ &\quad + \left\{ Q(U_{n-k,n}) + n \int_{U_{n-k-1,n}}^{1-(k+1)/n} \left(G_n(s) - \frac{n-k-1}{n} \right) dQ(s) \right\}\end{aligned} \tag{1.4}$$

for any integers $m, k \geq 0$ such that $m + 1 < n - k$.

Relations (1.1) and (1.4) suggest the feasibility of a probabilistic approach to the problem of the asymptotic distribution of sums of independent, identically distributed random variables, or more generally, of the corresponding trimmed sums, to be based on some properties of Q and the asymptotic behaviour of G_n, rather than on characteristic functions or other transforms of F. Naturally enough, the analytic conditions that this method yields are all expressed in terms of the quantile function Q.

On the technical side, we are completely free to choose the underlying space $(\Omega, \mathcal{F}, \mathcal{P})$. This will be the one described in [3, 4]. It carries two independent sequences $\{Y_n^{(j)}, n \geq 1\}$, $j = 1, 2$, of independent, exponentially distributed random variables with mean 1 and a sequence $\{B_n(s),\ 0 \le s \le 1; n \ge 1\}$ of Brownian bridges with the properties that we describe now. For each $n \ge 2$, let

$$Y_j(n) = \begin{cases} Y_j^{(1)} & , j = 1, \ldots, [n/2], \\ Y_{n+2-j}^{(2)} & , j = [n/2]+1, \ldots, n+1, \end{cases}$$

and for $k = 1, \ldots, n+1$, write

$$S_k(n) = \sum_{j=1}^{k} Y_j(n).$$

Then the ratios $U_{k,n} = S_k(n)/S_{n+1}(n),\ k = 1, \ldots, n$, have the same joint distribution as the order statistics of n independent uniform $(0,1)$ random variables, and for the corresponding (left-continuous version of the) empirical distribution function

$$G_n^{(1)}(s) = n^{-1} \sum_{j=1}^{n} I(U_{j,n} < s),\ 0 \le s \le 1,$$

where $I(\cdot)$ is the indicator function, and the empirical quantile function

$$U_n(s) = \begin{cases} U_{k,n} & , (k-1)/n < s \le k/n;\ k = 1, \ldots, n, \\ U_{1,n} & , s = 0, \end{cases}$$

we have

$$\text{(1.5)} \qquad \sup_{1/n \le s \le 1-1/n} \frac{|n^{1/2}(G_n(s) - s) - B_n(s)|}{(s(1-s))^{1/2-\nu}} = O_p(n^{-\nu})$$

for any fixed $\nu \in [0, 1/4)$ and

$$\text{(1.6)} \qquad \sup_{1/n \le s \le 1-1/n} \frac{|n^{1/2}(s - U_n(s)) - B_n(s)|}{(s(1-s))^{1/2-\nu}} = O_p(n^{-\nu})$$

for any fixed $\nu \in [0, 1/2)$ as $n \to \infty$. Moreover, the independent standard Poisson processes

$$N_j(t) = \sum_{k=1}^{\infty} I(S_k^{(j)} < t),\ 0 \le t < \infty,\ j = 1, 2,$$

associated with the two independent jump-point sequences $S_k^{(j)} = Y_1^{(j)} + \cdots + Y_k^{(j)}$, $j = 1, 2$, will be close enough for the present purposes to the random functions $nG_n^{(j)}(t/n)$, $j = 1, 2$, respectively as $n \to \infty$, where

$$G_n^{(2)}(s) = n^{-1} \sum_{j=1}^{n} I(1 - U_{n+1-j,n} < s), \; 0 \le s \le 1,$$

is the empirical distribution function obtained by "counting down" from 1.

The quantile-transform method based on (1.2) has long been in use in statistical theory and scattered applications of it can be found also in probability. Here we don't aim at giving any bibliography of this method, a good source for its earlier use is the book [53]. It is the approximation results in (1.5) and (1.6) in combination with Poisson approximation techniques for extremes that has made this old method especially feasible for the handling of problems of the asymptotic distribution of various sums of order statistics.

The approach touched upon above was first used in [3] and [4] to obtain probabilistic proofs of the sufficiency parts of the normal and stable convergence criteria, respectively, for whole sums $\sum_{j=1}^{n} X_j$. The effect on the asymptotic distribution of trimming off a fixed number m of the smallest and and a fixed number k of the largest summands. i.e. the investigation of the lightly trimmed sums $T_n(m, k) = \sum_{j=m+1}^{n-k} X_{j,n}$, was already considered in [4] under the (quantile equivalent of the) classical stable convergence criterion. This line of research goes back to Darling [19] and Arov and Bobrov [1], with later contributions by Hall [36], Teugels [55], Maller [43], Mori [49], Egorov [21] and Vinogradov and Godovan'chuk [56]. (Again, we don't intend to compile full bibliographies of the problems considered.) The earlier literature is concentrated almost exclusively on trimmed sums where summands with largest absolute values are discarded. We shall refer to this kind of trimming as modulus or magnitude trimming in the sequel, as opposed to our natural-order, or simply natural, trimming described above.

The paper [13] has initiated the study of two problems. One was the problem of the asymptotic distribution of moderately trimmed sums $T_n(k_n, k_n)$, where $k_n \to \infty$ as

$n \to \infty$ such that $k_n/n \to 0$, the other one was the same problem for the corresponding extreme sums $T_n(0, n-k_n)$ and $T_n(n-k_n, 0)$. The first problem was looked at under the restrictive initial assumption that F belonged to the domain of attraction of a normal or a non-normal stable distribution, while the second one only in the non-normal stable domain. Later, the second problem concerning extreme sums was solved in [15] for all F with regularly varying tails and, extending a result in [14], Lo [40] determined the asymptotic distribution of extreme sums for all F which are in the domain of attraction of a Gumbel distribution in the sense of extreme value theory. All these papers use the probabilistic method.

The method itself has been perfected in the three papers [9,10,11], where a general pattern of necessity proofs has also been worked out, which together constitute a general unified theory of the asymptotic distribution of sums of order statistics. The next three sections are devoted to a very brief sketch of this theory according to [10], [9] and [11], respectively. Some related matters are taken up even more briefly in Section 5.

In these lectures we shall only deal with problems in probability. The same method has already been used to tackle some problems in asymptotic statistics. Some of these applications, where the mathematics is most closely related to things discussed here, can be found in [5], [8], [17] and [18].

An earlier and much shorter survey of the method is in [16]. In comparison, the present survey may be looked upon as a progress report covering the last two years.

2. Full and lightly trimmed sums [10]

The aim is to determine all possible limiting distributions of the suitably centered and normalized sequence

$$T_n(m,k) = \sum_{j=m+1}^{n-k} X_{j,n},$$

where $m \geq 0$ and $k \geq 0$ are fixed, along subsequences of $\{n\}$ under the broadest possible conditions.

Choose the integers l and r such that $m \le l \le r \le n-r \le n-l \le n-k$, and write

$$\begin{aligned} T_n(m,k) - \mu_n(m+1, n-(k+1)) &= \left\{ \sum_{j=m+1}^{l} X_{j,n} - \mu_n(m+1, l+1) \right\} \\ &+ \left\{ \sum_{j=l+1}^{r} X_{j,n} - \mu_n(l+1, r+1) \right\} \\ &+ \left\{ \sum_{j=r+1}^{n-r} X_{j,n} - \mu_n(r+1, n-(r+1)) \right\} \\ &+ \left\{ \sum_{j=n-r+1}^{n-l} X_{j,n} - \mu_n(n-(r+1), n-(l+1)) \right\} \\ &+ \left\{ \sum_{j=n-l+1}^{n-k} X_{j,n} - \mu_n(n-(l+1), n-(k+1)) \right\} \\ &= v_m^{(1)}(l,n) + \delta_1(l,r,n) + \overline{m}(r,n) + \\ &+ \delta_2(l,r,n) + v_k^{(2)}(l,n). \end{aligned}$$

Now if we introduce

$$\varphi_n^{(1)}(s) = \begin{cases} \frac{1}{A_n} Q\left(\frac{s}{n}+\right) & , 0 < s \le n - n\alpha_n, \\ \frac{1}{A_n} Q((1-\alpha_n)+) & , n - n\alpha_n < s < \infty, \end{cases}$$

and

$$\varphi_n^{(2)}(s) = \begin{cases} -\frac{1}{A_n} Q\left(1 - \frac{s}{n}\right) & , 0 < s \le n - n\alpha_n, \\ -\frac{1}{A_n} Q(\alpha_n) & , n - n\alpha_n < s < \infty, \end{cases}$$

where $A_n > 0$ is some potential normalizing sequence and $\alpha_n \to 0$ as $n \to \infty$ such that $n\alpha_n \to 0$ (so that $P\{\alpha_n \le U_{1,n} \le U_{n,n} \le 1 - \alpha_n\} \to 1$ as $n \to \infty$), and also

$$Z_{q,n}^{(j)} = \begin{cases} nU_{q,n} & , j = 1, \\ n(1 - U_{n+1-q,n}) & , j = 2, \end{cases}$$

then, using (1.2) and integration by parts, for

$$V_h^{(j)}(l,n) = \frac{1}{A_n} v_h^{(j)}(l,n), \;\; h = m, k; \;\; j = 1, 2,$$

we can write

$$
\begin{aligned}
V_h^{(j)}(l,n) \stackrel{\mathcal{D}}{=} (-1)^{j+1}\Bigg\{ & \int_{Z_{h+1,n}^{(j)}}^{Z_{l+1,n}^{(j)}} \left(s - nG_n^{(j)}\left(\frac{s}{n}\right)\right) d\varphi_n^{(j)}(s) \\
& + \int_{h+1}^{Z_{h+1,n}^{(j)}} (s-(h+1))d\varphi_n^{(j)}(s) \\
& + \varphi_n^{(j)}\left(Z_{h+1,n}^{(j)}\right) \\
& + \int_{Z_{l+1,n}^{(j)}}^{l+1} (s-(l+1))d\varphi_n^{(j)}(s) \\
& - \varphi_n^{(j)}\left(Z_{l+1,n}^{(j)}\right)\Bigg\}.
\end{aligned}
$$

If we now assume that there exists a subsequence $\{n'\}$ of the positive integers such that for two non-decreasing, right-continuous functions φ_1 and φ_2 we have

(2.1) $\varphi_{n'}^{(j)}(s) \to \varphi_j(s)$, as $n' \to \infty$, at every continuity point $s \in (0,\infty)$ of φ_j, $j = 1,2$,

then it turns out that the right-side of the last distributional limit converges in probability to a limit as $n' \to \infty$ for each fixed l, and these limits converge, if we let $l \to \infty$, in probability to

$$
\begin{aligned}
V_h^{(j)} = (-1)^{j+1}\Bigg\{ & \int_{S_{h+1}^{(j)}}^{\infty} (s - N_j(s))d\varphi_j(s) + \int_1^{S_{h+1}^{(j)}} s d\varphi_j(s) \\
& + \varphi_j(1) - h\varphi_j\left(S_{h+1}^{(j)}\right) + \int_1^{h+1} \varphi_j(s)ds\Bigg\},
\end{aligned}
$$

$h = m,k$; $j = 1,2$. These limits are well-defined random variables because condition (2.1) implies that

$$
\int_{\varepsilon}^{\infty} \varphi_j^2(s)ds < \infty \quad \text{for any} \quad \varepsilon > 0,\ j = 1,2. \tag{2.2}
$$

Also, it turns out that the terms $\delta_j(l,r,n')$, $j = 1,2$, above only play the role of "sanitary cordons" in the sense that under (2.1), $\delta_j(l,r,n')/A_{n'}$ converge to some limits in probability as $n' \to \infty$, $j = 1, 2$, and if we let $l \to \infty$ (forcing $r \to \infty$) then both of these limit sequences converge to zero in probability. This fact shows that these two strips do not contribute to the limit and they only separate $V_m^{(1)}$ and $V_m^{(2)}$ from a possibly vanishing normal component of the limit coming from the middle term

$$M(r, n') = \frac{1}{A_{n'}} \overline{m}(r, n'),$$

which component by later appropriate choices of $l = l_{n'}$ and $r = r_{n'}$ and by an application of a result of Rossberg [52] will be independent of the vector $(V_m^{(1)}, V_k^{(2)})$, the two components of the latter being independent by construction.

Finally, using (1.2) and the representation (1.4) with $m = k = r$, and (1.5), it can be shown that for any sequence $r_{n'} \to \infty$, $r_{n'}/n' \to 0$, we have

$$M(r_{n'}, n') \stackrel{\mathcal{D}}{=} \frac{a_{n'}}{A_{n'}} \sigma_{n'} N_{n'}(0,1) + o_p(1),$$

as $n' \to \infty$, where

$$0 \le \sigma_{n'} = \frac{\sigma((r_{n'}+1)/n')}{\sigma(1/n')} \le 1$$

and $a_{n'} = \sqrt{n'}\sigma(1/n')$, where for $0 < s < 1$,

$$\sigma^2(s) = \int_s^{1-s} \int_s^{1-s} (\min(u,v) - uv) dQ(u) dQ(v)$$

and where

$$N_{n'}(0,1) = \int_{(r_{n'}+1)/n'}^{1-(r_{n'}+1)/n'} B_{n'}(s) dQ(s) \Big/ \sigma((r_{n'}+1)/n')$$

is a standard normal ($N(0,1)$) random variable for each n'.

This is the way we arrive at the direct half (i) of the following result which comprises the essential elements of Theorems 1-5 in [10].

RESULT. (i) *Assume (2.1) and that*

(2.3) $$a_{n'}/A_{n'} \to \delta < \infty,$$

where δ is some non-negative constant. If $\delta = 0$, then

$$\frac{1}{A_{n'}}\left\{\sum_{j=m+1}^{n'-k} X_{j,n'} - \mu_{n'}(m+1, n'-(k+1))\right\} \xrightarrow{\mathcal{D}} V_m^{(1)} + V_k^{(2)}$$

as $n' \to \infty$, where, necessarily, $\varphi_j(s) = 0$ if $s \geq 1$ $j = 1, 2$. If $\delta > 0$, then for any subsequence $\{n''\}$ of $\{n'\}$ for which $\sigma_{n''} \to \sigma$ as $n'' \to \infty$, where $0 \leq \sigma \leq 1$, we have

$$\frac{1}{A_{n''}}\left\{\sum_{j=m+1}^{n''-k} X_{j,n''} - \mu_{n''}(m+1, n''-(k+1))\right\} \xrightarrow{\mathcal{D}} V_m^{(1)} + \delta\sigma N(0,1) + V_k^{(2)}.$$

as $n'' \to \infty$, where the three terms in the limit are independent. In both cases $V_m^{(1)}$ is non-degenerate if $\varphi_1 \not\equiv 0$ and $V_k^{(2)}$ is non-degenerate if $\varphi_2 \not\equiv 0$.

(ii) *If there exist two sequences of constants $A_n > 0$ and C_n and a sequence $\{n'\}$ of positive integers such that*

$$\frac{1}{A_{n'}}\left\{\sum_{j=m+1}^{n'-k} X_{j,n'} - C_{n'}\right\} \tag{2.4}$$

converges in distribution to a non-degenerate limit, then there exist a subsequence $\{n''\}$ of $\{n'\}$ and non-decreasing, non-positive, right-continuous functions φ_1 and φ_2 defined on $(0, \infty)$ satisfying (2.2) and a constant $0 \leq \delta < \infty$ such that (2.1) and (2.3) hold true for $A_{n''}$ along $\{n''\}$. The limiting random variable of the sequence in (2.4) is necessarily of the form $V_m^{(1)} + \delta\sigma N(0,1) + V_k^{(2)} + d$ with independent terms, where

$$d = \lim_{n''' \to \infty} d_{n'''} = \lim_{n''' \to \infty} \{\mu_{n'''}(m+1, n'''-(k+1)) - C_{n'''}\}/a_{n'''} \tag{2.5}$$

for some subsequence $\{n'''\}$ of $\{n''\}$. If $\delta > 0$ then either $\sigma > 0$ or at least one of φ_1 and φ_2 is not identically zero. If $\delta = 0$ then $\varphi_j = 0$ on $[1, \infty)$, $j = 1, 2$, but at least one of them is not identically zero.

In the proof of the converse half (ii), the case when

$$\limsup_{n' \to \infty} \frac{A_{n'}}{a_{n'}} |\varphi_{n'}^{(j)}(s)| < \infty, \; 0 < s < \infty; \; j = 1, 2, \tag{2.6}$$

is trivial, for then by Helly-Bray selection and the convergence of types theorem there exist a subsequence $\{n'''\}$ such that (2.1), (2.3) and (2.4) all hold along it and $\delta > 0$ in (2.3), and we can apply the direct half with $\sigma \geq 0$.

When, contrary to (2.6), there exists $\{n''\} \subset \{n'\}$ such that

$$\lim_{n''\to\infty} \frac{A_{n''}}{a_{n''}} \varphi^{(1)}_{n''}(s) = -\infty \tag{2.7}$$

for some $s > 0$, for which one can show that necessarily $s < 1$, then the sequence in (2.4) is equal in distribution to

$$\frac{a_{n'}}{A_{n'}} \left\{ R^{(1)}_{n'} + W_{n'} + R^{(2)}_{n'} \right\} + d_{n'} \tag{2.8}$$

where $R^{(1)}_n$, W_n and $R^{(2)}_n$ result from dividing by $a_{n'}$ the three terms on the right- side of (1.4), respectively. Then again Rossberg's [52] result implies that the two sequences $|R^{(1)}_n|$ and $|R^{(2)}_n|$ are asymptotically independent, and we can show that

$$\lim_{M\to\infty} \liminf_{n\to\infty} P\left\{ |R^{(j)}_n| < M \right\} > 0, \; j = 1, 2,$$

which is somewhat less than the stochastic boundedness of the sequences of $R^{(j)}_n$, $j = 1, 2$. However, it can be shown that these two facts and the stochastic boundedness of the sequence in (2.8) (which holds since by assymption it has a limiting distribution) already imply that both sequences

$$D^{(j)}_{n'} = H_{n'} |R^{(j)}_{n'}| = a_{n'} |R^{(j)}_{n'}| / \max(a_{n'}, A_{n'}), \tag{2.9}$$
$$j = 1, 2, \text{ are stochastically bounded.}$$

However, on the event $\{U_{m+1,n''} < s/n''\}$ with positiv limiting probability of $P\{S_{m+1} < s\}$, where s is as in (2.7), we have

$$D^{(1)}_{n''} \geq |Q\left(\frac{s}{n''}+\right)| / \max(a_{n''}, A_{n''}) = H_{n''} \left| \frac{A_{n''}}{a_{n''}} \varphi^{(1)}_{n''}(s) \right|$$

because the integral term in $R_n^{(1)}$ is non-positive for large enough n and $Q(s/n'')$ is non-positive for large enough n'' (otherwise (2.7) could not happen). This fact, (2.7) and (2.9) imply that $a_{n''}/A_{n''} \to 0$ as $n'' \to \infty$ and that

$$\limsup_{n''\to\infty} |\varphi_{n''}^{(1)}(s)| < \infty, \ 0 < s < \infty.$$

By repeating this proof if necessary one can choose a further subsubsequence $\{n'''\} \subset \{n''\}$ to arrive at

$$\limsup_{n'''\to\infty} |\varphi_{n'''}^{(2)}(s)| < \infty, \ 0 < s < \infty,$$

and hence by a final application of a Helly-Bray selection we are done again.

Noting that the integral term in $R_n^{(2)}$ is non-negative for large enough n, the subcase when (2.6) fails for $j = 2$ is entirelly analogous.

The special case $m = k = 0$ of the result above gives an equivalent version of the classical theory of the asymptotic distribution of independent, identically distributed random variables (see, e.g. [25]) with a condition formulated in terms of the quantile function. In this case, the limiting random variable in the direct half (i) is in general $V_{0,0} = V_0^{(1)} + \rho N(0,1) + V_0^{(2)}$, where $\rho = \delta\sigma \geq 0$ and

$$\begin{aligned} V_0^{(j)} &= (-1)^{j+1}\left\{\int_{S_1^{(j)}}^{\infty} (s - N_j(s))d\varphi_j(s) + \int_1^{S_1^{(j)}} s\,d\varphi_j(s) + \varphi_j(1)\right\} \\ &= (-1)^{j+1}\left\{\int_1^{\infty} (s - N_j(s))d\varphi_j(s) - \int_0^1 N_j(s)d\varphi_j(s) + \varphi_j(1)\right\}, \end{aligned}$$

for $j = 1, 2$. This is an infinitely divisible random variable with characteristic function

$$\begin{aligned} Ee^{itV_{0,0}} = \exp\Big(it\gamma - \frac{1}{2}\rho^2 t^2 &+ \int_{-\infty}^{0} \left(e^{itx} - 1 - \frac{itx}{1+x^2}\right) dL(x) \\ &+ \int_0^{\infty} \left(e^{itx} - 1 - \frac{itx}{1+x^2}\right) dR(x)\Big), \end{aligned} \tag{2.10}$$

$t \in \mathbf{R}$, where $\gamma = \gamma_1 + \gamma_2$ with

$$\gamma_j = (-1)^{j+1} \left\{ \int_0^1 \frac{\varphi_j(s)}{1+\varphi_j^2(s)} ds - \int_1^\infty \frac{\varphi_j^3(s)}{1+\varphi_j^2(s)} ds \right\}, \; j = 1, 2,$$

and $L(x) = \inf\{s > 0 : \varphi_1(s) \geq x\}$, $-\infty < x < 0$, and $R(x) = \inf\{s < 0 : -\varphi_2(-s) \geq x\}$, $0 < x < \infty$, and any infinitely divisible random variable can be represented as $V_{0,0}$ plus a constant (Theorem 3 in [10]) by reversing the definitions of the inverse functions if a pair (L, R) of left and right Lévy measures is given. (See Section 5.3 below.) So the result above shows rather directly how these measures arise, while the sketched proof indicates which portions of the whole sum contribute these extreme parts of the limiting infinitely divisible law.

The direct and converse halfs of the result above are used in [10] to derive necessary and sufficient conditions for full or lightly trimmed sums to be in the domain of attraction of a normal law (the normal convergence criterion) or to be in the domain of partial attraction of a normal law, for full sums to be in the domain of attraction of a non-normal stable law (the stable convergence criterion; stable laws of exponent $0 < \alpha < 2$ arise with the functions $\varphi_j(s) = -c_j s^{-1/\alpha}$, $0 < s < \infty$, $j = 1, 2$, where $c_1, c_2 \geq 0$ are constants such that $c_1 + c_2 > 0$) or to be in the domain of partial attraction of a non-normal stable law. The domains of normal attraction of these laws are also characterized. Analogous characterization results are derived for the domain of partial attraction of some infinitely divisible law or its lightly trimmed version $V_{m,k} = V_m^{(1)} + \rho N(0,1) + V_k^{(2)}$, and necessary and sufficient conditions are derived for the stochastic compactness and subsequential compactness of lightly trimmed or full sums, together with a Pruitt-type [50] quantile description of the arising subsequential limiting laws in the compact case. All these results are deduced from (i) and (ii) above, independently of the existing literature. all the obtained necessary and sufficient conditions are expressed in terms of the quantile function and hence are of independent interest, and most of the results are effectively new as far as light trimming is concerned.

3. Moderately and heavily trimmed sums [9]

We call the sum

$$T_n = T_n(m_n, k_n) = \sum_{j=m_n+1}^{n-k_n} X_{j,n}$$

moderately trimmed if the integers m_n and k_n are such that, as $n \to \infty$,

$$m_n \to \infty,\ k_n \to \infty,\ m_n/n \to 0,\ k_n/n \to 0. \tag{3.1}$$

Now, fixing these two sequences $\{m_n\}$ and $\{k_n\}$, with

$$\mu_n = \mu_n(m_n, n-k_n) = n \int_{m_n/n}^{1-k_n/n} Q(s)ds$$

the equality (1.4) simplifies to

$$\begin{aligned}
\frac{1}{A_n}\{T_n - \mu_n\} &\stackrel{\mathcal{D}}{=} \frac{n}{A_n}\int_{m_n/n}^{U_{m_n,n}} \left(G_n(s) - \frac{m_n}{n}\right) dQ(s) \\
&\quad + \frac{n}{A_n}\int_{m_n/n}^{1-k_n/n} (s - G_n(s))dQ(s) \\
&\quad + \frac{n}{A_n}\int_{U_{n-k_n,n}}^{1-k_n/n} \left(G_n(s) - \frac{n-k_n}{n}\right) dQ(s) \\
&= R_{1,n} + Y_n + R_{2,n},
\end{aligned} \tag{3.2}$$

where $A_n > 0$ is some potential norming sequence, and $R_{1,n} \leq 0,\ R_{2,n} \geq 0$.

First, using (1.5) it turns out that

$$Y_n = \frac{a_n}{A_n}(Z_n + o_p(1)),$$

where now a_n is defined to be $a_n = \sqrt{n}\sigma(m_n/n, 1 - k_n/n)$, where for $0 \leq s \leq t \leq 1$.

$$\sigma^2(s,t) = \int_s^t \int_s^t (\min(u,v) - uv)dQ(u)dQ(v), \tag{3.3}$$

and where

$$Z_n = -\frac{1}{\sigma(m_n/n, 1 - k_n/n)} \int_{m_n/n}^{1-k_n/n} B_n(s)dQ(s)$$

is a standard normal variable for each n.

Concerning $R_{1,n}$, it is easy to see from (1.6) that

$$\frac{n}{m_n^{1/2}}\left(U_{m_n,n}-\frac{m_n}{n}\right)+Z_{1,n}=O_p(m_n^{-\nu})$$

for any $0<\nu<1/4$, where $Z_{1,n}=(n/m_n)^{1/2}B_n(m_n/n)$. Using this in conjunction with (1.5), it can be shown that $R_{1,n}$ behaves asymptotically as

$$\begin{aligned}
&\frac{n^{1/2}}{A_n}\int_{m_n/n}^{(m_n-m_n^{1/2}Z_{1,n})/n}\left\{B_n(s)+n^{1/2}\left(s-\frac{m_n}{n}\right)\right\}dQ(s)\\
&=\frac{n^{1/2}}{A_n}\int_0^{-Z_{1,n}}\left\{B_n\left(\frac{m_n}{n}+x\frac{m_n^{1/2}}{n}\right)+x\left(\frac{m_n}{n}\right)^{1/2}\right\}dQ\left(\frac{m_n}{n}+x\frac{m_n^{1/2}}{n}\right)\\
&=\int_0^{-Z_{1,n}}\left\{\left(\frac{n}{m_n}\right)^{1/2}B_n\left(\frac{m_n}{n}+x\frac{m_n^{1/2}}{n}\right)+x\right\}d\psi_n^{(1)}(x),
\end{aligned}$$

which in turn behaves asymptotically as

$$\int_0^{-Z_{1,n}}(Z_{1,n}+x)d\psi_n^{(1)}(x)=\int_{-Z_{1,n}}^0\psi_n^{(1)}(x)dx,$$

provided the sequence of functions

$$\psi_n^{(1)}(x)=\begin{cases}\psi_n^{(1)}\left(-\frac{m_n^{1/2}}{2}\right) & ,\ -\infty<x<-\frac{m_n^{1/2}}{2},\\ \frac{m_n^{1/2}}{A_n}\left\{Q\left(\frac{m_n}{n}+x\frac{m_n^{1/2}}{n}\right)-Q\left(\frac{m_n}{n}\right)\right\} & ,\ |x|\le\frac{m_n^{1/2}}{2},\\ \psi_n^{(1)}\left(\frac{m_n^{1/2}}{2}\right) & ,\ \frac{m_n^{1/2}}{2}<x<\infty,\end{cases}$$

is at least bounded. Similarly, it can be shown that $R_{2,n}$ behaves asymptotically as

$$\int_{-Z_{2,n}}^0(Z_{2,n}+x)d\psi_n^{(2)}(x)=\int_0^{-Z_{2,n}}\psi_n^{(2)}(x)dx,$$

where $Z_{2,n}=(n/k_n)^{1/2}B_n(1-k_n/n)$ and

$$\psi_n^{(2)}(x)=\begin{cases}\psi_n^{(2)}\left(-\frac{k_n^{1/2}}{2}\right) & ,\ -\infty<x<-\frac{k_n^{1/2}}{2},\\ \frac{k_n^{1/2}}{A_n}\left\{Q\left(1-\frac{k_n}{n}+x\frac{k_n^{1/2}}{n}\right)-Q\left(1-\frac{k_n}{n}\right)\right\} & ,\ |x|\le\frac{k_n^{1/2}}{2},\\ \psi_n^{(2)}\left(\frac{k_n^{1/2}}{2}\right) & ,\ \frac{k_n^{1/2}}{2}<x<\infty.\end{cases}$$

For each n, $(Z_{1,n}, Z_n, Z_{2,n})$ is a trivariate normal vector with covariance matrix

$$\begin{pmatrix} 1-\frac{m_n}{n} & r_{1,n} & \left(\frac{m_n k_n}{n^2}\right)^{1/2} \\ r_{1,n} & 1 & r_{2,n} \\ \left(\frac{m_n k_n}{n^2}\right)^{1/2} & r_{2,n} & 1-\frac{k_n}{n} \end{pmatrix}$$

where

$$-\left(1-\frac{m_n}{n}\right)^{1/2} \le r_{1,n} = -\left(\frac{m_n}{n}\right)^{1/2} \int_{m_n/n}^{1-k_n/n} (1-s)dQ(s) \Big/ \sigma\left(\frac{m_n}{n}, 1-\frac{k_n}{n}\right) \le 0$$

and

$$-\left(1-\frac{k_n}{n}\right)^{1/2} \le r_{2,n} = -\left(\frac{k_n}{n}\right)^{1/2} \int_{m_n/n}^{1-k_n/n} s dQ(s) \Big/ \sigma\left(\frac{m_n}{n}, 1-\frac{k_n}{n}\right) \le 0.$$

If we brake up Z_n as

$$\begin{aligned} Z_n &= -\frac{1}{\sigma(m_n/n, 1-k_n/n)} \int_{m_n/n}^{1/2} B_n(s)dQ(s) \\ &\quad - \frac{1}{\sigma(m_n/n, 1-k_n/n)} \int_{1/2}^{1-k_n/n} B_n(s)dQ(s) \\ &= W_{1,n} + W_{2,n}, \end{aligned}$$

then

$$EW_{1,n}^2 = \sigma_{1,n}^2 = \sigma^2\left(\frac{m_n}{n}, \frac{1}{2}\right) \Big/ \sigma^2\left(\frac{m_n}{n}, 1-\frac{k_n}{n}\right),$$

$$EW_{2,n}^2 = \sigma_{2,n}^2 = \sigma^2\left(\frac{1}{2}, 1-\frac{k_n}{n}\right) \Big/ \sigma^2\left(\frac{m_n}{n}, 1-\frac{k_n}{n}\right),$$

where $\sigma_{1,n}^2 + \sigma_{2,n}^2 = 1$ for each n, and it can be shown that if $EX^2 = \infty$ then the three covariances $\mathrm{Cov}(Z_{1,n}, W_{2,n})$, $\mathrm{Cov}(W_{1,n}, W_{2,n})$ and $\mathrm{Cov}(Z_{2,n}, W_{1,n})$ all converge to zero as $n \to \infty$.

This is the way we arrive at the direct half (i) of the following main result of [9], where this result is formulated somewhat differently. The proof of the converse half (ii) goes along the same line as that of the proof of the converse half in the preceding section, the last step being technically different but the same in spirit.

RESULT. (i) *Assume that there exists a subsequence $\{n'\}$ of the positive integers such that for two non-decreasing, left-continuous functions ψ_1 and ψ_2 satisfying $\psi_j(0) \leq 0$, $\psi_j(0+) \geq 0$, $j = 1, 2$ we have*

$$\psi_{n'}^{(j)}(x) \to \psi_j(x), \text{ at every continuity point } x \in \mathbf{R} \text{ of } \psi_j,\ j = 1, 2, \tag{3.4}$$

and that

$$a_{n'}/A_{n'} \to \delta < \infty, \tag{3.5}$$

where $a_n = n^{1/2}\sigma(m_n/n, 1 - k_n/n)$ and δ is some non-negative constant. If $\delta = 0$, then necessarily $\psi_1(x) = \psi_2(-x) = 0$ for all $x > 0$, and, with $\mu_n = \mu_n(m_n, n - k_n)$ given above

$$\frac{1}{A_{n'}}\left\{\sum_{j=m_{n'}+1}^{n'-k_{n'}} V_{j,n'} - \mu_{n'}\right\} \xrightarrow{\mathcal{D}} V_1 + V_2,$$

where

$$V_j = V_j(\psi_j) = (-1)^{j+1}\int_{-Z_j}^{0} \psi_j(x)dx,\ j = 1, 2,$$

where Z_1 and Z_2 are independent standard normal random variables. If $\delta > 0$, then for any subsequence $\{n''\}$ of $\{n'\}$ for which $r_{j,n''} \to r_j$, $j = 1, 2$, where $-1 \leq r_1, r_2 \leq 0$. we necessarily have $\psi_1(x) \leq -r_1$ and $\psi_2(x) \geq r_2$ for all $x \in \mathbf{R}$, and

$$\frac{1}{A_{n''}}\left\{\sum_{j=m_{n''}+1}^{n''-k_{n''}} X_{j,n''} - \mu_{n''}\right\} \xrightarrow{\mathcal{D}} V_1 + \delta Z + V_2,$$

where, with Z_1 and Z_2 figuring in V_1 and V_2, (Z_1, Z, Z_2) is s trivariate normal random vector with zero mean and covariance matrix

$$\begin{pmatrix} 1 & r_1 & 0 \\ r_1 & 1 & r_2 \\ 0 & r_2 & 1 \end{pmatrix}.$$

Moreover, if Var$(X) = \infty$ *and, if necessary,* $\{n'''\}$ *is a further subsequence of* $\{n''\}$ *such that for some positive constants* σ_1 and σ_2 *with* $\sigma_1^2 + \sigma_2^2 = 1$. $\sigma_{j,n'''} \to \sigma_j$, $j = 1, 2$, *then*

$$\frac{1}{A_{n'''}} \left\{ \sum_{j+m_{n'''}+1}^{n'''-k_{n'''}} X_{j,n'''} - \mu_{n'''} \right\} \xrightarrow{\mathcal{D}} V_1 + \delta(W_1 + W_2) + V_2,$$

where (Z_1, W_1, W_2, Z_2) *is a quadrivariate normal vector with mean zero and covariance matrix*

$$\begin{pmatrix} 1 & r_1 & 0 & 0 \\ r_1 & \sigma_1^2 & 0 & 0 \\ 0 & 0 & \sigma_2^2 & r_2 \\ 0 & 0 & r_2 & 1 \end{pmatrix}.$$

(ii) *If there exist two sequences of constants* $A_n > 0$ *and* C_n *and a sequence* $\{n'\}$ *of positive integers such that*

$$\frac{1}{A_{n'}} \left\{ \sum_{j=m_{n'}+1}^{n'-k_{n'}} X_{j,n'} - C_{n'} \right\} \tag{3.6}$$

converges in distribution to a non-degenerate limit, then there exist a subsequence $\{n''\}$ *of* $\{n'\}$ *and non-decreasing, left-continuous functions* ψ_1 *and* ψ_2 *satisfying* $\psi_j(0) \le 0$ *and* $\psi_j(0+) \ge 0$, $j = 1, 2$, *and a constant* $0 \le \delta < \infty$ *such that (3.4) and (3.5) hold true for* $A_{n''}$ *along* $\{n''\}$, *where at least one of* ψ_1 *and* ψ_2 *is not identically zero if* $\delta = 0$, *in which case* $\psi_1(x) = \psi_2(-x) = 0$ *for all* $x > 0$. *The limiting random variable of the sequence in (3.6) is necessarily of the form* $V_1 + \delta Z + V_2 + d$, *with* $V_1 . Z$ *and* V_2 *described above, where*

$$d = \lim_{n''' \to \infty} (\mu_{n'''} - C_{n'''})/A_{n'''}$$

for some subsequence $\{n'''\} \subset \{n''\}$.

This result implies as a corollary (by showing that the possible limits can only be normal if $\psi_1 \equiv 0 \equiv \psi_2$) that the sequence in (3.6) converges in distribution to a standard normal random variable if and only if $\varphi_{n'}^{(j)}(x) \to 0$, as $n' \to \infty$, for all $x \in \mathbf{R}$, $j = 1, 2$.

In this case $A_{n'}$ can be chosen to be $a_{n'}$ and $C_{n'}$ to be $\mu_{n'}$. Here $\{n'\}$ is arbitrary and can of course be $\{n\}$.

For a discussion of the conditions (3.4) and (3.5) we refer to the original paper [9] and [31]. Subsequent to [9], Griffin and Pruitt [27] used the more classical characteristic function methodology to prove mathematically equivalent versions of the above results and showed that all possible subsequential limits indeed arise.. In another paper, [26], they deal with the analogous problem of moderately trimmed sums when $k_n (k_n \to \infty,\ k_n/n \to 0)$ of the summands largest in absolute value are discarded at each step (see also [51]), assuming that the underlying distribution is symmetric about zero. (As a demonstration of the present approach, a part of their results is rederived in [12].) A comparison of the two sets of results shows that (perhaps contrary to intuition) even if we assume symmetry and $m_k = k_n$ above, the two trimming problems are wholly different. For further results on various interesting versions of moderate trimming see Kuelbs and Ledoux [38], Hahn, Kuelbs and Samur [33], Hahn and Kuelbs [32], Hahn, Kuelbs and Weiner [34,35] and Maller [44].

Finally, we turn to heavy trimming assuming instead of (3.1) that

$$m_n = [n\alpha] \quad \text{and} \quad k_n = n - [\beta n],\ 0 < \alpha < \beta < 1. \tag{3.7}$$

This is the case of the classical trimmed sum, for which Stigler [54] completely solved the problem of asymptotic distribution. Suppose that $\sigma(\alpha, \beta) > 0$, where $\sigma(\cdot, \cdot)$ is as in (3.3). The proof sketched above produces the following version of Stigler's theorem (Theorem 5 in [9]), where a_n and μ_n are defined in terms of the present m_n and k_n: For any underlying distribution,

$$\frac{1}{a_n}\left\{\sum_{j=[n\alpha]+1}^{[n\beta]} X_{j,n} - \mu_n\right\} \xrightarrow{\mathcal{D}} V_1(\psi_1) + Z + V_2(\psi_2),$$

as $n \to \infty$, where

$$\psi_1(x) = \begin{cases} 0 & , x \le 0, \\ \frac{\sqrt{\alpha}}{\sigma(\alpha,\beta)}(Q(\alpha+) - Q(\alpha)) & , x > 0, \end{cases}$$

and

$$\psi_2(x) = \begin{cases} 0 & , \quad x \le 0, \\ \frac{\sqrt{1-\beta}}{\sigma(\alpha,\beta)}(Q(\beta+) - Q(\beta)) & , \quad x > 0, \end{cases}$$

so that

$$V_1(\psi_1) = \frac{\sqrt{\alpha}}{\sigma(\alpha,\beta)}(Q(\alpha+) - Q(\alpha)) \min(0, Z_1)$$

and

$$V_2(\psi_2) = \frac{\sqrt{1-\beta}}{\sigma(\alpha,\beta)}(Q(\beta+) - Q(\beta)) \max(0, -Z_2),$$

where (Z_1, Z, Z_2) is a trivariate normal random vector with mean zero and covariance matrix

$$\begin{pmatrix} 1-\alpha & r_1 & (\alpha(1-\beta))^{1/2} \\ r_1 & 1 & r_2 \\ (\alpha(1-\beta))^{1/2} & r_2 & \beta \end{pmatrix},$$

where

$$r_1 = -\sqrt{\alpha} \int_\alpha^\beta (1-s)dQ(s) \text{ and } r_2 = -\sqrt{1-\beta} \int_\alpha^\beta s dQ(s).$$

This version puts Stigler's theorem (giving asymptotic normality if and only if Q is continuous both at α and β) into a broader picture. Substituting α and β for m_n/n and $1 - k_n/n$ in the arguments of $\psi_n^{(1)}$ and $\psi_n^{(2)}$, the proof also works for more general m_n and k_n sequences, provided $\sqrt{n}(m_n/n - \alpha) \to 0$ and $\sqrt{n}(1 - k_n/n - \beta) \to 0$ as $n \to \infty$.

For rates of convergence in Stigler's theorem in the case when the limit is normal, see Egorov and Nevzorov [22], who in [23] also investigate the related problem in the case of magnitude trimming.

4. Extreme sums [11]

Here we are interested in the sums of extreme values

$$E_n = E_n(k_n) = \sum_{j=1}^{k_n} X_{n+1-j,n} = \sum_{j=n-k_n+1}^{n} X_{j,n},$$

where $k_n \to \infty$ and either $k_n/n \to 0$ as $n \to \infty$, or $k_n = [n\alpha]$ with $0 < \alpha < 1$. (We shall refer to the first case when $k_n/n \to 0$ as the case $\alpha = 0$.) It is more convenient to work with the function

$$H(s) = -Q((1-s)-), \ 0 \le s < 1,$$

instead of Q itself, for which we have

$$(X_{1,n}, \ldots, X_{n,n}) \stackrel{\mathcal{D}}{=} (-H(U_{n,n}), \ldots, -H(U_{1,n}))$$

instead of (1.2). The natural centering sequence now turns out to be

$$\mu_n = \mu_n(k_n) = -n \int_{1/n}^{k_n/n} H(s)ds - H\left(\frac{1}{n}\right),$$

and with a potential normalizing sequence $A_n > 0$ the role of (1.4) or (3.2) is taken over by the decomposition

$$\frac{1}{A_n}\{E_n - \mu_n\} \stackrel{\mathcal{D}}{=} \Delta_n^{(1)}(m_n) + \Delta_n^{(2)}(m_n, l_n) + \Delta_n^{(3)}(l_n, k_n),$$

where $1 \le m_n \le l_n \le k_n$, $A_n > 0$, and

$$\begin{aligned}
\Delta_n^{(1)}(m_n) = &\int_{nU_{1,n}}^{m_n} \left(nG_n\left(\frac{u}{n}\right) - u\right) d\frac{H(u/n) - H(1/n)}{A_n} \\
&+ \int_{nU_{1,n}}^{1} (u-1) d\frac{H(u/n) - H(1/n)}{A_n} - \frac{H(nU_{1,n}/n) - H(1/n)}{A_n}, \\
\Delta_n^{(2)}(m_n, l_n) = &\int_{m_n}^{l_n} \left(nG_n\left(\frac{u}{n}\right) - u\right) d\frac{H(u/n) - H(1/n)}{A_n},
\end{aligned}$$

and

$$\Delta_n^{(3)}(l_n, k_n) = \int_{l_n/n}^{k_n/n} n(G_n(u) - u) d\frac{H(u)}{A_n} + \int_{k_n/n}^{U_{k_n,n}} n(G_n(u) - u) d\frac{H(u)}{A_n}.$$

The numbers m_n and l_n (not necessarily integers) will be appropriately chosen such that $m_n \to \infty$, $l_n/m_n \to \infty$ and $k_n/l_n \to \infty$ as $n \to \infty$, and we see that the term $\Delta_n^{(1)}$ is very similar to $V_0^{(1)}(l,n)$ in Section 2, while the term $\Delta_n^{(3)}$ presents a "trimmed-sum problem" considered in Section 4 with only one R_n-like term. Therefore, we need both φ_n and ψ_n type functions, which are presently defined as

$$\varphi_n(s) = \begin{cases} \frac{1}{A_n}\left\{H\left(\frac{s}{n}\right) - H\left(\frac{1}{n}\right)\right\} & , 0 < s \le n - n\alpha_n, \\ \frac{1}{A_n}\left\{H(1-\alpha_n) - H\left(\frac{1}{n}\right)\right\} & , n - n\alpha_n < s < \infty, \end{cases}$$

where α_n is as in Section 2, and

$$\psi_n(x) = \begin{cases} \psi_n\left(-\frac{k_n^{1/2}}{2}\right) & , -\infty < x < -\frac{k_n^{1/2}}{2}, \\ \frac{k_n^{1/2}}{A_n}\left\{H\left(\frac{k_n}{n} + x\frac{k_n^{1/2}}{n}\right) - H\left(\frac{k_n}{n}\right)\right\} & , |x| \le \frac{k_n^{1/2}}{2}, \\ \psi_n\left(\frac{k_n^{1/2}}{2}\right) & , \frac{k_n^{1/2}}{2} < x < \infty. \end{cases}$$

Again, the middle term $\Delta_n^{(2)}$ turns out to be a "sanitary cordon" converging to zero in probability, and redefining

$$\sigma^2(s,t) = \int_s^t \int_s^t (\min(u,v) - uv)dH(u)dH(v), \ 0 \le s \le t \le 1,$$

and

$$a_n = \begin{cases} n^{1/2}\sigma(1/n, k_n/n) & , \text{if } \sigma(1/n, k_n/n) > 0, \\ n^{1/2} & , \text{otherwise}, \end{cases}$$

and introducing

$$0 \le r_n = \left(\frac{n}{k_n}\right) \frac{1 - k_n/n}{\sigma(l_n/n, k_n/n)} \int_{l_n/n}^{k_n/n} s dH(s) \le \left(1 - \frac{k_n}{n}\right)^{1/2}$$

and

$$N(t) = \sum_{k=1}^{\infty} I(S_k^{(1)} \le t), \ 0 \le t < \infty,$$

the right-continuous version of the Poisson process $N_1(\cdot)$ in Sections 1 and 2, the proof of the following main result (Theorems 1 and 2 in [11]) is obtained by an involved combination of the techniques of [9] and [10], i.e., those of the preceding two sections.

RESULT. (i) *Assume that there exist a subsequence $\{n'\}$ of the positive integers, a left-continuous, non-decreasing function φ defined on $(0,\infty)$ with $\varphi(1) \le 0$ and $\varphi(1+) \ge 0$, a left- continuous, non-decreasing function ψ defined on $(-\infty,\infty)$ with $\psi(0) \le 0$ and $\psi(0+) \ge 0$, and a constant $0 \le \delta < \infty$ such that, as $n' \to \infty$,*

$$\varphi_{n'}(s) \to \varphi(s) \quad \text{at every continuity point} \quad s \in (0,\infty) \text{ of } \varphi, \tag{4.1}$$

$$\psi_{n'}(x) \to \psi(x) \quad \text{at every continuity point} \quad x \in \mathbf{R} \text{ of } \psi, \tag{4.2}$$

$$a_{n'}/A_{n'} \to \delta. \tag{4.3}$$

Then, necessarily, $\varphi(s) \le \delta$ for all $s \in (0,\infty)$,

$$\int_{\varepsilon}^{\infty} (\varphi(s) - \varphi(\infty))^2 ds < \infty \quad \text{for all} \quad \varepsilon > 0, \tag{4.4}$$

and there exist a subsequence $\{n''\} \subset \{n'\}$ and a sequence of positive numbers $l_{n''}$ satisfying $l_{n''} \to \infty$ and $l_{n''}/k_{n''} \to 0$, as $n'' \to \infty$, such that either $\sigma(l_{n''}/n'',\ k_{n''}/n'') > 0$ for all n'', in which case for some $0 \le b \le \delta$ and $0 \le r \le (1-\alpha)^{1/2}$, $\sqrt{n''}\sigma(l_{n''}/n'', k_{n''}/n'')/A_{n''} \to b$, $r_{n''} \to r$, or $\sigma(l_{n''}/n'',\ k_{n''}/n'') = 0$ for all n'', in which case we put $b = r = 0$, and

$$\frac{1}{A_{n''}} \left\{ \sum_{j=1}^{k_{n''}} X_{n''+1-j.n''} - \mu_{n''} \right\} \xrightarrow{\mathcal{D}} V(\varphi, \psi, b, r, \alpha)$$

as $n'' \to \infty$, where α is zero or positive according to the two cases, ψ necessarily satisfies

$$\psi(x) \ge -\delta r/(1-\alpha), \ -\infty < x < \infty, \tag{4.5}$$

and

$$V(\varphi,\psi,b,r,\alpha) = \int_1^{\infty} (N(t)-t)d\varphi(t) + \int_0^1 N(t)d\varphi(t) + bZ_1 + \int_{-Z(r,\alpha)}^{0} \psi(x)dx.$$

where $Z(r,\alpha) = -rZ_1 + (1-\alpha-r^2)^{1/2}Z_2$, where Z_1 and Z_2 are standard normal random variables such that Z_1, Z_2 and $N(\cdot)$ are independent. Moreover, if $\varphi \equiv 0$ then $b = \delta$, while if $\delta = 0$ then $\varphi(s) = 0$ for all $s \geq 1$.

(ii) If there exist a subsequence $\{n'\}$ of the positive integers and two sequences $A_{n'} > 0$ and $C_{n'}$ along it such that

$$\frac{1}{A_{n'}}\left\{\sum_{j=1}^{k_{n'}} X_{n'+1-j,n'} - C_{n'}\right\} \tag{4.6}$$

converges in distribution to a non-degenerate limit, then there exists a subsequence $\{n''\} \subset \{n'\}$ such that conditions (4.1), (4.2) and (4.3) hold along the sequence $\{n''\}$ for $A_{n''}$ in (4.6) and for appropriate functions φ and ψ with the properties listed above (4.1) and for some constant $0 \leq \delta < \infty$, with φ satisfying (4.4) and ψ satisfying (4.5) with an $r \in (0,(1-\alpha)^{1/2})$ arising along a possible further subsequence. The limiting random variable of the sequence in (4.6) is necessarily of the form $V(\varphi,\psi,b,r,\alpha)+d$ for appropriate constants $0 \leq b \leq \delta$, $0 \leq r \leq (1-\alpha)^{1/2}$ and

$$d = \lim_{n''' \to \infty} (\mu_{n'''} - C_{n'''})/A_{n'''},$$

for some subsequence $\{n'''\} \subset \{n''\}$. Moreover, either $\varphi \not\equiv 0$ or $\psi \not\equiv 0$ or $b > 0$.

Just as in the case of full sums in Section 2, it is possible to see the effect on the limiting distribution of deleting a finite number $k \geq 0$ of the largest summands from the extreme sums $E_n(k_n)$. Replacing $E_n(k_n)$ by $\sum_{j=n-k_n+1}^{n-k} X_{j,n}$, μ_n by

$$\mu_n(k) = -n\int_{(k+1)/n}^{k_n/n} H(u)du - H\left(\frac{k+1}{n}\right),$$

and the first two integrals in $V(\varphi,\psi,b,r,\alpha)$ by

$$\int_{S_{k+1}^{(1)}}^{\infty} (N(t)-t)d\varphi(t) - \int_1^{S_{k+1}^{(1)}} td\varphi(t) + k\varphi(S_{k+1}^{(1)}) - \int_1^{k+1} \varphi(t)dt,$$

the result above remains true word for word.

The result can be formulated for the sum of lower extremes $\sum_{j=1}^{m_n} X_{j,n}$, where $m_n \to \infty$ and $m_n/n \to 0$ or $m_n = [n\beta]$ with $0 < \beta < 1$. The limiting random variable is of the form $-V(\varphi, \psi, b, r, \beta)$ with appropriate ingredients. In fact, if at least one of α and β is zero then the two convergence statements hold jointly with the limiting random variables being independent.

One corollary of the result above is that the sequence in (4.6) converges in distribution to a non-degenerate normal variable if and only if (4.1) and (4.2) are satisfied with $A_{n'} \equiv a_{n'}$, $\varphi \equiv 0$ and $\psi \equiv 0$, in which case, choosing $A_{n'} \equiv a_{n'}$ and $C_{n'} \equiv \mu_{n'}$ in (4.6) the limit is standard normal.

Another exhaustive corollary is the convergence in distribution of $(E_n - \mu_n)/a_n$, along the whole sequence $\{n\}$, when the underlying distribution is in the domain of one of the three possible limiting extremal distributions in the sense of extreme value theory. The details are contained in Corollary 2 in [11], being a common generalization of results from [13], [15] and [40] mentioned in the introduction.

5. Related results

Here we mention some closely related developments obtained by the same probabilistic approach. References are given only if they are directly relevant to this approach, further references can be found in the cited papers.

5.1 A generalization: L-statistics. Mason and Shorack [46,47,48] consider linear combinations of order statistics of the form

$$\sum_{j=m_n+1}^{n-k_n} c_{jn} X_{j,n} \stackrel{\mathcal{D}}{=} \sum_{j=m_n+1}^{n-k_n} c_{jn} Q(U_{j,n}),$$

or more generally

$$T_n^\star = \sum_{j=m_n+1}^{n-k_n} c_{jn} g(U_{j,n}),$$

where g is some function and

$$c_{jn} = n \int_{(j-1)/n}^{j/n} J(t)dt, \ 1 \leq j \leq n,$$

with some function J regularly varying at 0 and 1. In the moderate trimming case they use a reduction principle showing that the asymptotic distribution problem for T_n^* is the same as that for

$$\overline{T}_n = \sum_{j=m_n+1}^{n-k_n} K(U_{j,n}),$$

where K, as a measure, is defined by $dK = Jdg$. Thus, under certain conditions on g and J, in [48] they obtain results parallel to those in [9] sketched in Section 3 above. Even for fixed (light) trimming (or no trimming), in [46,47] they still obtain a theory parallel to that in [10], sketched in Section 2 above.

5.2. Extreme and self-normalized sums in the domain of attraction of a stable law. The paper [6] gives a unified theory of such sums based on the preliminary results in [4]. (A somewhat incomplete such theory was given earlier by LePage, Wordroofe and Zinn [39].) The idea is that properly centered whole sums $\sum_{j=1}^{n} Q(U_j)$ and the individual extrems $Q(U_{1,n}), \ldots, Q(U_{m,n}), Q(U_{n-k,n}, \ldots, Q(U_{n,n})$ converge jointly with the same normalizing factor. This is trivial in our approach, where convergence is in fact in probability. Paralleling a result of Hall [36], an approximation of an arbitrary stable law by suitably centered sums being the asymptotic representations of the sum of a finite number of extremes is given. (For a generalization, see the next subsection.) The self-normalized sums are those considered by Logan, Mallows, Rice and Shepp [41]: $L_n(p) = (\sum_{j=1}^{n} X_i)/(\sum_{j=1}^{n} |X_j|^p)^{1/p}$. The emphasis in [6] concerning this is the investigation of the properties of the limiting distribution using a representation arising out of our approach. The extremes have a definite role in these properties. In [16], we used our quantile approach to prove half of a conjecture in [41] stating that if F is in the domain of attraction of a normal law and $EX = 0$, then $L_n(2) \xrightarrow{\mathcal{D}} N(0,1)$. This was proved earlier by Maller [42]. The converse half of the conjecture is still open. For further results see [35].

5.3. An "extreme-sum" approximation of infinitely divisible laws without a normal component. Given an infinitely divisible distribution by its characteristic function of the form of the right-side of (2.10) with γ replaced by a general constant θ, where L and R are left- and right-continuous Lévy measures, respectively, so that $L(-\infty) = 0 = R(\infty)$ and

$$\int_{-\varepsilon}^{0} x^2 dL(x) + \int_{0}^{\varepsilon} x^2 dR(x) < \infty \quad \text{for any} \quad \varepsilon > 0,$$

we see that forming $\varphi_1(s) = \inf\{x < 0 : L(x) > s\}$, $0 < s < \infty$ and $\varphi_2(s) = \inf\{x < 0 : -R(-x) > s\}$, $0 < s < \infty$, so that (2.2) is satisfied, the random variable $V_{0,0} + \theta - \gamma$ has the given infinitely divisible distribution. From now on suppose that $\rho = 0$, and let $F_0(\cdot) = F_0(\varphi_1, \varphi_2, \theta; \cdot)$ be the distribution function of $V_{0,0} + \theta - \gamma$. The approach sketched in Section 2 implies that under the said conditions

$$\frac{1}{A_n}\left(\left(\sum_{j=1}^{n} X_j - \mu_n(1, n-1)\right), \sum_{j=1}^{m} X_{j,n} + \sum_{j=n-k+1}^{n} X_{j,n}\right)$$

converges in distribution along some subsequence to

$$\left(V_{0,0}, \sum_{j=1}^{m} \varphi_1(S_j^{(1)}) - \sum_{j=1}^{k} \varphi_2(S_j^{(2)})\right).$$

So the second component here represents the asymptotic contribution of the extrems in the limiting infinitely divisible law of the full sum. Hence it is conceivable that a suitably centered form of this second component can approximate now $V_{0,0}$ if $m, k \to \infty$.

Let $L_{m,k}$ be the Lévy distance between F_0 and the distribution function of

$$\sum_{j=1}^{m} \varphi_1(S_j^{(1)}) - \sum_{j=1}^{k} \varphi_2(S_j^{(2)}) - \left(\int_1^m \varphi_1(s)ds - \int_1^k \varphi_2(s)ds\right) + \theta - \gamma.$$

Then it is shown in [7] that $L_{m,k} \to 0$ as $m, k \to \infty$, and, depending on how fast $\varphi_1(s)$ and $\varphi_2(s)$ converge up to zero as $s \to \infty$, rates of this convergence are also provided which rates are sometimes amazingly fast. In the special case of a stable distribution with exponent $0 < \alpha < 2$, given by $\varphi_j(s) = -c_j s^{-1/\alpha}$, $s > 0$, $c_1, c_2 \geq 0$, $c_1 + c_2 > 0$, when $L_{m,k}$ can be replaced by the supremum distance $K_{m,k}$, we obtain

$$K_{m,k} = o\left(\max\left(c_1 m^{-\varepsilon\left(\frac{1}{\alpha}-\frac{1}{2}\right)}, c_2 k^{-\varepsilon\left(\frac{1}{\alpha}-\frac{1}{2}\right)}\right)\right) \text{ as } m, k \to \infty,$$

where $0 < \varepsilon < 1$ is as close to 1 as we wish.

5.4. Almost sure behaviour: stability and the law of the iterated logarithm. Since this topic is so broad that it could be reviewed only in a separate survey, here we restrict ourselves to mention that Haeusler and Mason [29,30,31], Einmahl, Haeusler and Mason [24] and Haeusler [28] use the quantile transform - empirical process method to investigate the almost sure behaviour of sums considered in [13] and [15], and when the underlying distribution has a slowly varying tail. Here the basic approach should be combined with techniques which go back to Kiefer [37] and Csáki [2] and some further strong developments such as Deheuvels [20]. The same method is used to prove a universal liminf law by Mason [45].

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