ON THE CENTRAL LIMIT THEOREMS
FOR THE RENEWAL AND CUMULATIVE PROCESSES

by

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In our dissertation we derive the Central Limit Theorems (CLTs) with the remainder terms for the renewal and cumulative processes. A renewal process based on the sequence of random variables \( \{X_n\} \) is defined in one of the following ways:

1) \( N_1(t) = \sup\{n: S_j \leq t, j = 1,2,\ldots,n\} \),
2) \( N_2(t) = \text{number of } n \text{ such that } S_n \leq t \),
3) \( N_3(t) = \sup\{n: S_n \leq t\} \),

where \( S_n = X_1 + \ldots + X_n \). If \( X_n > 0 \) then all three definitions are identical. In this case the renewal process is denoted \( N(t) \).

It is well-known that a CLT for the \( S_n \)'s allows an expansion if \( E|X_1|^k < \infty \) for some \( k \geq 3 \). The order of the remainder term in the CLT depends on \( k \). We prove a similar CLT for \( N(t) \) when the \( X_n \)'s are i.i.d. and positive. Next we allow the \( X_n \)'s to have different distributions and to take negative values. We prove a CLT with the remainder term of the order of \( O\left(\frac{1}{n^{1/3}}\right) \) for each \( N_i(t) \), \( i = 1,2,3 \), when the \( X_n \)'s have the uniformly bounded third moments and satisfy other "natural" conditions.

When all \( X_n \)'s are i.i.d. and positive, a cumulative process can be defined as \( W(t) = \sum_{j=1}^{N(t)+1} Y_j \), where \( Y_1, Y_2, \ldots \) are such random variables that pairs \( (X_1,Y_1), (X_2,Y_2), \ldots \) are i.i.d. Under certain assumptions on the \( X \)'s and \( Y \)'s, a CLT is obtained for \( W(t) \). The order of the remainder term depends on the finiteness of the absolute moments of the \( Y \)'s. An even better estimate is found for an average error in the CLT approximations for \( W(t) \).
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CHAPTER I
INTRODUCTION

1.1. OVERVIEW

A well-known Central Limit Theorem (CLT) for the independent, identically distributed (i.i.d.) random variables $X_1, X_2, \ldots$ requires only the existence of their second moments. It would be natural to investigate if an improvement could be made when more information is available about the random variables. It has been known for many years that when the $k^{th}$ moment of the $X$'s is finite, the CLT allows an asymptotic expansion with the error term of the order of $1/n^k$.

Similar results were obtained for the independent non-identically distributed random variables. We will state the precise theorems in Chapter II.

It would be interesting to obtain such results for the renewal processes. A renewal process based on a sequence of random variables $\{X_n\}_{n=1}^{\infty}$ is defined for $t > 0$ in one of the following ways:

1) $N_1(t) = \text{Sup}\{n: S_j \leq t, \ j = 1,2,\ldots,n\},$

2) $N_2(t) = \text{number of } n \text{ such that } S_n \leq t,$

3) $N_3(t) = \text{Sup}\{n: S_n \leq t\},$

where $S_n = X_1 + X_2 + \ldots + X_n.$

It is clear that when $X_n > 0$, all three definitions are identical. In this case we will use the $N(t)$ notation to describe the renewal process.

When $X_n$'s are i.i.d. positive random variables with a finite mean $\mu$ and variance $\sigma^2 > 0$, the Central Limit Theorem for the renewal process associated
with these $X_n$'s can be derived with relative ease from the classical CLT for the sequences of random variables. The situation becomes much more complicated when the random variables are not identically distributed and, especially, when they take negative values. Our objective is to bring the theory of the Central Limit Theorems with the remainder terms for the renewal processes to the same level where our knowledge of the CLT's for the sequences of random variables is.

Another area of interest is a cumulative process $W(t)$. Here is its formal definition. Let $\{(X_n, Y_n)\}, n = 1, 2, \ldots$, be a sequence of the i.i.d. bivariate random variables with a known joint distribution function $F(x,y)$. Suppose that the $X_n$'s are strictly positive with finite mean $\mu$ and variance $\sigma^2 > 0$ and that the $Y_n$'s have finite mean $\lambda$ and variance $\tau^2 > 0$. Then

$$W(t) = \sum_{j=1}^{N(t)+1} Y_j$$

is a cumulative process. This process is known as a type A cumulative process. A type B cumulative process is defined almost like $W(t)$ in (1.1.2) except that the summation ends at $j = N(t)$. We will be interested in a type A process only.

Smith (1955) proved an analogue of the CLT for $W(t)$ assuming only that $X_1$ and $Y_1$ have finite second moments. We will consider possible generalizations of Smith's CLT when more is known about $X$'s and $Y$'s. Again, our goal is to obtain the improvements in the Central Limit Theorem for the cumulative processes similar to those obtained in the CLT for the sequences of random variables.

1.2. SUMMARY

In Chapter II we put in a proper perspective the existing Central Limit Theorems for the sequences of independent random variables, the renewal and
cumulative processes. Our investigation shows that much more advanced CLTs were proved for the sequences of random variables than for the processes defined in (1.1.1) and (1.1.2). That provides the motivation to learn more about the renewal and cumulative processes and to prove more sophisticated Central Limit Theorems for them. We also present some other limit theorems discussed in recent papers on this subject.

In Chapter III we concentrate on the renewal processes. At first we consider a renewal process $N(t)$ based on the independent identically distributed positive (a.s.) $X_n$'s with the $r$th finite absolute moment ($r \geq 3$). We obtain a Central Limit Theorem for $N(t)$ with the remainder term of the same order as one would get in the CLT for the sequence of random variables with the same condition on the finiteness of their moments. The term that approximates a distribution function of the (normalized) random variables $N(t)$ is more complex than that used in the classical Central Limit Theorem for the sequences of independent random variables. We give an explicit formula for this term.

Next we turn to the $X_n$'s that are not identically distributed and that may take negative values, thus making all three definitions in (1.1.1) different. By imposing certain requirements on the first three moments of the $X_n$'s we obtain (in Section 3.3) for $N_1(t)$, $N_2(t)$ and $N_3(t)$ a Central Limit Theorem with the remainder term. Prior to that, in Section 3.2, we prove several lemmas that allow us to extend the theorem to cover $N_2(t)$ and $N_3(t)$, once it is proved for $N_1(t)$. In Section 3.4 we investigate the necessity of the assumptions in the Central Limit Theorem.

Chapter IV is dedicated to the cumulative process. We impose the necessary assumptions on the distributions of $X$'s and $Y$'s and prove (in Sections 4.2 - 4.5) that these assumptions lead to certain properties that $X$'s and $Y$'s satisfy and that are critical in the proof of the Central Limit Theorems.
In Sections 4.6 and 4.7 we prove several different CLTs for $W(t)$. The size of the remainder term depends on the order of the finite absolute moment of $Y_1$ (and hence of any $Y_n$).

Our next step is to estimate an average (over all possible values of $t$) error in the Central Limit Theorem approximations for $W(t)$. Such results do not follow immediately from the estimates of the remainder terms obtained earlier in that chapter. We end Chapter IV with a discussion of possible directions for the future research in this area.

The most common application of a cumulative process is when a machine is repaired at time zero at the cost of $Y_1$, then it runs for the $X_1$ units of time, then gets repaired at the cost of $Y_2$, etc. Random variables $X_j$ and $Y_j$ may be dependent. Then $W(t)$ is the total cost of repairs by time $t$.

If we considered a type B process instead then the machine would first run for time $X_1$, then get repaired at the cost of $Y_1$, and so on.
CHAPTER II
BACKGROUND

2.1. INTRODUCTION

In this chapter we review the literature on the Central Limit Theorem with the remainder term for the sequences of independent random variables, for the renewal processes and for the cumulative processes. Most of the research previously done concentrated on the limiting distributions of the sums of independent random variables. We describe those results in Section 2.2.

The first systematic approach to this problem was developed by Cramér in his 1937 book. Later, Gnedenko and Kolmogorov (1954) wrote a monograph with a much deeper analysis of the Central Limit Theorem and other limit theorems for the sequences of independent random variables. By that time it was noticed (Feller, 1949) that similar techniques could be applied to obtain the CLTs for the renewal processes. In the 1950's Smith obtained for the first time a Central Limit Theorem for the Cumulative Processes. In the 1960's and 1970's much more has become known about the limiting distributions for the sequences of random variables. Petrov's book (1975) provides an excellent summary of these results. Among other things, Petrov and others developed the estimates for the probabilities of large deviations; these results will help us to obtain a Central Limit Theorem for the Cumulative Processes.

In the 1980's, a Central Limit Theorem with a remainder term was proved for the renewal processes (see Ahmad, 1981, and also Niculescu and Omey, 1985). These papers, however, contain errors. The results stated there are not true; they
only hold with more restricted conclusions than the authors suggest. In Section 2.3 we give a more detailed description of these papers and in Chapter 3 we prove much more general CLT's for the renewal processes which include those stated by Ahmad, Niculescu and Omey as special cases.

After more than 30 years since Smith's original paper on the CLT for the cumulative processes was published, there was another publication on this subject. Murphee and Smith (1986) proved a CLT for a so-called transient cumulative process. See Section 2.4 for details.

2.2. SUMS OF INDEPENDENT RANDOM VARIABLES

The very first result improving the Central Limit Theorem for the sums of independent, identically distributed random variables with the finite higher moments was apparently obtained by Cramér (see his book, 1937) who showed that if $EX_1 = 0$, $E|X_1|^k < \infty$ ($k \geq 3$) and the characteristic function $\phi$ of $X_1$ satisfies the following (C) condition

$$(2.2.1) \quad \limsup_{|\theta| \to \infty} |\phi(\theta)| < 1,$$

then

$$(2.2.2) \quad \left\| P \left\{ \frac{S_n}{\sigma \sqrt{n}} \leq x \right\} - \Phi(x) - \sum_{\nu=1}^{k-3} \frac{Q_{\nu}(x)}{n^{\nu/2}} \right\| = O\left(\frac{1}{n^{(k-2)/2}}\right),$$

where $S_n = X_1 + \ldots + X_n$. Here $\Phi(x)$ is the distribution function of a standard normal random variable and $\|A(x)\|$ is defined for any function $A(x)$ as

$$\sup_{-\infty < x < \infty} |A(x)|.$$ Functions $Q_{\nu}(x)$ are explicitly defined and can be expressed in terms of the Hermite polynomials of $x$ and the cumulant moments of $X_1$. See Appendix A for the precise definitions of functions $Q_{\nu}(x)$.

Formula (2.2.2) is often referred to as the 'Edgeworth expansion'. In
particular, for \( k = 3 \), (2.2.2) means that if the third moment of \( X_1 \) is finite then
\[
\| P \left\{ \frac{S_n}{\sigma \sqrt{n}} \leq x \right\} - \Phi(x) \| = O(\frac{1}{\sqrt{n}}).
\]

Various modifications of (2.2.2) were discussed by Gnedenko and Kolmogorov (1954). Ibragimov (1967) addressed the issue of not only the sufficient but also the necessary conditions for equalities like (2.2.2) to hold. He also considered a case of fractional moments and showed that if \( EX_1 = 0 \) and condition (2.2.1) holds then for \( 0 < \delta < 1 \),
\[
(2.2.3) \quad \| P \left\{ \frac{S_n}{\sigma \sqrt{n}} \leq x \right\} - \Phi(x) - \sum_{\nu=1}^{k-2} \frac{Q_{\nu}(x)}{n^{\nu/2}} \| = O\left(\frac{1}{n^{(k-2+\delta)/2}}\right),
\]
if and only if \( \int_{|x|>z} |x|^k dF(x) = O(z^{-\delta}) \), where \( F \) is the distribution function of \( X_1 \).

Deeper results for the case of the i.i.d. \( X_n \)'s were obtained by Osipov (1967, 1969, 1972). He considered a further improvement in the quality of the estimator of the remainder term when the value of variable \( x \) in (2.2.2) is taken into account. Here is one of his theorems (not the most general one).

**Theorem 2.2.1 (Osipov).**

Suppose that (2.2.1) holds, \( EX_1 = 0 \) and \( E|X_1|^k \leq \infty \) for some integer \( k \geq 3 \). Then
\[
(2.2.4) \quad \| (1 + |x|)^k \{ P \left\{ \frac{S_n}{\sigma \sqrt{n}} \leq x \right\} - \Phi(x) - \sum_{\nu=1}^{k-2} \frac{Q_{\nu}(x)}{n^{\nu/2}} \} \| = o\left(\frac{1}{n^{(k-2)/2}}\right).
\]

In addition, Osipov obtained an estimate for the expression in the left hand side of (2.2.4) when the (2.2.1) condition does not hold. That last estimate is non-trivial only when the value of \( |x| \) is very large.

It is important to note that if the theorems mentioned above were stated for \( k = 3 \) only, they would not require the (2.2.1) condition to hold. This is related to
the method of proof, where some estimates of the characteristic functions $\phi_n(\theta)$ of certain random variables must be obtained. The method of estimation varies depending on the value of the parameter $\theta$. It turns out, however, that for $k = 3$ there are no values of $\theta$ such that the estimation of $\phi_n(\theta)$ requires the (2.2.1) condition to hold. That will be the case in our research also, even though our methods are different from those used by Cramér, Ibragimov and Osipov and the problems we consider are different from those considered by them. In Chapter III, the (2.2.1) condition will be required for the Edgeworth expansion, but will not be necessary in our analysis of the renewal process with the underlying random variables all having a finite third absolute moment. In Chapter IV, assumption (2.2.1) in the Central Limit Theorems for the cumulative processes is not necessary to achieve the required precision in the remainder term since we do not assume the existence of the absolute moment of $Y_1$ of the order higher than 3.

Another way to generalize (2.2.2) is to allow the independent random variables $X_1, X_2, \ldots$ to have different distributions. This problem was confronted by several authors including Petrov (1960), Statulevičius (1965), Bikelis (1966) and Pipiras (1970).

For our purposes it is sufficient to present here the following result due to Bikelis (1966).

**Theorem 2.2.2 (Bikelis).**

Let $\{X_n\}$, $n=1,2,\ldots$, be a sequence of independent random variables with zero means and finite absolute third moments. Denote $B_n^2 = \frac{1}{n} \sum_{j=1}^n \sigma_j^2$, where $\sigma_j^2 = \text{Var}X_j$. Then there exists an absolute constant $C$ such that for all $n \geq 1$,

\begin{equation}
(2.2.5) \quad \left\| (1+|x|)^3 \left\{ \mathbb{P}\left\{ \frac{S_n}{B_n} \leq x \right\} - \Phi(x) \right\} \right\| \leq \frac{C \sum_{j=1}^n \mathbb{E}|X_j|^3}{B_n^{3/2}}.
\end{equation}
Petrov (1960) considered a case of \( X_n \)'s having the \( k^{th} \) finite absolute moments. He proved that under certain conditions on the characteristic functions of \( X_n \)'s, the following equality holds:

\[
(2.2.6) \quad \left\| \frac{d^k}{dx^k} \{ P \{ \frac{S_n}{B_n} \leq x \} - \Phi(x) - \sum_{\nu=1}^{k-2} \frac{Q_{\nu n}(x)}{n^{\nu/2}} \} \right\| = o\left( \frac{1}{n^{(k-2)/2}} \right),
\]

where \( \ell = 0, 1, \ldots, L \) and the value of \( L \geq 0 \) depends upon the behavior of the characteristic functions of \( X_1, \ldots, X_n \). Functions \( Q_{\nu n}(x) \) are defined similarly to functions \( Q_{\nu}(x) \). See Petrov's book (1975) for detail.

Note that, unlike (2.2.5), the estimate in (2.2.6) does not take into account the size of \( x \). As far as we know, nobody has been able to combine (2.2.5) and (2.2.6) even for \( L = 0 \), i.e., there is no known estimate for

\[
\left\| (1 + |x|)^k \left\{ P \left\{ \frac{S_n}{\sigma \sqrt{n}} \leq x \right\} - \Phi(x) - \sum_{\nu=1}^{k-2} \frac{Q_{\nu n}(x)}{n^{\nu/2}} \right\} \right\|
\]

under some reasonable assumptions on \( X_n \)'s.

Let \( X_1, X_2, \ldots \) again be a sequence of independent, identically distributed random variables with zero means.

Another interesting problem that was first stated and solved for the sequences of random variables and that we will try to extend into the area of the cumulative processes is to find an estimate of an average error in the asymptotic expansion of the CLT. Of course, we are not interested in the trivial consequences of (2.2.2) or (2.2.6). Suppose \( E|X_1|^k + \delta < \infty \) for an integer \( k \) and \( 0 \leq \delta < 1 \). The object in question is the convergence of the series

\[
(2.2.7) \quad \sum_{n=1}^{\infty} \frac{k+\delta}{2} - 2 \left\| P \left\{ \frac{S_n}{\sigma \sqrt{n}} \leq x \right\} - \Phi(x) - \sum_{\nu=1}^{k-2} \frac{Q_{\nu}(x)}{n^{\nu/2}} \right\|,
\]

Galstyan (1971) proved the following theorem.
Theorem 2.2.3 (Galstyan).

Suppose that the condition (2.2.1) is satisfied and also that

\[
\begin{align*}
E\{|X_1|^k \log(1 + |X_1|)\} &< \infty \quad \text{for } \delta = 0, \\
E|X_1|^{k+\delta} &< \infty \quad \text{for } 0 < \delta < 1.
\end{align*}
\] (2.2.8)

Then the expression in (2.2.7) has a finite value.

Galstyan also showed that when \( \delta \neq 0 \) or when \( \delta = 0 \) and \( k \) is even, condition (2.2.8) is necessary for the convergence of the series in (2.2.7). A result equivalent to Theorem 2.2.3 was obtained by Heyde and Leslie (1972). In both papers a method originally developed by Heyde (1967) for \( k = 2 \) was used.

Later in this dissertation we will prove a theorem for a cumulative process similar to Theorem 2.2.3.

There is another topic in the theory of the limiting distributions of the sums of independent random variables that we would like to mention here. This is called in the literature "the probabilities of large deviations." While we do not extend any of the results from this theory into the area of the renewal and cumulative processes, we will use the theorems that give the estimates for the probabilities of large deviations.

The quantities of interest here are

\[
(2.2.9) \quad \frac{1 - F_n(x)}{1 - \Phi(x)} \quad \text{and} \quad \frac{F_n(-x)}{\Phi(-x)},
\]

where \( X_1, X_2, \ldots \) are independent identically distributed random variables with zero means and \( F_n(x) = P\{\frac{S_n}{\sigma\sqrt{n}} \leq x\} \). We are interested in the behavior of the ratios in (2.2.9) when \( x \) depends on \( n \) and tends to \(+\infty\) as \( n \to \infty \). Petrov (1968) proved a rather general theorem that gives the estimates of the above ratios as \( n \to \infty \). We will present his result in a form which is most useful for our purposes.
Theorem 2.2.4 (Petrov).

If $EE^{gX_1} < \infty$ for some $g > 0$ and if $x \geq 0$, $x = o(n^{r/2(r+2)})$ for some positive integer $r$, then

\begin{equation}
\frac{1 - F_n(x)}{1 - \Phi(x)} = \exp\left(\frac{x^3}{n^{1/2}} \lambda^{[r]}\left(\frac{x}{n^{1/2}}\right)\right) \left(1 + O\left(\frac{x+1}{n^{1/2}}\right)\right),
\end{equation}

\begin{equation}
\frac{F_n(-x)}{\Phi(-x)} = \exp\left(-\frac{x^3}{n^{1/2}} \lambda^{[r]}\left(-\frac{x}{n^{1/2}}\right)\right) \left(1 + O\left(\frac{x+1}{n^{1/2}}\right)\right),
\end{equation}

where $\lambda^{[r]}(t) = \sum_{k=0}^{r-1} a_k t^k$, and the coefficients $a_0, a_1, \ldots, a_{r-1}$ depend only on the cumulants of the random variable $X_1$.

We will use this theorem in Chapter IV to obtain a CLT for the cumulative process.

2.3. RENEWAL PROCESSES

There have been a number of attempts to apply the knowledge of the asymptotic distributions for the sums of independent random variables to obtain a Central Limit Theorem for the renewal process. Feller (1949) noticed that for the positive (a.s.) random variables $X_1, X_2, \ldots$, the following identity holds:

\begin{equation}
P\{N(t) \geq n\} = P\{S_n \leq t\},
\end{equation}

for any $t > 0$ and any $n = 0, 1, \ldots$. When used in conjunction with the classical Central Limit Theorem, (2.3.1) easily leads to the following CLT for a renewal process based on the sequence of the independent identically distributed positive (a.s.) random variables:

\begin{equation}
P\left\{\frac{N(t) - t/\mu}{\sigma t^{1/2}/\mu^{3/2}} \leq x\right\} \rightarrow \Phi(x) \quad \text{as } t \rightarrow \infty,
\end{equation}

for any real $x$.  

Until the 1970's, no successful efforts were made to extend (2.3.2) to get a remainder term in the Central Limit Theorem, or to allow $X_n$'s to have different distributions or take negative values, so that all three definitions of the renewal process given in (1.1.1) would have to be considered. Instead, the researchers concentrated on the analysis of the behavior of the renewal function $H(t) = E N(t)$ and of $\text{Var} \ N(t)$, as $t \to \infty$. Feller (1941) and Doob (1948), using different methods, proved a so-called "elementary renewal theorem":

\begin{equation}
(2.3.3) \quad H(t)/t \to \mu^{-1} \quad \text{as} \quad t \to \infty.
\end{equation}

Here $\mu = EX_1$ and the theorem applies even when $\mu = \infty$. Blackwell (1948) showed that for a continuous renewal process,

\begin{equation}
(2.3.4) \quad H(t + \alpha) - H(t) \to \alpha \mu^{-1} \quad \text{as} \quad t \to \infty,
\end{equation}

for any fixed $\alpha > 0$. Smith (1954) showed that if the second moment $\mu_2$ of $X_1$ is finite then

\begin{equation}
(2.3.5) \quad H(t) - t/\mu \to \frac{\mu_2}{2\mu_1} - 1 \quad \text{as} \quad t \to \infty.
\end{equation}

Smith then used his methods to examine the asymptotic behavior of $\text{Var} \ N(t)$. He showed that if $\mu_2 < \infty$ then

\[
\text{Var} \ N(t) = \frac{\mu_2 - \mu_1^2}{\mu_1^3} t + o(t) \quad \text{as} \quad t \to \infty.
\]

To obtain a more precise result he had to assume that $F$ belongs to a class of $\mathcal{G}$, which means that there exists an integer $k$ such that $P \{ S_k \leq x \}$ is a distribution function with a non-null absolutely continuous component. When $F \in \mathcal{G}$ and a third absolute moment $\mu_3$ of $X_1$ is finite, then (Smith, 1959)

\begin{equation}
(2.3.6) \quad \text{Var} \ N(t) = \frac{\mu_2 - \mu_1^2}{\mu_1^3} t + \left( \frac{5\mu_2^2}{4\mu_1^4} - \frac{2\mu_3}{3\mu_1^3} - \frac{\mu_2}{2\mu_1^2} \right) + o(1) \quad \text{as} \quad t \to \infty.
\end{equation}
The issue of the Central Limit Theorem for the renewal processes was raised again by Hunter (1974) who considered a case of a two-dimensional renewal process, i.e., when \( \{X_n^{(1)}, X_n^{(2)}\} \), \( n=1,2,\ldots \), is a sequence of independent identically distributed bivariate positive random variables and \( N(t_1,t_2) \) is defined for any pair of real numbers \( t_1 \) and \( t_2 \) as \( \text{Sup} \{n: X_1^{(1)} + \ldots + X_n^{(1)} \leq t_1, X_1^{(2)} + \ldots + X_n^{(2)} \leq t_2\} \). Hunter obtained a Central Limit Theorem for it. In this dissertation we will only consider a one-dimensional renewal process and while many authors have generalized the CLTs to cover the multi-dimensional cases, we will state their theorems (with appropriate notes) as they apply to the one-dimensional renewal processes.

Csenki (1979) considered a k-dimensional case. He also allowed \( X_n \)'s to take negative values which created a difference between the three definitions of a renewal process which we gave in Chapter I. Csenki proved a Central Limit Theorem for \( N_1(t) \).

In his 1981 paper, Ahmad stated several Central Limit Theorems with remainder terms for the two-dimensional renewal processes. He allowed the random variables not to be necessarily positive. The following theorem summarizes his results as they appear in a one-dimensional form.

**Theorem 2.3.1 (Ahmad).**

If \( X_n \)'s are independent identically distributed random variables with mean \( \mu > 0 \), variance \( \sigma^2 > 0 \) and such that \( E|X_1|^3 < \infty \), then the following equality holds:

\[
(2.3.7) \quad \left\| P \left\{ \frac{N_1(t) - t/\mu}{\sigma t^{1/2}/\mu^{3/2}} \leq x \right\} - \Phi(x) \right\| = O(t^{-1/2}) \quad \text{as } t \to \infty.
\]
Ahmad's proof contains an error. On page 121 of his paper he claims that

\[ |x| \left| 1 - \frac{1}{\sqrt{1 + \frac{\sigma x}{t^\mu}}} \right| \leq Ct^{-1/2}, \]

for \( x \geq -\frac{(t^\mu)^{1/2}}{2\sigma} \) and some constant C. He later proceeds with an assumption that C does not depend on x. What is true, however, is that

\[ |x| \left| 1 - \frac{1}{\sqrt{1 + \frac{\sigma x}{t^\mu}}} \right| \leq Cx^2t^{-1/2}, \]

for some C independent of x. This mistake in the proof does not allow one to obtain a uniform (in x) bound for the difference between \( P \{ \frac{N_1(t) - t/\mu}{\sigma t^{1/2}/\mu^{3/2}} \leq x \} \) and \( \phi(x) \). All that one can get is that for any real x,

\[ \left| P \{ \frac{N_1(t) - t/\mu}{\sigma t^{1/2}/\mu^{3/2}} \leq x \} - \phi(x) \right| = O(t^{-1/2}) \quad \text{as } t \to \infty, \]

where a constant in O may depend on x.

We should note that even (2.3.9) represents a significant step forward from what was done before. A remainder term in the Central Limit Theorem for the renewal process was obtained.

In the same paper Ahmad stated another theorem similar to Theorem 2.3.1, except that \( N_3(t) \) was used instead of \( N_1(t) \). The proof of that other theorem is wrong too. It is based on the erroneous formula \( P \{ S_n \leq t \} = P \{ N_3(t) \geq n \} \). (See page 119 of Ahmad's paper.) So for \( N_3(t) \) he did not get even a non-uniform version of the Central Limit Theorem.

Niculescu and Omey (1985) extended Ahmad's result to the case of a k-dimensional renewal process. Their methods are the same as Ahmad's and the errors are the same too. Hence, in fact, they only correctly obtained a result for
$N_1(t)$ with the bound for the error that is not uniform in $x$. In the same paper they also covered the case of fractional moments between 2 and 3, i.e., when $E|X_1|^{2+\delta} < \infty$, $0 < \delta \leq 1$. Their result in this case, when written in a one-dimensional form, would be

\begin{equation}
(2.3.10) \quad \left| P\left\{ \frac{N_1(t) - t/\mu}{\sigma t^{1/2} / \mu^{3/2}} \leq x \right\} - \Phi(x) \right| = O(t^{-\delta/2}),
\end{equation}

where a constant in $O$ may depend on $x$. Niculescu had previously written a paper (1984) where (2.3.10) was stated and correctly proved. In that paper he did not make any effort to extend (2.3.10) to cover $N_3(t)$ or to get a uniform error bound. In the very same paper he also proved an asymptotic CLT for $X_n$'s that are not identically distributed, but he did not get a remainder term in this case.

Woodroofe and Keener (1987) used a completely different approach to prove a result that includes (2.3.9) as a special case. We discuss their paper in greater detail in Section 2.4.

As we can see from the above discussion, the Central Limit Theorem with a remainder term for a renewal process is proved only when the renewal process is based on a sequence of independent identically distributed random variables with a finite absolute third moment and the estimate of the remainder term is not uniform. Also, only one of the three possible definitions of the renewal process is covered. There is at present no result similar to the Edgeworth expansion (2.2.2) under the assumption of the finiteness of the higher moments of $X_n$'s. In Chapter III, we will prove the theorems that expand the existing Central Limit Theorems for the renewal processes to cover the Edgeworth expansions. We will also allow $X_n$'s to be non-positive and not identically distributed random variables and consider all three definitions of the renewal process. That will bring the theory
of the CLTs for the renewal processes to the level where such theory for the sums
of independent random variables stands.

2.4. CUMULATIVE PROCESSES

Not many papers dealt with the CLT for the cumulative processes. Smith
(1955) achieved the following result.

Theorem 2.4.1 (Smith).

Suppose $X_1$ has a finite first moment and $Y_1$ has a finite second moment.
Assume also that $\lambda = \text{E}Y_1 = 0$. Then for any real $x$,

$$
P \left\{ \frac{W(t)}{\tau(t/\mu)^{1/2}} \leq x \right\} \rightarrow \Phi(x) \quad \text{as } t \rightarrow \infty.
$$

In the same paper, Smith has shown that the restriction $\lambda = 0$ can be avoided.
The corollary to Theorem 2.4.1 shows what happens when $\lambda \neq 0$.

Corollary 2.4.1 (Smith).

If, as in Theorem 2.4.1, $\text{E}X_1$ and $\text{E}Y_1^2$ are both finite, then for any real $x$,

$$
P \left\{ \frac{W(t) - \lambda N(t)}{\tau(t/\mu)^{1/2}} \leq x \right\} \rightarrow \Phi(x) \quad \text{as } t \rightarrow \infty.
$$

If, in addition, $\text{Var} \: X_1 < \infty$ then

$$
P \left\{ \frac{W(t) - \lambda t}{\gamma(t/\mu)^{1/2}} \leq x \right\} \rightarrow \Phi(x) \quad \text{as } t \rightarrow \infty,
$$

where $\gamma^2 = \text{Var}(Y_1 - \frac{1}{\mu}X_1)$ is assumed not to equal to zero.

We like (2.4.3) better than (2.4.2) because it does not involve any random
variables other than $W(t)$.

Smith then extended his result to obtain a CLT for a multi-dimensional
cumulative process. Murphee and Smith (1986) obtained results similar to (2.4.1),

(2.4.2) and (2.4.3) for the cumulative processes with an improper distribution function $F$ of $X_1$, i.e., when $F(\infty) < 1$. They called them the "transient regenerative processes."

Woodroofe and Keener (1987) took a somewhat different approach to this problem. They considered a distribution of

\begin{equation}
W_o(t) = \sum_{n=1}^{j_t} Y_n,
\end{equation}

where $j_t$ is a "stopping time" random variable satisfying certain conditions together with a sequence of $Y_n$’s. It turns out that if $j_t = N(t) + 1$, as it is in our definition of the cumulative process, then the value of $Y_n$ is uniquely determined (as a result of those conditions on $j_t$ and $Y_n$’s) by the value of $X_n$, $n = 1, 2, \ldots$. Hence $W_o(t) = \sum_{n=1}^{N(t)+1} h(X_n)$, where $h$ is a certain function of a real argument. That reduces a cumulative process to a variation of a renewal process. For their $W_o(t)$, Woodroofe and Keener obtained a Central Limit Theorem with a remainder term of the order of $o(t^{-1/2})$.

As in a case of the renewal processes, there are results that describe the asymptotic behavior of $EW(t)$. Smith (1955) showed that $EW(t) = \frac{1}{\mu} t + o(t)$ as $t \to \infty$. He also proved the following so-called "ergodic theorem."

**Theorem 2.4.2** (Smith, 1955).

If $\lambda$, $\mu$ are both finite then

\[ P \left\{ \lim_{t \to \infty} \frac{W(t)}{t} = \frac{1}{\mu} \right\} = 1. \]

In this paper we are only concerned with improvements in the Central Limit Theorem for $W(t)$. 
CHAPTER III
CENTRAL LIMIT THEOREMS FOR THE RENEWAL PROCESSES

3.1. AN EDGEWORTH EXPANSION

In this section we prove a new Central Limit Theorem for a renewal process. By assuming the finiteness of the higher moments, we can achieve an appropriate improvement in the order of the remainder term, as it was done for the sums of independent random variables. Throughout this section we assume that $X_1, X_2, \ldots$ are independent identically distributed positive (a.s.) random variables with mean $\mu$, variance $\sigma^2 > 0$ and such that $E|X_1|^r < \infty$ for some integer $r \geq 3$.

Theorem 3.1.1.

If (2.2.1) holds for the characteristic function $\phi$ of $X_1$ then for $x \geq -(t\mu)^{1/2}/\sigma$,

$$P\left\{ \frac{N(t) - t/\mu}{\sigma^{1/2}/\mu^{3/2}} \leq x \right\} - \phi(-u(x,t)) + \sum_{\nu=1}^{r-2} \frac{Q_\nu(u(x,t))}{\{k(x,t)\}^{\nu/2}} = o\left(\frac{1}{t^{(r-2)/2}}\right)$$

uniformly in $x$, where

$$k(x,t) = \left[t/\mu + x\sigma t^{1/2}/\mu^{3/2}\right] + 1,$$

$$u(x,t) = \frac{t - \mu k(x,t)}{\sigma \{k(x,t)\}^{1/2}},$$

functions $Q_\nu(x)$ are defined in the Appendix A and $[a]$, as usual, stands for an integer part of $a$.

Note 1. It is not of any interest to consider a situation when $x < -(t\mu)^{1/2}/\sigma$ because then $P\left\{ \frac{N(t) - t/\mu}{\sigma^{1/2}/\mu^{3/2}} \leq x \right\} = 0$. 
Note 2. It turns out that for \( r > 3 \) in order to achieve the required precision in the error term estimate, we need to use functions \( k(x,t) \) and \( u(x,t) \) defined in the theorem instead of the variables \( x \) and \( t \) which one would expect to use looking at (2.2.2).

Proof of Theorem 3.1.1.

The following equalities hold:

\[
\begin{align*}
P\left\{ \frac{N(t)-t/\mu}{\sigma t^{1/2} / \mu^{3/2}} \leq x \right\} & = 1 - P\{N(t) > t/\mu + x\sigma t^{1/2} / \mu^{3/2}\} \\
& = 1 - P\{N(t) \geq [t/\mu + x\sigma t^{1/2} / \mu^{3/2}] + 1\} \\
& = 1 - P\{N(t) \geq k(x,t)\} \\
& = 1 - P\{S_k(x,t) \leq t\} \quad \text{(according to (2.3.1))} \\
& = 1 - P\left\{ \sum_{j=1}^{k(x,t)} X_j \leq t \right\}.
\end{align*}
\]

(3.1.2)

We will consider two separate cases.

Case 1: \( x \geq -(t\mu)^{1/2} / 2\sigma \).

In this case, \( k(x,t) \geq [t/2\mu] \), hence \( 1/k(x,t) = O(t^{-1}) \) uniformly in \( x \). For \( j = 1,2,\ldots \), let \( Z_j = (X_j - \mu) / \sigma \). Evidently,

\[
1 - P\left\{ \sum_{j=1}^{k(x,t)} X_j \leq t \right\} \\
= 1 - P\left\{ \frac{1}{\{k(x,t)\}^{1/2}} \sum_{j=1}^{k(x,t)} Z_j \leq \frac{t - \mu k(x,t)}{\sigma k(x,t)^{1/2}} \right\} \\
= 1 - P\left\{ \frac{1}{\{k(x,t)\}^{1/2}} \sum_{j=1}^{k(x,t)} Z_j \leq u(x,t) \right\} \\
= 1 - \left\{ \Phi(u(x,t)) + \sum_{\nu=1}^{r-2} \frac{Q_{\nu}(u(x,t))}{\{k(x,t)\}^{\nu/2}} + o\left( \frac{1}{\{k(x,t)\}^{(r-2)/2}} \right) \right\}
\]

(3.1.3)
(according to Theorem 2.2.1, which is applicable since condition (2.2.1) is satisfied for the characteristic function of $Z_1$. Note that the rate of convergence to zero in $o$ depends only on $r$ and on the distribution of $X_1$.)

\[ \Phi(-u(x,t)) = \sum_{\nu=1}^{r-2} \frac{Q_\nu(u(x,t))}{(k(x,t))^{\nu/2}} + o\left(\frac{1}{t^{(r-2)/2}}\right) \quad \text{as } t \to \infty, \]

uniformly in $x$. That proves (3.1.1) in this case.

**Case 2:** $-(t\mu)^{1/2}/\sigma \leq x < -(t\mu)^{1/2}/2\sigma$.

In this case, $u(x,t) > Ct^{1/2}$ for some $C > 0$. Since $Q_\nu(y) = \exp(-y^2/2)H_\nu(y)$, where $H_\nu(y)$ is a polynomial of degree $3\nu - 1$ with coefficients depending on the first $(\nu + 2)$ moments of $X_1$ only, we have

\[ \left| \sum_{\nu=1}^{r-2} \frac{Q_\nu(u(x,t))}{(k(x,t))^{\nu/2}} \right| \leq \sum_{\nu=1}^{r-2} \left| Q_\nu(u(x,t)) \right| = o\left(\frac{1}{t^{(r-2)/2}}\right). \]

(3.1.4)

Also, $\Phi(-u(x,t)) = o\left(\frac{1}{t^{(r-2)/2}}\right)$ when $u(x,t) > Ct^{1/2}$. Hence in order to prove (3.1.1) in this case, we only need to show that for $x \leq -(t\mu)^{1/2}/2\sigma$,

\[ \mathbb{P}\left\{ \frac{N(t) - \mu}{\sqrt{\frac{t}{\mu}}} \leq x \right\} = o\left(\frac{1}{t^{(r-2)/2}}\right). \]

(3.1.5)

Denote $y = -(t\mu)^{1/2}/2\sigma$. Since $x < y$,

\[ \mathbb{P}\left\{ \frac{N(t) - \mu}{\sqrt{\frac{t}{\mu}}} \leq x \right\} \leq \mathbb{P}\left\{ \frac{N(t) - \mu}{\sqrt{\frac{t}{\mu}}} \leq y \right\} \]

\[ \quad \text{(case 1 now applies because } y \text{ satisfies its restrictions.)} \]

\[ = \Phi(-u(y,t)) - \sum_{\nu=1}^{r-2} \frac{Q_\nu(u(y,t))}{(k(y,t))^{\nu/2}} + o\left(\frac{1}{t^{(r-2)/2}}\right) \]
\[ = o\left(\frac{1}{t^{(r-2)/2}}\right) \quad \text{as } t \to \infty, \]

uniformly in \( x \). The last equality holds since \( k(y,t) = \lfloor t/2\mu \rfloor + 1 \) and \( u(y,t) > Ct^{1/2} \), for some \( C > 0 \).

That completes the proof of Theorem 3.1.1. \( \square \)

In the remaining portion of this chapter we will concentrate on the renewal processes based on the sequences of the independent random variables with finite third absolute moments only \((r = 3)\). It turns out that for such renewal processes we can obtain a Central Limit Theorem with the remainder term of the order of \( O(t^{-1/2}) \), while not using functions \( k(x,t) \) and \( u(x,t) \) introduced earlier in this section. We will also allow \( X_n \)'s to have different distribution functions and to take negative values. The Central Limit Theorem will be proved for all three definitions of the renewal process: \( N_1(t), N_2(t) \) and \( N_3(t) \).

3.2. AUXILIARY LEMMAS

In this section we will prove four lemmas that will allow us later to derive the Central Limit Theorem for \( N_1(t) \) and then to use it in order to obtain one for \( N_3(t) \). From the definitions it follows immediately that \( N_1(t) \leq N_3(t) \). We want to show that \( N_1(t) \) and \( N_3(t) \) do not differ very much in some sense. Lemma 3.2.1 allows us to relate the estimation of \( P\{N_1(t) \geq k\} \) to that of \( P\{S_k \leq t\} \). Lemma 3.2.4 shows the connection between \( P\{N_3(t) \geq k\} \) and \( P\{S_k \leq t\} \). Lemma 3.2.2 and Lemma 3.2.3 are needed to prove Lemma 3.2.4.

Throughout this section the following notation and assumptions will apply: \( X_1, X_2, \ldots \) is a sequence of random variables (not necessarily identically distributed and not necessarily positive). We denote

\[ Z_n = X_n - \mu_n, \]
\[
B_n^2 = \sum_{j=1}^{n} \sigma_j^2.
\]

We will assume that there exist positive constants \( \mu, \alpha, D, g \) and \( M \) such that the following conditions hold for all \( n \geq 1 \):

(i) \( \mu_n > \alpha \),

(ii) \( \left| \sum_{j=1}^{n} \mu_j - n \mu \right| < D \),

(iii) \( B_n^2 \geq gn \) and

(iv) \( E|Z_n|^3 < M \).

These conditions allow us to apply Lemma 2.1 from Ahmad's 1981 paper. Condition (iv) also implies that there exists \( M_1 > 0 \) such that \( \sigma_n^2 < M_1 \) for all \( n \). We will be making references to \( M_1 \) later in the text.

Lemma 3.2.1.

The following equality holds:

\[
\left\| P \{ S_n \leq x \} - P \left\{ \max_{1 \leq k \leq n} S_k \leq x \right\} \right\| = O(n^{-1/2}).
\]

The proof of this lemma is given by Ahmad (1981); although we have mentioned that there are errors in his paper, this result is correct. He proves it as a part of the proof of Lemma 2.1 in his paper. See formula (2.7) there.

Lemma 3.2.2.

For any \( \epsilon > 0 \) and any \( j \geq 0 \),

\[
P \left\{ \frac{Z_{j+1} + \cdots + Z_{j+n}}{n} > \epsilon \right\} = O(n^{-2}),
\]

where a constant in \( O \) does not depend on \( j \).
Proof.

\[ P \left\{ \frac{Z_{j+1} + \ldots + Z_{j+n}}{n} > \epsilon \right\} \]

\[ = P \left\{ \frac{Z_{j+1} + \ldots + Z_{j+n}}{(\sigma_{j+1}^2 + \ldots + \sigma_{j+n}^2)^{1/2}} > \frac{n \epsilon}{(\sigma_{j+1}^2 + \ldots + \sigma_{j+n}^2)^{1/2}} \right\} \]

\[ = 1 - P \left\{ \frac{Z_{j+1} + \ldots + Z_{j+n}}{(\sigma_{j+1}^2 + \ldots + \sigma_{j+n}^2)^{1/2}} \leq \frac{n \epsilon}{(\sigma_{j+1}^2 + \ldots + \sigma_{j+n}^2)^{1/2}} \right\} \]

\[
\leq \left| 1 - \Phi \left( \frac{n \epsilon}{(\sigma_{j+1}^2 + \ldots + \sigma_{j+n}^2)^{1/2}} \right) \right| \\
+ \left| P \left\{ \frac{Z_{j+1} + \ldots + Z_{j+n}}{(\sigma_{j+1}^2 + \ldots + \sigma_{j+n}^2)^{1/2}} > \frac{n \epsilon}{(\sigma_{j+1}^2 + \ldots + \sigma_{j+n}^2)^{1/2}} \right\} \right| \\
- \Phi \left( \frac{n \epsilon}{(\sigma_{j+1}^2 + \ldots + \sigma_{j+n}^2)^{1/2}} \right) \right| 
\]

\[ = A + B, \text{ say.} \]

Since

\[ \frac{n \epsilon}{(\sigma_{j+1}^2 + \ldots + \sigma_{j+n}^2)^{1/2}} > \frac{n \epsilon}{(nM_1)^{1/2}} = \epsilon_1 n^{1/2}, \]

where \( \epsilon_1 = \epsilon/M_1^{1/2} \) does not depend on \( j \) or \( n \),

\[ A < \Phi(-\epsilon_1 n^{1/2}) = O(n^{-2}), \]

uniformly in \( j \).

Let us now estimate \( B \). The conditions of Theorem 2.2.2 are satisfied.

Applying this theorem to the sequence \( \{Z_\ell\}_{\ell=j+1}^{j+n} \) with \( x = \frac{n \epsilon}{(\sigma_{j+1}^2 + \ldots + \sigma_{j+n}^2)^{1/2}}, \)
we get
\[ B \leq \frac{C(E|Z_{j+1}|^3 + \ldots + E|Z_{j+n}|^3)}{(\sigma_{j+1}^2 + \ldots + \sigma_{j+n}^2)^{3/2}} \cdot \frac{(\sigma_{j+1}^2 + \ldots + \sigma_{j+n}^2)^{3/2}}{(nc)^3} \]
(where \( C \) is an absolute constant)
\[ \leq \frac{C_n M}{n^3 c^3} = O(n^{-2}), \]
where the constant in \( O \) does not depend on \( j \).

Lemma 3.2.2 is proved. \( \square \)

**Lemma 3.2.3.**

For any \( \epsilon > 0 \) and \( j \geq 0 \),
\[ \mathbb{P}\left\{ \inf_{k \geq n} \frac{Z_{j+1} + \ldots + Z_{j+k}}{k} < -\epsilon \right\} = O(n^{-2}), \]  
(3.2.5)
uniformly in \( j \).

**Proof.**

Denote \( V_n = -Z_n \), \( n = 1, 2, \ldots \). Then (3.2.5) becomes
\[ \mathbb{P}\left\{ \sup_{k \geq n} \frac{V_{j+1} + \ldots + V_{j+k}}{k} < \epsilon \right\} = O(n^{-2}), \]  
(3.2.6)
uniformly in \( j \). We will prove (3.2.6).

For a given \( n \), choose \( m = m(n) \) such that
\[ 2^m \leq n < 2^{m+1}. \]

Then
\[ \mathbb{P}\left\{ \sup_{k \geq n} \frac{V_{j+1} + \ldots + V_{j+k}}{k} > \epsilon \right\} \leq \mathbb{P}\left\{ \sup_{k \geq 2^m} \frac{V_{j+1} + \ldots + V_{j+k}}{k} > \epsilon \right\} \]
\[ \leq \sum_{i=m}^{\infty} \mathbb{P}\left\{ \max_{2^i \leq k < 2^{i+1}} \frac{V_{j+1} + \ldots + V_{j+k}}{k} > \epsilon \right\}. \]
\( \sum_{i=m}^{2n} p_{j+1+i} \leq \frac{c}{2m+1} \) (since \( 2m+1 > n \)).

(by Lemma 3.2.2, constant \( c \) does not depend on \( j \) and \( n \)).

\[ \sum_{i=m}^{2n} p_{j+1+i} \leq \frac{c}{2m+1} \]

so the value of the sums of the probabilities in the right-hand side of (3.2.7) does not exceed

\[ \sum_{i=m}^{2n} p_{j+1+i} \leq \frac{c}{2m+1} \]


\[ \sum_{i=m}^{2n} p_{j+1+i} \leq \frac{c}{2m+1} \]

(3.2.7)

(3.2.8)
where $c_1$ does not depend on $j$ and $n$.

This completes the proof of Lemma 3.2.3. \qed

Finally, we need to prove a lemma that will use the result of Lemma 3.2.3 to tie the estimation of $P\{N_3(t) \geq k\}$ to that of $P\{S_k \leq t\}$.

Lemma 3.2.4.

If four conditions (3.2.1) are satisfied, then

\[(3.2.10) \quad \|P\{\text{there exists } k \geq n \text{ such that } S_k \leq x\} - P\{S_n \leq x\}\| = O(n^{-1/2}),\]

where a constant in $O$ does not depend on $n$.

Proof.

For any real $x$,

\[
0 \leq P\{\text{there exists } k \geq n \text{ such that } S_k \leq x\} - P\{S_n \leq x\} = P\{S_n > x, \text{ there exists } k > n \text{ such that } S_k \leq x\} \leq P\{\max_{0 \leq j \leq n} S_j > x, \text{ there exists } k > n \text{ such that } S_k \leq x\}
\]

\[= \sum_{j=0}^{n} P\{S_0 \leq x, ..., S_{j-1} \leq x, S_j > x, \text{ there exists } k > n \text{ such that } S_k \leq x\} \leq \sum_{j=0}^{n} P\{S_0 \leq x, ..., S_{j-1} \leq x, S_j > x, \inf_{k>n} (X_{j+1} + ... + X_k) < 0\} = \sum_{j=0}^{n} P\{S_0 \leq x, ..., S_{j-1} \leq x, S_j > x\} \cdot P\{\inf_{k>n} (X_{j+1} + ... + X_k) < 0\}.
\]

The last equality holds because of the independence of $X_1, X_2, ...$.

Let us show that
where a constant in $O$ does not depend on $n$ or $j$.

Set $m = n - j$, $\ell = k - j$. Then

\[
P\{ \inf_{k > n} (X_{j+1} + \ldots + X_k) < 0 \} = P\{ \inf_{\ell > m} (X_{j+1} + \ldots + X_{j+\ell}) < 0 \}
\]

\[
= P\left\{ \inf_{\ell > m} \left( (Z_{j+1} + \ldots + Z_{j+\ell}) + (\mu_{j+1} + \ldots + \mu_{j+\ell}) \right) < 0 \right\}
\]

\[
= P\left\{ \inf_{\ell > m} \left( \frac{Z_{j+1} + \ldots + Z_{j+\ell}}{\ell} + \left( \frac{\mu_{j+1} + \ldots + \mu_{j+\ell}}{\ell} - \mu \right) + \mu \right) < 0 \right\}
\]

\[
= P\left\{ \inf_{\ell > m} \left( \frac{Z_{j+1} + \ldots + Z_{j+\ell}}{\ell} + \frac{\mu_{j+1} + \ldots + \mu_{j+\ell} - \ell}{\ell} \right) < -\mu \right\}
\]

Since

\[
\left| \frac{\mu_{j+1} + \ldots + \mu_{j+\ell} - \ell}{\ell} \right| \leq \left| \frac{\mu_1 + \ldots + \mu_{j+\ell} - \mu(j+\ell)}{\ell} \right| + \left| \frac{\mu_1 + \ldots + \mu_j - \mu_j}{\ell} \right|
\]

\[
\leq \frac{2D}{\ell} \quad \text{(by condition (3.2.1, ii))},
\]

the probability in the right hand side of (3.2.13) does not exceed

\[
P\left\{ \inf_{\ell > m} \left( \frac{Z_{j+1} + \ldots + Z_{j+\ell}}{\ell} - \frac{2D}{\ell} \right) < -\mu \right\}
\]

\[
\text{(3.2.14)}
\]

For $m > 4D/\mu$, the value of the probability in (3.2.14) does not exceed

\[
P\left\{ \inf_{\ell > m} \left( \frac{Z_{j+1} + \ldots + Z_{j+\ell}}{\ell} - \frac{\mu}{2} \right) < -\mu \right\}
\]

\[
= P\left\{ \inf_{\ell > m + 1} \frac{Z_{j+1} + \ldots + Z_{j+\ell}}{\ell} < -\frac{\mu}{2} \right\}
\]
where a constant in $O$ does not depend on $n$ or $j$. Since the value of the probability is not greater than 1, by choosing a constant in $O$ appropriately, we can assure that (3.2.15) holds for $m \leq 4D/\mu$ too, while that constant in $O$ is independent of $n$ and $j$. Combining (3.2.13), (3.2.14) and (3.2.15) and substituting $n - j$ for $m$, we prove that (3.2.12) holds.

Therefore the expression in the right hand side of (3.2.11) does not exceed

\[
\begin{align*}
&c \sum_{j=0}^{n} P\{S_0 \leq x, \ldots, S_{j-1} \leq x, S_j > x\} (n-j+1)^{-2} \\
&= c \left\{ \sum_{j=0}^{\lfloor n/2 \rfloor} P\{S_0 \leq x, \ldots, S_{j-1} \leq x, S_j > x\} (n-j+1)^{-2} \\
&\quad + \sum_{j=\lceil n/2 \rceil}^{n} P\{S_0 \leq x, \ldots, S_{j-1} \leq x, S_j > x\} (n-j+1)^{-2} \right\} \\
&= c(I_1 + I_2), \text{ say,}
\end{align*}
\]

where $c$ does not depend on $n$ or $j$. We have

\[
(3.2.17) \quad I_1 \leq \max_{1 \leq j \leq \lfloor n/2 \rfloor} (n-j+1)^{-2} = O(n^{-2}).
\]

Now we need to estimate $I_2$. Ahmad (1981, see formulae (2.14) and (2.15) in his paper) proved correctly that the following estimate holds:

\[
(3.2.18) \quad P\{S_{j-1} \leq x, S_j > x\} \leq c(j-1)^{-1/2},
\]

where a constant $c$ is independent of $x$ and $j$. For $j > \lceil n/2 \rceil$, the expression in the right hand side of (3.2.18) does not exceed $c_2 n^{-1/2}$, where $c_2$ is independent of $x$
and j. Therefore

\[
I_2 \leq \left\{ \max_{[n/2]+1 \leq j \leq n} P\{S_0 \leq x, \ldots, S_{j-1} \leq x, S_j > x\} \right\} \cdot \sum_{j=\lfloor n/2 \rfloor + 1}^{n} (n-j+1)^{-2}
\]

(3.2.19) \[
\leq \left\{ \max_{[n/2]+1 \leq j \leq n} P\{S_{j-1} \leq x, S_j > x\} \right\} \cdot \sum_{j=1}^{\infty} j^{-2}
\]

\[
\leq 2c_2 [n/2]^{-1/2} = O(n^{-1/2}).
\]

By combining (3.2.11), (3.2.16), (3.2.17) and (3.2.19), we complete the proof of the lemma.

3.3. A CENTRAL LIMIT THEOREM FOR THE DIFFERENT TYPES OF THE RENEWAL PROCESSES

In this section we state and prove a rather general Central Limit Theorem for a renewal process based on a sequence of independent random variables \(X_1, X_2, \ldots\), not necessarily identically distributed and not necessarily positive. That theorem will cover all three different definitions of the renewal process: \(N_1(t), N_2(t),\) and \(N_3(t)\). The same approximation and the same remainder term will apply in each case.

We will use the notation consistent with that used in Sections 3.1 and 3.2:

\[
Z_n = X_n - \mu, \quad n = 1, 2, \ldots,
\]

\[
B_n^2 = \sigma_1^2 + \ldots + \sigma_n^2,
\]

also,

\[
k(x,t) = \left[ t/\mu + xB_{[t/\mu]/\mu} \right] + 1.
\]

The following theorem is the main result of this chapter.
Theorem 3.3.1.

If conditions (3.2.1, (i) – (iv)) are satisfied then the following three formulae are true:

(3.3.1) \[ \left\| P \left\{ \frac{N_1(t) - t/\mu}{\frac{t}{\mu} B[t/\mu]} \leq x \right\} - \Phi(x) \right\| = O(t^{-1/2}), \]

(3.3.2) \[ \left\| P \left\{ \frac{N_2(t) - t/\mu}{\frac{t}{\mu} B[t/\mu]} \leq x \right\} - \Phi(x) \right\| = O(t^{-1/2}) \]

and

(3.3.3) \[ \left\| P \left\{ \frac{N_3(t) - t/\mu}{\frac{t}{\mu} B[t/\mu]} \leq x \right\} - \Phi(x) \right\| = O(t^{-1/2}), \]

as \( t \to \infty. \)

Proof.

Since \( N_1(t) \leq N_2(t) \leq N_3(t), \) it is sufficient to prove (3.3.1) and (3.3.3) only.

Let us evaluate the probability involved in (3.3.1). We will use the following identity:

(3.3.4) \[ P\{N_1(t) \geq k\} = P\{\max_{1 \leq n \leq k} S_k \leq t\}. \]

It generalizes the Feller’s identity (2.3.1) which was only applicable to positive \( X_n \)'s. We obtain

\[
P \left\{ \frac{N_1(t) - t/\mu}{\frac{t}{\mu} B[t/\mu]} \leq x \right\} = 1 - P \left\{ \frac{N_1(t) - t/\mu}{\frac{t}{\mu} B[t/\mu]} > x \right\}
\]

\[
= 1 - P \{ N_1(t) > t/\mu + xB[t/\mu]/\mu \}
\]

(3.3.5) \[ = 1 - P \{ N_1(t) \geq [t/\mu + xB[t/\mu]/\mu] + 1 \} \]
\[
1 - P\{N_1(t) \geq k(x,t)\} \\
= 1 - P\{\max_{1 \leq n \leq k(x,t)} S_n \leq t\}.
\]

By performing calculations similar to the above and using the identity

\[P\{N_3(t) \geq k\} = P\{\text{there exists } n \geq k \text{ such that } S_n \leq t\},\]

which follows from the definition of \(N_3(t)\), we get

\[
P\left\{\frac{N_3(t) - t/\mu}{\frac{1}{\mu} B[t/\mu]} \leq x\right\} = 1 - P\{N_3(t) \geq k(x,t)\} \\
= 1 - P\{\text{there exists } n \geq k(x,t) \text{ such that } S_n \leq t\}.
\]

(3.3.7)

We will consider two main cases.

**Case 1:** \(x < -t/2B[t/\mu]\) and

**Case 2:** \(x \geq -t/2B[t/\mu]\).

In Case 1, we will show that, uniformly in \(x\),

(3.3.8) \[1 - P\{N_1(t) \geq k(x,t)\} = O(t^{-1/2}) \text{ as } t \to \infty\]

and

(3.3.9) \[\Phi(x) = O(t^{-1/2}) \text{ as } t \to \infty.\]

That will prove the first part of the theorem, (3.3.1), since as we saw in (3.3.5),

\[
P\left\{\frac{N_3(t) - t/\mu}{\frac{1}{\mu} B[t/\mu]} \leq x\right\} = 1 - P\{N_1(t) \geq k(x,t)\}.
\]

Moreover, since \(N_3(t) \geq N_1(t)\), it will follow from (3.3.8) that

\[1 - P\{N_3(t) \geq k(x,t)\} = O(t^{-1/2}) \text{ as } t \to \infty,\]
uniformly in $x$. This result together with (3.3.7) shows that

\[
(3.3.10) \quad P\left\{ \frac{N_3(t) - t/\mu}{B[t/\mu]} \leq x \right\} = O(t^{-1/2}) \quad \text{as } t \to \infty,
\]

uniformly in $x$. Together with (3.3.7), equation (3.3.10) proves the theorem for $N_3(t)$.

Thus by proving (3.3.8) and (3.3.9), we will complete the proof of the theorem when the Case 1 condition applies. First, (3.3.9) holds since

\[
\Phi(x) < \Phi(-t/2B[t/\mu]) \\
< \Phi\left( -\frac{t}{2 \langle t/\mu \rangle M_1 \rangle^{1/2}} \right) \quad (M_1 \text{ is such that } \sigma_n^2 \leq M_1 \text{ for all } n) \\
\leq \Phi(-ct^{1/2}) \quad \text{(for some constant } c > 0) \\
= O(t^{-1/2}).
\]

Let us prove (3.3.8). Under the condition of Case 1, $k(x,t) \leq (t/2\mu) + 1$, so

\[
(3.3.11) \quad P\{N_1(t) \geq k(x,t)\} = P\{ \max_{1 \leq n \leq k(x,t)} S_n \leq t \} \geq P\{ \max_{1 \leq n \leq (t/2\mu) + 1} S_n \leq t \}.
\]

According to Lemma 3.2.1, the probability in the right hand side of (3.3.11) is equal to

\[
P\{S_{[t/2\mu]+1} \leq t\} + O\left( \frac{1}{([t/2\mu]+1)^{1/2}} \right)
\]

\[
(3.3.12) \quad = P\left\{ \frac{1}{B[t/2\mu]+1} \sum_{j=1}^{[t/2\mu]+1} Z_j \leq \frac{t - \sum_{j=1}^{[t/2\mu]+1} \mu_j}{B[t/2\mu]+1} \right\} + O(t^{-1/2}).
\]

The constant in $O$ may depend only on parameters $\mu$, $\alpha$, $D$, $g$ and $M$ from
conditions (3.2.1). From the theorem proved by Bikelis (1966), it follows that the value of the above expression is equal to

\[(3.3.13) \quad \phi \left( \frac{t - \sum_{j=1}^{[t/2\mu]+1} \mu_j}{B^{-1}_{[t/2\mu]+1}} \right) + O\left( B^{-1}_{[t/2\mu]+1} + O(t^{-1/2}) \right). \]

Because of condition (3.2.1, ii),

\[ O\left( B^{-1}_{[t/2\mu]+1} \right) = O(t^{-1/2}) \quad \text{as} \quad t \to \infty. \]

Also,

\[
\frac{t - \sum_{j=1}^{[t/2\mu]+1} \mu_j}{B^{-1}_{[t/2\mu]+1}} > \frac{t - \{[t/2\mu]+1\} \mu - D}{M_1([t/2\mu]+1)^{1/2}} > c_1 t^{1/2},
\]

for some \( c_1 > 0 \). Therefore, the value of the expression in (3.3.13) is greater than \( \phi(c_1 t^{1/2}) + O(t^{-1/2}) \).

Hence by combining (3.3.11), (3.3.12) and (3.3.13), we obtain

\[
P\{N_1(t) \geq k(x,t)\} > \phi(c_1 t^{1/2}) + O(t^{-1/2})
\]

and thus

\[
1 - P\{N_1(t) \geq k(x,t)\} < 1 - \phi(c_1 t^{1/2}) + O(t^{-1/2}) = O(t^{-1/2})
\]

as \( t \to \infty \).

This proves (3.3.8) and therefore, under the condition of Case 1, the theorem is proved.

Let us now consider Case 2, i.e., when \( x \geq -t/2B_{[t/\mu]} \). In this case,

\[
k(x,t) = \frac{t}{\mu} + xB_{[t/\mu]/\mu} + 1 > \frac{t}{2\mu},
\]

so

\[
\{k(x,t)\}^{-1/2} = O(t^{-1/2}), \quad \text{uniformly in} \ x.
\]
From (3.3.5) and Lemma 3.2.1 we get

$$P \left\{ \frac{N_1(t) - t/\mu}{\mu B[t/\mu]} \leq x \right\} = 1 - P \left\{ \max_{1 \leq n \leq k(x,t)} S_n \leq t \right\}$$

(3.3.14)

$$= 1 - P\{S_k(x,t) \leq t\} + O(\{k(x,t)\}^{-1/2})$$

$$= 1 - P\{S_k(x,t) \leq t\} + O(t^{-1/2}) \quad \text{as } t \to \infty,$$

uniformly in $x$.

Similarly, from (3.3.7) and Lemma 3.2.4 we deduce that

$$P \left\{ \frac{N_3(t) - t/\mu}{\mu B[t/\mu]} \leq x \right\} = 1 - P\{\text{there exists } n \geq k(x,t) \text{ such that } S_n \leq t\}$$

(3.3.15)

$$= 1 - P\{S_k(x,t) \leq t\} + O(\{k(x,t)\}^{-1/2})$$

$$= 1 - P\{S_k(x,t) \leq t\} + O(t^{-1/2}) \quad \text{as } t \to \infty,$$

uniformly in $x$.

Formulae (3.3.14) and (3.3.15) show that in order to prove the theorem for $N_1(t)$ and $N_3(t)$ (and hence for $N_2(t)$ too), we only need to show that

(3.3.16) $$1 - P\{S_k(x,t) \leq t\} = \Phi(x) + O(t^{-1/2}) \quad \text{as } t \to \infty,$$

uniformly in $x$. Obviously,

(3.3.17) $$P\{S_k(x,t) \leq t\} = P \left\{ \frac{1}{B_k(x,t)} \sum_{j=1}^{k(x,t)} Z_j \leq \frac{t - \sum_{j=1}^{k(x,t)} \mu_j}{B_k(x,t)} \right\}.$$

According to Theorem 2.2.2, the above probability is equal to

(3.3.18) $$\Phi \left( \frac{t - \sum_{j=1}^{k(x,t)} \mu_j}{B_k(x,t)} \right) + O(\frac{\sum_{j=1}^{k(x,t)} E|Z_j|^3}{B_k(x,t)^{3/2}}),$$

with an absolute constant in $O$. 

Since

\[
\frac{\sum_{j=1}^{k(x,t)} E[Z_j] \mu_j}{B_{k(x,t)}} \leq \frac{M k(x,t)}{\{g_k(x,t)\}^{3/2}} = O(\{k(x,t)\}^{-1/2}) = O(t^{-1/2}),
\]

uniformly in x, it follows from (3.3.17) and (3.3.18) that

\[
1 - P\{S_{k(x,t)} \leq t\} = 1 - \Phi\left(\frac{t - \sum_{j=1}^{k(x,t)} \mu_j}{B_{k(x,t)}}\right) + O(t^{-1/2})
\]

\[
= \Phi\left(\frac{\sum_{j=1}^{k(x,t)} \mu_j - t}{B_{k(x,t)}}\right) + O(t^{-1/2}).
\]

Hence, in order to obtain a complete proof of the theorem, all we have to do is to show that

\[
\left|\Phi\left(\frac{\sum_{j=1}^{k(x,t)} \mu_j - t}{B_{k(x,t)}}\right) - \Phi(x)\right| = O(t^{-1/2}) \quad \text{as } t \to \infty,
\]

uniformly in x.

By using conditions (3.2.1, (ii) and (iii)), we obtain

\[
\frac{\sum_{j=1}^{k(x,t)} \mu_j - t}{B_{k(x,t)}} = \frac{k(x,t)\mu - t}{B_{k(x,t)}} + \frac{\sum_{j=1}^{k(x,t)} \mu_j - k(x,t)\mu}{B_{k(x,t)}}
\]

\[
= \frac{k(x,t)\mu - t}{B_{k(x,t)}} + O(t^{-1/2}),
\]

uniformly in x. Since \(|\Phi(y_1) - \Phi(y_2)| \leq |y_1 - y_2|\) for any real \(y_1, y_2\), in order to prove (3.3.21) it will suffice to show that under the condition of Case 2,
\[(3.3.23) \quad \left| \Phi \left( \frac{k(x,t)\mu - t}{B_{k(x,t)}} \right) - \Phi(x) \right| = O(t^{-1/2}) \quad \text{as } t \to \infty, \]

uniformly in \(x\).

We will break Case 2 into the following subcases:

**Case 2.1:** \(x > t^{1/6}\),

**Case 2.2:** \(0 \leq x \leq t^{1/6}\)

and

**Case 2.3:** \(-\frac{t}{2B[t/\mu]} \leq x < 0\).

First, let us consider Case 2.1. Under its condition, one can see that

\[1 - \Phi(x) = O(t^{-1/2}),\]

uniformly in \(x\). So we need to show that

\[(3.3.24) \quad 1 - \Phi \left( \frac{k(x,t)\mu - t}{B_{k(x,t)}} \right) < ct^{-1/2},\]

for some constant \(c\). Let us evaluate the argument of \(\Phi\) in (3.3.24). We have

\[(3.3.25) \quad \frac{k(x,t)\mu - t}{B_{k(x,t)}} = \frac{\left\{t/\mu + xB[t/\mu]/\mu + 1\right\} \mu - t}{B_{k(x,t)}}\]

\[\geq \frac{x B[t/\mu]}{B_{k(x,t)}} = x \sqrt{\frac{\sum_{j=1}^{[t/\mu]} \sigma_j^2}{\sum_{j=1}^{k(x,t)} \sigma_j^2}}\]

\[= x \sqrt{\frac{\sum_{j=1}^{[t/\mu]} \sigma_j^2}{\sum_{j=[t/\mu]}^{k(x,t)} \sigma_j^2 + \sum_{j=[t/\mu]+1}^{\sigma_j^2}}} = x \sqrt{\frac{1}{1 + A_1}},\]

where
\[ A_1 = \frac{\sum_{j=\lceil t/\mu \rceil + 1}^{k(x,t)} \sigma_j}{\sum_{j=1}^{\lceil t/\mu \rceil} \sigma_j^2} \leq \frac{M_1(k(x,t) - \lceil t/\mu \rceil)}{g(t/\mu)} \]

\[ \leq \frac{M_1\left(\frac{x^B t/\mu}{\mu} + 1\right)}{g(t/\mu)} \text{ (from the definition of } k(x,t)) \]

\[ \leq \frac{M_1\left(\frac{x^{M_1 t/\mu}}{\mu} + 1\right)}{g(t/\mu)} \leq \frac{c_1 x}{t^{1/2}}, \]

for some constant \( c_1 \) independent of \( x \) and \( t \). Hence the value of the expression in the right hand side of (3.3.25) is greater than or equal to

\[ x \sqrt{\frac{1}{1 + \frac{c_1 x}{t^{1/2}}}} = \sqrt{\frac{1}{1 + \frac{c_1}{x^{1/2}}}}. \]

Since \( x > t^{1/6} \),

\[ \frac{1}{x^2} + \frac{c_1}{xt^{1/2}} < c_2 t^{-1/3}, \text{ for some } c_2. \]

Therefore,

\[ \sqrt{\frac{1}{1 + \frac{c_1}{xt^{1/2}}}} > c_3 t^{1/6}, \]

for some \( c_3 \) independent of \( x \) and \( t \). Combining (3.3.25), (3.3.27) and (3.3.28), we prove that

\[ \frac{k(x,t)\mu - t}{B_k(x,t)} > c_3 t^{1/6} \]

and that proves (3.3.24). Thus in Case 2.1, the theorem is proved.

Now, let us consider Case 2.2, i.e., when \( 0 \leq x \leq t^{1/6} \). In this case,
\[
\frac{x - \frac{k(x,t)\mu - t}{B_k(x,t)}}{B_k(x,t)} = \frac{x B_k(x,t) - \left\{ t/\mu + x B_{\lceil t/\mu \rceil}/\mu \right\} + 1}{B_k(x,t)} \mu + t \\
= \frac{x \left\{ B_k(x,t) - B_{\lceil t/\mu \rceil} \right\} t}{B_k(x,t)} + O(t^{-1/2}) \\
= x \left\{ 1 - \frac{B_{\lceil t/\mu \rceil}}{B_k(x,t)} \right\} + O(t^{-1/2}) \\
= x \left\{ 1 - \sqrt{\frac{\sum [t/\mu]}{\Sigma_{j=1}^{k(x,t)} \sigma_j^2}} \right\} + O(t^{-1/2}) \\
= x \left\{ 1 - \sqrt{\frac{1}{\sum_{j=1}^{k(x,t)} \sigma_j^2 + \sum_{j=\lceil t/\mu \rceil+1}^{x} \sigma_j^2}} \right\} + O(t^{-1/2}) ,
\]

where

\[
A_2 = \frac{\sum_{j=\lceil t/\mu \rceil+1}^{x} \sigma_j^2}{\sum_{j=1}^{\lceil t/\mu \rceil} \sigma_j^2} .
\]

By performing the calculations similar to those in (3.3.26), we find that

\[
A_2 \leq \frac{x}{g(t/\mu)} \leq \frac{c(1+x)}{t^{1/2}} ,
\]

for some c that does not depend on x and t. Hence the right hand side of (3.3.30) does not exceed

\[
x \left\{ 1 - \sqrt{\frac{1}{1 + \frac{c(1+x)}{t^{1/2}}}} \right\} = x \left\{ 1 - \left( 1 + O\left( \frac{1+x}{t^{1/2}} \right) \right) \right\} \\
= O\left( \frac{x+x^2}{t^{1/2}} \right) = O\left( \frac{1+x^2}{t^{1/2}} \right) .
\]
Therefore, it follows from (3.3.30) that

\begin{equation}
(3.3.33) \quad x - \frac{k(x,t)\mu - t}{B_{k(x,t)}} \leq \frac{c_1(1 + x^2)}{t^{1/2}},
\end{equation}

for some constant \(c_1\).

Direct calculations show that

\begin{equation}
(3.3.34) \quad x - \frac{k(x,t)\mu - t}{B_{k(x,t)}} \geq - \frac{c_2}{t^{1/2}},
\end{equation}

for some \(c_2 > 0\), independent of \(x\) and \(t\). Thus, if \(x \leq \frac{k(x,t)\mu - t}{B_{k(x,t)}}\) then

\begin{equation}
(3.3.35) \quad \left| \phi\left(\frac{k(x,t)\mu - t}{B_{k(x,t)}}\right) - \phi(x) \right| \leq \left| \frac{k(x,t)\mu - t}{B_{k(x,t)}} - x \right| = O(t^{-1/2}),
\end{equation}

so (3.3.23) holds. Similarly, if \(x \leq 1\) then

\begin{equation}
\left| \phi\left(\frac{k(x,t)\mu - t}{B_{k(x,t)}}\right) - \phi(x) \right| \leq \left| \frac{k(x,t)\mu - t}{B_{k(x,t)}} - x \right|
\end{equation}

\begin{equation}
(3.3.36) \quad < \frac{c_1(1 + x^2)}{t^{1/2}} \quad \text{(according to (3.3.33))}
\end{equation}

= \(O(t^{-1/2})\), since \(0 \leq x \leq 1\).

So we can assume that \(x > 1\) and \(x > \frac{k(x,t)\mu - t}{B_{k(x,t)}}\). Suppose that \(t\) is large enough so that \(\frac{c_1(1 + x^2)}{t^{1/2}} < x/2\). (This can be done since \(1 < x \leq t^{1/6}\).) This implies that

\begin{equation}
x > \frac{k(x,t)\mu - t}{B_{k(x,t)}}
\end{equation}

\begin{equation}
> x - \frac{c_1(1 + x^2)}{t^{1/2}} \quad \text{(by (3.3.33))}
\end{equation}

\begin{equation}
> x/2 > 0.
\end{equation}
Hence

$$\phi(x) - \Phi\left(\frac{k(x,t)\mu - t}{B_{k(x,t)}}\right) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{x} e^{-y^2/2} \, dy$$

$$\leq \frac{1}{(2\pi)^{1/2}} \exp\left\{ -\frac{1}{2} \left( \frac{k(x,t)\mu - t}{B_{k(x,t)}} \right)^2 \right\} \left( x - \frac{k(x,t)\mu - t}{B_{k(x,t)}} \right)$$

$$\leq \frac{1}{(2\pi)^{1/2}} \exp\left\{ -\frac{1}{2} (x - \frac{c_1(1+x^2)}{t^{1/2}})^2 \right\} \cdot \frac{c_1(1+x^2)}{t^{1/2}}$$

(3.3.36)

$$\leq \frac{1}{(2\pi)^{1/2}} \exp\left\{ -\frac{1}{2} (x/2)^2 \right\} \cdot \frac{c_1(1+x^2)}{t^{1/2}} \cdot x^2 \exp(-x^2/8)$$

$$= O(t^{-1/2}) \quad \text{as } t \to \infty,$$

uniformly in $x$.

This proves (3.3.23). So the theorem is proved under the condition of Case 2.2 too.

Finally, consider Case 2.3, i.e., when $-\frac{t}{2B_{[t/\mu]}} \leq x < 0$.

In this case, $k(x,t) \leq [t/\mu] + 1$. If

(3.3.37) \quad k(x,t) = [t/\mu] + 1,

then $\frac{|x|B_{[t/\mu]}}{\mu} < 1$, so $|x| = O(t^{-1/2})$. Also, if (3.3.37) holds then $\frac{k(x,t)\mu - t}{B_{k(x,t)}} = O(t^{-1/2})$. This shows that
\[ x - \frac{k(x,t)\mu - t}{B_k(x,t)} = O(t^{-1/2}) \quad \text{as} \quad t \to \infty, \]

uniformly in \( x \). This proves (3.3.23) when condition (3.3.37) holds.

Suppose now that \( k(x,t) \leq [t/\mu] \). Then

\[
x - \frac{k(x,t)\mu - t}{B_k(x,t)} = \frac{x B_k(x,t) - [t/\mu + x B_{[t/\mu]}/\mu] \mu - \mu + t}{B_k(x,t)} \\
\geq \frac{x B_k(x,t) - (t/\mu + x B_{[t/\mu]}/\mu) \mu - \mu + t}{B_k(x,t)} \\
= \frac{(-x)(B_{[t/\mu]} - B_k(x,t))}{B_k(x,t)} - \frac{\mu}{B_k(x,t)} \\
\geq - \frac{\mu}{B_k(x,t)} > -ct^{-1/2},
\]

for some \( c > 0 \). So if \( x \leq \frac{k(x,t)\mu - t}{B_k(x,t)} \) then

\[ 0 \leq \Phi \left( \frac{k(x,t)\mu - t}{B_k(x,t)} \right) - \Phi(x) < ct^{-1/2}, \]

so (3.3.23) holds in this situation too. So we only have to consider the case when

\[ x > \frac{k(x,t)\mu - t}{B_k(x,t)}. \]

Performing the calculations similar to those in (3.3.25) and (3.3.30), we obtain:

\[
0 < x - \frac{k(x,t)\mu - t}{B_k(x,t)} = \frac{x(B_k(x,t) - B_{[t/\mu]})}{B_k(x,t)} + O(t^{-1/2}) \\
= x \left( 1 - \frac{B_{[t/\mu]}}{B_k(x,t)} \right) + O(t^{-1/2})
\]
(3.3.39) \quad x \left\{ 1 - \sqrt{\frac{\sum_{j=1}^{[t/\mu]} \sigma_j^2}{\sum_{j=1}^{[t/\mu]} \sigma_j^2 - \sum_{j=k(x,t)+1}^{[t/\mu]} \sigma_j^2}} \right\} + O(t^{-1/2})

= |x| \left\{ \frac{1}{1 - A_3} - 1 \right\} + O(t^{-1/2}),

where

\begin{align*}
A_3 &= \frac{\sum_{j=k(x,t)+1}^{[t/\mu]} \sigma_j^2}{\sum_{j=1}^{[t/\mu]} \sigma_j^2} \\
&\leq \frac{M_1 \{ [t/\mu] - k(x,t) \}}{g[t/\mu]}
\end{align*}

(3.3.40)

\begin{align*}
&= \frac{|x| M_1 B_{[t/\mu]}}{\mu g[t/\mu]} + O(t^{-1}) \\
&\leq \frac{|x| M_1^{3/2} [t/\mu]^{1/2}}{\mu g[t/\mu]} + O(t^{-1}) \leq \frac{c(1 + |x|)}{t^{1/2}},
\end{align*}

for some $c > 0$ that does not depend on $x$ and $t$. Therefore the right hand side of (3.3.39) does not exceed

\begin{align*}
|x| \left\{ \frac{1}{1 - \frac{c(1 + |x|)}{t^{1/2}}} - 1 \right\} = |x| \left\{ \left(1 + O\left(\frac{1 + |x|}{t^{1/2}}\right)\right) - 1 \right\}
\end{align*}

(3.3.41)

\begin{align*}
&= \frac{c_1 (1 + x^2)}{t^{1/2}},
\end{align*}

for an appropriate $c_1 > 0$. We showed, therefore, that

\[ 0 < x \frac{k(x,t)\mu - t}{B_{k(x,t)}} \leq \frac{c_1 (1 + x^2)}{t^{1/2}}. \]
Hence

\[ \Phi(x) - \Phi\left( \frac{k(x,t)\mu - t}{B_k(x,t)} \right) = \frac{1}{(2\pi)^{1/2}} \int_{x}^{\infty} e^{-y^2/2} dy \]

\[ \leq \frac{1}{(2\pi)^{1/2}} \exp(-x^2/2) \left( x - \frac{k(x,t)\mu - t}{B_k(x,t)} \right) \]

\[ \leq \frac{1}{(2\pi)^{1/2}} \exp(-x^2/2) \cdot \frac{c_1(1+x^2)}{t^{1/2}} \]

\[ = O(t^{-1/2}), \]

uniformly in \( x \). This proves (3.3.23) under the condition of Case 2.3.

Theorem 3.3.1 is now completely proved. \( \square \)

3.4. THE NECESSITY OF CONDITIONS IN THE CENTRAL LIMIT THEOREM

In Section 3.3 we proved the Central Limit Theorem for a renewal process. That renewal process was based on a sequence of independent random variables \( X_1, X_2, \ldots \) with the finite absolute third moments. Also, we required conditions (3.2.1, (i)--(iv)) to hold. In this section, we raise a question of the necessity of these conditions.

First, let us note that parts (iii) and (iv) of (3.2.1) are not unusual. Very similar conditions are imposed by Petrov (1960) in his Central Limit Theorem for the sequences of independent non-identically distributed random variables with zero means and by Ahmad (1981) in the Central Limit Theorem for the renewal processes.

Condition (3.2.1, (i)) was also assumed by Ahmad in that portion of his 1981 paper where he proved a lemma for the non-identically distributed \( X_n \)'s. This
condition assures the (a.s.) finiteness of \( N_i(t), \ i = 1, 2, 3 \).

The only new condition that we introduce is (3.2.1, (ii)). It may look too restrictive. For example, Cox and Smith (1953) were able to achieve significant results in renewal theory by assuming that a different condition on the \( \mu_n \)'s is satisfied. They introduced the so-called "stable sequences."

Definition (Cox and Smith, 1953).

A sequence \( \{\mu_n\} \) such that

\[
\lim_{p \to \infty} \frac{1}{(p+1)} \sum_{i=n}^{n+p} \mu_i = \mu \quad \text{uniformly in} \quad n
\]

is called stable with average \( \mu \).

Our condition (3.2.1, (ii)), instead, requires the existence of positive numbers \( \mu \) and \( D \) such that

\[
\left| \sum_{j=1}^{n} \mu_j - n\mu \right| < D \quad \text{for all} \quad n \geq 1.
\]

The following proposition provides a comparison between the stable sequences and those satisfying (3.4.2).

Proposition 3.4.1.

(i) If a sequence \( \{\mu_n\} \) satisfies condition (3.4.2), then it is stable.

(ii) There exist stable sequences that do not satisfy (3.4.2).

Proof.

(i) Suppose \( \{\mu_n\} \) satisfies (3.4.2) with some \( D \) and \( \mu > 0 \). Then
\[ \left| \frac{1}{p+1} \sum_{i=n}^{n+p} \mu_i - \mu \right| \]

\[ = \left| \frac{1}{p+1} \left( \left( \sum_{i=1}^{n+p} \mu_i - (n+p)\mu \right) - \left( \sum_{i=1}^{n-1} \mu_i - (n-1)\mu \right) \right) \right| \]

\[ \leq \left| \sum_{i=1}^{n+p} \mu_i - (n+p)\mu \right|/(p+1) + \left| \sum_{i=1}^{n-1} \mu_i - (n-1)\mu \right|/(p+1) \]

\[ \leq 2D/(p+1), \]

hence

\[ \lim_{p \to \infty} \frac{1}{p+1} \sum_{i=n}^{n+p} \mu_i = \mu \text{ uniformly in } n. \]

That proves (i).

(ii). Consider the following sequence: \( \mu_n = \mu + n^{1/2} - (n-1)^{1/2}, n \geq 1, \) where \( \mu > 0 \) is fixed. Sequence \( \{\mu_n\} \) is stable since for any \( n, \)

\[ \mu < \frac{1}{p+1} \sum_{i=n}^{n+p} \mu_i = \frac{1}{p+1} \left\{ (p+1)\mu + (n+p)^{1/2} - (n-1)^{1/2} \right\} \]

\[ = \mu + \frac{(n+p)^{1/2} - (n-1)^{1/2}}{p+1} \leq \mu + \frac{1}{(p+1)^{1/2}}, \]

which proves that (3.4.1) holds. Let us now suppose that (3.4.2) also holds for \( \{\mu_n\}, \) i.e., there exist \( \mu', D > 0 \) such that for all \( n, \) \( |\sum_{j=1}^{n} \mu_i - n\mu'| < D. \) Evidently,

\[ \left| \sum_{j=1}^{n} \mu_j - n\mu' \right| = \left| n\mu + n^{1/2} - n\mu' \right|. \]

That shows that \( |\sum_{j=1}^{n} \mu_i - n\mu'| \to \infty \) as \( n \to \infty \) for any choice of \( \mu'. \) Therefore, \( \{\mu_n\} \) does not satisfy (3.4.2) and that completes the proof of the proposition. \( \square \)
In the paper mentioned above, Cox and Smith achieved some significant results by only assuming that the sequence \( \{ \mu_n \} \) of means of the random variables \( X_1, X_2, \ldots \) is a stable sequence. In Theorem 3.3.1, we require that \( \{ \mu_n \} \) satisfies a stronger condition. The following theorem shows that even if we wanted to prove Theorem 3.3.1 for \( N_1(t) \) only and even if we didn't want a uniform (in \( x \)) bound for

\[
\left| P \left\{ \frac{N_1(t) - t/\mu}{\hat{\mu} B(t/\mu)} \leq x \right\} - \phi(x) \right|,
\]

we would still need condition (3.2.1, (ii)) (or (3.4.2), as it was restated in this section) to hold.

**Theorem 3.4.1.**

Suppose that conditions (3.2.1, (i), (iii) and (iv)) are satisfied with some positive \( \alpha, \beta \) and \( M \) and that for some \( \mu > 0 \),

\[
(3.4.6) \quad P \left\{ \frac{N_1(t) - t/\mu}{\hat{\mu} B(t/\mu)} \leq x \right\} - \phi(x) = O(t^{-1/2}),
\]

where a constant in \( O \) may depend on \( x \). Then there exists \( D > 0 \) such that for all \( n \geq 1 \),

\[
\left| \sum_{j=1}^{n} \mu_j - n\mu \right| \leq D,
\]

i.e., condition (3.2.1, (ii)) holds.

**Proof.**

It has been shown in the course of the proof of Theorem 3.3.1 that with conditions (3.2.1, (i), (iii) and (iv)) satisfied,

\[
(3.4.7) \quad P \left\{ \frac{N_1(t) - t/\mu}{\hat{\mu} B(t/\mu)} \leq x \right\} = \phi \left( \frac{\Sigma_{j=1}^{k(x,t)} \mu_j - t}{B_{k(x,t)}} \right) + O(t^{-1/2}).
\]
From (3.4.6) and (3.4.7) it follows that

\[
(3.4.8) \quad \left| \Phi \left( \frac{\sum_{j=1}^{k(x,t)} \mu_j - t}{B_{k(x,t)}} \right) - \Phi(x) \right| < \frac{c(x)}{t^{1/2}},
\]

where \(c(x)\) is a constant that may depend on \(x\) but not on \(t\).

Set \(x = 0\) and suppose with no loss of generality that \(t\) tends to \(+\infty\) through such values that \(t/\mu\) is an integer. Then we have \(\Phi(x) = \frac{1}{2}\) and \(k(x,t) = t/\mu + 1\).

Set \(c_1 = 2(2\pi)^{1/2}c(0)\) and suppose \(t\) is large enough so that

\[
(3.4.9) \quad \exp\left(-\frac{c_1^2}{2t}\right) > \frac{1}{2}.
\]

Denote \(z_t = \frac{\sum_{j=1}^{t/\mu+1} \mu_j - t}{B_{t/\mu+1}}\). Let us show that

\[
(3.4.10) \quad |z_t| \leq c_1 t^{-1/2}.
\]

Indeed, suppose (3.4.10) is not true. Set \(y_t = c_1 t^{-1/2}\). Evidently,

\[
c(0)t^{-1/2} > |\Phi(z_t) - \Phi(0)| \quad \text{(by (3.4.8) with } x = 0) \]

\[
> \Phi(y_t) - \Phi(0) \quad \text{(since } |z_t| > y_t) \]

\[
= y_t \Phi'(\xi_t) \quad \text{(for some } \xi_t \text{ such that } 0 \leq \xi_t \leq y_t) \]

\[
= y_t (2\pi)^{-1/2} \exp(-\xi_t^2/2)
\]

\[
(3.4.11) \quad \geq y_t (2\pi)^{-1/2} \exp(-y_t^2/2)
\]

\[
= \frac{c_1}{t^{1/2}} \cdot \frac{1}{(2\pi)^{1/2}} \exp(-c_1^2/2t)
\]

\[
> \frac{c_1}{2(2\pi)^{1/2}} t^{-1/2} \quad \text{(according to (3.4.9))}
\]

\[
= c(0)t^{-1/2}.
\]
This contradiction shows that (3.4.10) holds. Hence

\[(3.4.12) \quad \left| \sum_{j=1}^{t/\mu+1} \mu_j - t \right| \leq \frac{c_1B_{t/\mu+1}}{t^{1/2}} \leq \frac{c_1\sqrt{M_1(t/\mu+1)}}{t^{1/2}} < c_2,\]

where \( M_1 \) is such that \( \sigma_j^2 < M_1 \) for all \( j \) and \( c_2 \) is a constant independent of \( t \). Set \( n = t/\mu + 1 \). Inequalities (3.4.12) show that

\[c_2 > \left| \sum_{j=1}^{n} \mu_j - n\mu + \mu \right| \geq \left| \sum_{j=1}^{n} \mu_j - n\mu \right| - \mu,
\]

so condition (3.2.1, (ii)) has to be satisfied (with \( D = c_2 + \mu \)) for all sufficiently large \( n \). Therefore, it is satisfied for all \( n \geq 1 \) with an appropriate \( D \).

This completes the proof of Theorem 3.4.1. \( \square \)
CHAPTER IV
CENTRAL LIMIT THEOREMS FOR THE CUMULATIVE PROCESSES

4.1. INTRODUCTION

In this chapter we study properties of a cumulative process $W(t)$ defined in Section 1.1. Instead of referring to $F(x,y)$, a joint distribution function of $X_1$ and $Y_1$, we will use a conditional distribution function $G_x(y)$, which uniquely determines the distribution of $Y_1$ when $X_1$ is equal to $x$. The existence of $G_x(y)$ is guaranteed for any joint distribution function $F(x,y)$ (see Doob (1953)).

Throughout this chapter we will assume that $X_1$ (and hence every $X_j$, $j=1, 2, \ldots$) satisfies the following so-called Cramér condition: there exists $g > 0$ such that

$$E e^{g X_1} < \infty. \quad (4.1.1)$$

Condition (4.1.1) assures, in particular, that all moments of $X_1$ are finite. We also assume that there exists a real number $r > 2$ such that

$$E |Y_1|^r < \infty. \quad (4.1.2)$$

We are interested in obtaining Central Limit Theorems with a remainder term whose order depends on $r$.

We will use the following notation:

$\psi(\theta)$ — characteristic function of $Y_1$,

$\kappa(\theta)$ — cumulative function of $Y_1$ (see Appendix A for definition),

$$\theta_t = \frac{\theta \mu^{1/2}}{\nu_t^{1/2}}. \quad (4.1.3)$$
From the definitions it follows that $\psi(\theta) = \exp(\kappa(\theta))$ whenever $\psi(\theta) \neq 0$.

Define also

\[(4.1.4) \quad V(t) = \frac{W(t)\mu^{1/2}}{\tau t^{1/2}}.\]

$V(t)$ is a normalized cumulative process. Our Central Limit Theorems will be stated for $V(t)$, not for $W(t)$.

In all $O(\ )$ expressions the constants are independent of $t$ and $\theta$. Words “for all sufficiently large $t$” mean “for $t > t_0$, where $t_0$ depends on the joint distribution of $(X_1, Y_1)$ only.”

In Section 4.2 we examine the behavior of the distribution function of the random variable $R(t)$, where for a given $t$,

\[(4.1.5) \quad R(t) = \frac{N(t) + 1 - t/\mu}{\sigma t^{1/2}/\mu^{3/2}}.\]

It turns out that, because of the Cramér condition (4.1.1), the tails of the distribution function of $R(t)$ can be bounded uniformly in $t$ by an exponentially decreasing function.

In Section 4.3 we prove several lemmas that allow us to estimate the cumulative function of $Y_1$ under an assumption of finiteness of the higher moments of $Y_1$.

In Section 4.4, we prove a lemma that later helps to estimate an error in the characteristic function of $V(t)$ caused by an approximation of $N(t)$ by its mean. Then in Section 4.5 we make a necessary evaluation of the behavior of the characteristic function of $V(t)$ as $t$ tends to $\infty$.

Section 4.6 contains two Central Limit Theorems for $V(t)$ based on $Y_n$'s with zero means. The difference between the two is in the assumptions on the
moments of $Y_1$. The deepest of these results is Theorem 4.6.1. It provides the smallest remainder term. Both theorems in this section constitute a significant improvement over Theorem 2.4.1.

In Section 4.7 we drop an assumption of zero means and prove two CLTs that apply to arbitrary $Y$'s with a finite moment of a certain order.

Section 4.8 is dedicated to the estimation of average remainder terms in the Central Limit Theorems proved in the previous section. The main result of this section can be viewed as an extension of Theorem 2.2.3 into the area of the cumulative processes.

Finally, in Section 4.9 we discuss some further possible improvements in the Central Limit Theorems.

Results of this chapter bring the theory of the Central Limit Theorems for the cumulative processes closer to the level of where the theory of the CLTs for the renewal processes and for the sums of independent random variables stands.

4.2. ESTIMATES OF THE TAILS OF THE DISTRIBUTION FUNCTION OF $R(t)$

In this section we prove a lemma the result of which shows how close in probabilistic terms the value of $N(t)$ is to $t/\mu$. While this lemma does not refer to the random variables $Y_1, Y_2, \ldots$ at all, its result plays a critical role in estimating the characteristic function of $V(t)$.

A random variable $R(t)$ is defined by (4.1.5). Let $F_{R(t)}$ be a distribution function of $R(t)$, i.e., $F_{R(t)}(x) = P\{R(t) \leq x\}$. 

Lemma 4.2.1.

There exist positive real constants $\alpha$ and $\nu$ that do not depend upon $x$ or $t$ such that for all sufficiently large $t$, the following inequalities are satisfied:

(i) for any $x \geq 0$,

\[
1 - F_{R(t)}(x) \leq \alpha e^{-\nu x}
\]

(4.2.1)

and

(ii) for any $x \leq 0$,

\[
F_{R(t)}(x) \leq \alpha e^{-\nu |x|}.
\]

(4.2.2)

Proof.

We will first find $\alpha$ and $\nu$ that satisfy (4.2.1). The following three cases will be considered separately.

Case 1: $0 \leq x \leq t^{1/3}$,

Case 2: $t^{1/3} < x \leq Ct^{1/2}$, where $C$ is a constant that will be chosen later, and

Case 3: $x > Ct^{1/2}$.

We will consider Case 3 first. For all sufficiently large $t$,

\[
1 - F_{R(t)}(x) = P \left\{ \frac{N(t) + \frac{1 - t}{\mu}}{\sigma t^{1/2}/\mu^{3/2}} > x \right\}
\]

(4.2.3)

\[
= P \{ N(t) > x \sigma t^{1/2}/\mu^{3/2} + t/\mu - 1 \}
\]

\[
\leq P \{ N(t) \geq [x \sigma t^{1/2}/\mu^{3/2}] + 1 \}.
\]

Let us show that for any integer $k \geq 0$, ...
(4.2.4) \( P \{ N(t) \geq k \} \leq e^t \{ E e^{-X_1} \}^k. \)

Write \( S_k \) for \( X_1 + \ldots + X_k \) and let \( F_k \) be a distribution function of \( S_k \). Evidently,
\[
\{ E e^{-X_1} \}^k = E e^{-S_k} = \int_0^\infty e^{-x} dF_k(x)
\geq \int_0^t e^{-x} dF_k(x) \geq e^{-t} \int_0^t dF_k(x) = e^{-t} P \{ S_k \leq t \}
\]
\[
= e^{-t} P \{ N(t) \geq k \} \quad \text{(according to (2.3.1)).}
\]

That proves (4.2.4).

Taking \( k = [x\sigma^{1/2}/\mu^{3/2}] + 1 \) in (4.2.4), we deduce from (4.2.3) that
\[
(4.2.5) \quad 1 - F_{R(t)}(x) \leq e^t \{ E e^{-X_1} \}^{[x\sigma^{1/2}/\mu^{3/2}] + 1}.
\]

Since \( 0 < E e^{-X_1} < 1 \), there exists \( \delta > 0 \) such that
\[
(4.2.6) \quad e^{-\delta} = E e^{-X_1}.
\]

Hence
\[
1 - F_{R(t)}(x) \leq e^t e^{-\delta \{ x\sigma^{1/2}/\mu^{3/2} \} + 1}
\]
\[
(4.2.7) \quad < e^{t - x\delta \sigma^{1/2}/\mu^{3/2}}.
\]

Fix any \( \nu > 0 \). Assume that \( t \) is large enough so that
\[
(4.2.8) \quad \nu < \delta \sigma^{1/2}/2\mu^{3/2}.
\]

Choose
\[
(4.2.9) \quad C = 2\mu^{3/2}/\delta \sigma.
\]

That will be the choice of \( C \) that defines the condition of Case 3. Then
\[ x \delta \sigma t^{1/2} / \mu^{3/2} - \nu x = (\delta \sigma t^{1/2} / \mu^{3/2} - \nu)x \]
\[ > \delta \sigma t^{1/2} x / 2 \mu^{3/2} \quad \text{(because of (4.2.8))} \]
\[ > \delta \sigma C t / 2 \mu^{3/2} \quad \text{(since } x > C t^{1/2}) \]
\[ = t \quad \text{(by the choice of } C) \]

That shows that

\[ t - \delta \sigma x t^{1/2} / \mu^{3/2} < -\nu x, \]

so it follows from (4.2.7) that \( 1 - F_R(t)(x) < e^{-\nu x} \). Hence in Case 3, (4.2.1) is satisfied for any given \( \nu > 0 \) with \( \sigma = 1 \) as long as \( t \) is sufficiently large.

Let us now evaluate the left hand side of (4.2.1) when Case 1 or Case 2 apply. Denote \( Z_j = (X_j - \mu) / \sigma \) for \( j = 1, 2, \ldots \) and \( k(x,t) = [t/\mu + x \sigma t^{1/2} / \mu^{3/2}] \). We have

\[ 1 - F_R(t)(x) = P\{R(t) > x\} = P\left\{ \frac{N(t) + 1 - t / \mu}{\sigma t^{1/2} / \mu^{3/2}} > x \right\} \]
\[ = P\{N(t) > t/\mu - 1 + x \sigma t^{1/2} / \mu^{3/2}\} \]
\[ = P\{N(t) \geq [t/\mu + x \sigma t^{1/2} / \mu^{3/2}]\}. \]
\[ = P\{N(t) \geq k(x,t)\} \quad \text{(by the definition of } k(x,t)) \]

(4.2.12)
\[ = P\{S_{k(x,t)} \leq t\} \quad \text{(according to (2.3.1))} \]
\[ = P\left\{ \frac{k(x,t)}{\{k(x,t)\}^{1/2}} \sum_{j=1}^{k(x,t)} Z_j \leq \frac{t - k(x,t) \mu}{\sigma \{k(x,t)\}^{1/2}} \right\} \]
\[ \leq P \left\{ \frac{1}{\{k(x,t)\}^{1/2}} \sum_{j=1}^{k(x,t)} Z_j \leq \frac{t-(t/\mu+x\sigma t^{1/2}/\mu^{3/2}-1)/\mu}{\sigma\{k(x,t)\}^{1/2}} \right\} \]

\[ = P \left\{ \frac{1}{\{k(x,t)\}^{1/2}} \sum_{j=1}^{k(x,t)} Z_j \leq -x_1 \right\}, \]

where

\[ (4.2.13) \quad x_1 = \frac{x \sigma (t/\mu)^{1/2} - \mu}{\sigma \{k(x,t)\}^{1/2}}. \]

We can assume with no loss of generality that \( x_1 > 0 \). Indeed, if \( x_1 \leq 0 \) then \( x \) is bounded by a constant as \( t \to \infty \) (actually, \( x = O(t^{-1/2}) \), but we do not even need that), so \( x e^{-\nu x} \) can be made greater than 1 with an appropriate choice of \( \sigma \) and \( \nu > 0 \). Thus (4.2.1) holds in this case.

Suppose the condition of Case 1 applies, i.e., \( 0 \leq x \leq t^{1/3} \). Then for all sufficiently large \( t \),

\[ \frac{x_1}{\{k(x,t)\}^{1/2}} = \frac{x \sigma (t/\mu)^{1/2} - \mu}{\sigma k(x,t)} = \frac{x \sigma (t/\mu)^{1/2} - \mu}{\sigma [t/\mu + x\sigma t^{1/2}/\mu^{3/2}]} \]

\[ = \frac{x \sigma (t/\mu)^{1/2}}{\sigma [t/\mu + x\sigma t^{1/2}/\mu^{3/2}]} + O(t^{-1}) \]

\[ (4.2.14) \quad \leq \frac{x(t/\mu)^{1/2}}{t/\mu + x\sigma t^{1/2}/\mu^{3/2} - 1} + O(t^{-1}) \]

\[ = \frac{x}{(t/\mu)^{1/2} + x\sigma /\mu - (\mu/t)^{1/2}} + O(t^{-1}) \]

\[ \leq \frac{x}{(t/\mu)^{1/2}} + O(t^{-1/2}) = O(1/t^{1/6}) = O\left(\{k(x,t)\}^{-1/6}\right). \]
Hence

(4.2.15) \[ x_1 = O\left(\{k(x,t)\}^{1/3}\right) \]

with a constant in $O$ independent of $x$ as well as of $t$.

Formula (4.2.15) shows that we can use Theorem 2.2.4 to estimate the probability in the right hand side of (4.2.12). That theorem is applicable since (4.1.1) implies that $E\exp(gZ_1) < \infty$, i.e., the Cramér condition is satisfied for $Z_1$. Also $EZ_1 = 0$ and $\text{Var } Z_1 = 1$. From Theorem 2.2.4 (see formula (2.2.11)) we obtain (using $r = 4$, so that $r/2(r+2) = 1/3$)

\[
P\left\{\frac{1}{\{k(x,t)\}^{1/2}} \sum_{j=1}^{k(x,t)} Z_j \leq -x_1\right\}
\]

(4.2.16) \[ = \Phi(-x_1)\exp\left\{-\frac{x_1^3}{\{k(x,t)\}^{1/2}}\lambda^{[4]}\left(-\frac{x_1}{\{k(x,t)\}^{1/2}}\right)^3\right\}\left(1 + O\left(\frac{x_1+1}{\{k(x,t)\}^{1/2}}\right)\right)\]

From the definition of functions $\lambda^{[r]}(s)$ it follows that $\lambda^{[r]}(s)$ is bounded as $s \to 0$. Hence

(4.2.17) \[ P\left\{\frac{1}{\{k(x,t)\}^{1/2}} \sum_{j=1}^{k(x,t)} Z_j \leq -x_1\right\} \leq c_1 \Phi(-x_1)\exp\left\{-c_2\frac{x_1^3}{\{k(x,t)\}^{1/2}}\right\}.
\]

Here and later $c_1$, $c_2$, ... are some constants that do not depend on $x$ or $t$.

Since for $x_1 > 0$, $\Phi(-x_1)$ is bounded by $c_3 \exp(-x_1^2/2)$, the right hand side of (4.2.17) does not exceed

(4.2.18) \[ c_4 \exp\left(-x_1^2/2 + \frac{c_2x_1^3}{\{k(x,t)\}^{1/2}}\right).\]
However,

\[
\frac{x_1^3}{\{k(x,t)\}^{1/2}} = x_1^2 \cdot \frac{x_1}{\{k(x,t)\}^{1/2}}
\]

\[
= x_1^2 \cdot O\left(\{k(x,t)\}^{-1/6}\right) \quad \text{(from (4.2.15))}
\]

\[
= x_1 \cdot O(t^{-1/6}),
\]

so the value of the expression in (4.2.18) does not exceed

\[
c_4 \exp\left(-\frac{x_1^2}{2} + c_5 \frac{x_1^2}{t^{1/6}}\right)
\]

\[
\leq c_4 \exp\left(-c_6 x_1^2\right) = c_4 \exp\left(-\frac{c_6(x \sigma(t/\mu)^{1/2} - \mu)^2}{\sigma^2 k(x,t)}\right)
\]

\[
\leq c_7 \exp\left(-\frac{c_8 x_1^2 t}{k(x,t)}\right) \leq c_7 \exp\left(-\frac{c_8 x_1^2 t}{t/\mu + x \sigma t^{1/2}/\mu^{3/2}}\right)
\]

\[
\leq c_7 \exp(-c_9 x^2) \leq c_{10} e^{-c_{11} x}.
\]

Combining (4.2.12), (4.2.17), (4.2.18) and (4.2.20), we see that when conditions of Case 1 apply, there exist \(a, \nu > 0\) such that (4.2.1) holds.

Let us now consider Case 2, i.e., when \(t^{1/3} < x \leq C t^{1/2}\), where \(C\) is a constant defined in (4.2.9). In this case, for all sufficiently large \(t\) we have

\[
\frac{x_1}{\{k(x,t)\}^{1/2}} = \frac{x \sigma(t/\mu)^{1/2} - \mu}{\sigma k(x,t)} \geq \frac{x (t/\mu)^{1/2}}{2k(x,t)}
\]

\[
\geq \frac{x (t/\mu)^{1/2}}{2(t/\mu + x \sigma t^{1/2}/\mu^{3/2})} = \frac{x}{2(\frac{(t/\mu)^{1/2}}{\mu} + x \sigma/\mu)}.
\]

Since \(f(x) = \frac{x}{a + bx}\) is an increasing function of \(x\) for \(a, b, x > 0\), it follows that
(4.2.22) \[ \frac{x}{2\left((t/\mu)^{1/2} + x\sigma/\mu\right)} \geq \frac{t^{1/3}}{2\left((t/\mu)^{1/2} + t^{1/3}\sigma/\mu\right)} > c_1 t^{-1/6}. \]

(Here \(c_1, c_2, \ldots\) is a new set of constants.) From (4.2.21) and (4.2.22) we conclude that

(4.2.23) \[ x_1 > c_1 \{k(x,t)\}^{1/2} t^{-1/6}. \]

Denote \(x_2 = c_1 \{k(x,t)\}^{1/2} t^{-1/6}\). Let us estimate the probability in the right hand side of (4.2.12). Since \(x_1 > x_2\),

(4.2.24) \[ P \left\{ \frac{1}{\{k(x,t)\}^{1/2}} \sum_{j=1}^{k(x,t)} Z_j \leq -x_1 \right\} \leq P \left\{ \frac{1}{\{k(x,t)\}^{1/2}} \sum_{j=1}^{k(x,t)} Z_j \leq -x_2 \right\}. \]

Let us show that Theorem 2.2.4 can be used again. We need to verify that \(x_2 = O(\{k(x,t)\}^{1/3})\). That holds since

\[ \frac{x_2}{\{k(x,t)\}^{1/3}} = \frac{c_1 \{k(x,t)\}^{1/2}}{t^{1/6} \{k(x,t)\}^{1/2}} = c_1 \left(\frac{k(x,t)}{t}\right)^{1/6} \]

(4.2.25) \[ \leq c_1 \left(\frac{t/\mu + x\sigma t^{1/2}/\mu^{3/2}}{t}\right)^{1/6} = c_1 \left(1/\mu + x\sigma/(t\mu^{3/2})\right)^{1/6} = O(1), \text{ because } x \leq C t^{1/2}. \]

Therefore, Theorem 2.2.4 is applicable (with \(r = 4\)) and from it we obtain

\[ P \left\{ \frac{1}{\{k(x,t)\}^{1/2}} \sum_{j=1}^{k(x,t)} Z_j \leq -x_2 \right\} \]

\[ = \Phi(-x_2) \exp \left\{ -\frac{x_2^3}{\{k(x,t)\}^{1/2}} \right\} \left(1 + O\left(\frac{x_2}{\{k(x,t)\}^{1/2}}\right)\right) \]
\[ \leq c_2 \Phi(-x_2) \exp\left(\frac{c_3 x_2^3}{\{k(x,t)\}^{1/3}}\right) \]

(since the argument of \( \lambda^{[4]} \) tends to 0 as \( t \to \infty \))

\[ (4.2.26) \quad < c_4 \exp\left(-\frac{x_2^2}{2} + \frac{c_3 x_2^3}{\{k(x,t)\}^{1/3}}\right) \]

\[ = c_4 \exp\left(-\frac{c_1^2 k(x,t)}{2t^{1/3}} + \frac{c_3 k(x,t)}{t^{1/2}}\right) \quad (\text{we used the definition of } x_2 \text{ here}) \]

\[ < c_4 \exp(-c_6 k(x,t) t^{-1/3}) < c_4 \exp(-c_6(x t^{1/2}/\mu^{3/2}) t^{-1/3}) \]

\[ = c_4 \exp(-c_7 x t^{1/6}) < c_4 e^{-c_8 x}. \]

Hence when conditions of Case 2 apply, (4.2.1) holds for some \( \alpha, \nu > 0 \).

We have shown that (4.2.1) is satisfied in each of the three cases (with possibly different \( \alpha \) and \( \nu \)), i.e., if \( 0 \leq x \leq t^{1/3} \) then \( 1 - F_{R(t)}(x) \leq \alpha_1 e^{-\nu_1 x} \), if \( t^{1/3} < x \leq ct^{1/2} \) then \( 1 - F_{R(t)}(x) \leq \alpha_2 e^{-\nu_2 x} \) and if \( x > ct^{1/2} \) then \( 1 - F_{R(t)}(x) \leq \alpha_3 e^{-\nu_3 x} \).

Therefore (4.2.1) holds for all \( x \geq 0 \) and \( \alpha = \max(\alpha_1, \alpha_2, \alpha_3) \), \( \nu = \min(\nu_1, \nu_2, \nu_3) \).

The statement in part (i) of the lemma is valid for these \( \alpha \) and \( \nu \).

In order to find constants \( \alpha \) and \( \nu \) that will satisfy part (ii) of the lemma, we will consider the following three cases:

**Case 4:** \( 0 \leq |x| \leq t^{1/3} \),

**Case 5:** \( t^{1/3} < |x| \leq (t \mu)^{1/2}/\sigma \) and

**Case 6:** \( |x| > (t \mu)^{1/2}/\sigma \).

Case 6 is trivial since \( F_{R(t)}(x) = 0 \) for \( x < -(t \mu)^{1/2}/\sigma \), because \( N(t) \geq 0 \).
When conditions of Case 4 or Case 5 apply, we can write the following sequence of equalities:

\[
F_{R(t)}(x) = P\{R(t) \leq x\} = P\left\{\frac{N(t) + 1 - t/\mu}{\sigma t^{1/2}/\mu^{3/2}} \leq x\right\}
\]

\[
= P\{N(t) \leq t/\mu - 1 + x\sigma t^{1/2}/\mu^{3/2}\}
\]

(4.2.27) \quad = 1 - P\{N(t) > t/\mu - 1 + x\sigma t^{1/2}/\mu^{3/2}\}

= 1 - P\{N(t) \geq k(x,t)\}

(where \(k(x,t)\), as before, is set equal to \(t/\mu + x\sigma t^{1/2}/\mu^{3/2}\))

\[
= 1 - P\{S_{k(x,t)} \leq t\} = P\{S_{k(x,t)} > t\}.
\]

When Case 4 applies,

\[
P\{S_{k(x,t)} > t\} = P\left\{\frac{1}{\{k(x,t)\}^{1/2}} \sum_{j=1}^{k(x,t)} Z_j > \frac{t - k(x,t)\mu}{\sigma\{k(x,t)\}^{1/2}}\right\}
\]

(4.2.28) \quad \leq P\left\{\frac{1}{\{k(x,t)\}^{1/2}} \sum_{j=1}^{k(x,t)} Z_j > \frac{t - (t/\mu + x\sigma t^{1/2}/\mu^{3/2})\mu}{\sigma\{k(x,t)\}^{1/2}}\right\}

\[
= 1 - P\left\{\frac{1}{\{k(x,t)\}^{1/2}} \sum_{j=1}^{k(x,t)} Z_j \leq x_1\right\},
\]

where

(4.2.29) \quad x_1 = \frac{|x|(t/\mu)^{1/2}}{\{k(x,t)\}^{1/2}}.

Since \(|x| \leq t^{1/3}\), it follows that for all sufficiently large \(t\), we have \(c_1 t \leq k(x,t) \leq c_2 t\) for some \(c_1, c_2 > 0\) and hence
\[ x_1 = \{k(x,t)\}^{1/3} \cdot \frac{|x|(t/\mu)^{1/2}}{\{k(x,t)\}^{s/6}} \]

(4.2.30)

\[ \leq \{k(x,t)\}^{1/3} \cdot \frac{c_3 t^{5/6}}{\{k(x,t)\}^{s/6}} \leq c_4 \{k(x,t)\}^{1/3}. \]

Therefore Theorem 2.2.4 is again applicable. It yields (see formula (2.2.10))

\[
1 - P\left\{ \frac{1}{\{k(x,t)\}^{1/2}} \sum_{j=1}^{k(x,t)} Z_j \leq x_1 \right\}
\]

\[
= \{1 - \Phi(x_1)\} \exp\left( \frac{x_1^3}{\{k(x,t)\}^{1/3}} \lambda^{[4]} \right) \left( \frac{x_1}{\{k(x,t)\}^{1/2}} \right) \left( 1 + O\left( \frac{x_1}{\{k(x,t)\}^{1/2}} \right) \right)
\]

\[ \leq c_5 \{1 - \Phi(x_1)\} \exp\left( \frac{c_6 x_1^3}{\{k(x,t)\}^{1/3}} \right) \quad \text{(since } \lambda^{[4]}(s) = O(1) \text{ as } s \to 0) \]

(4.2.31)

\[ \leq c_7 \exp\left( -\frac{x_1^2}{2} + \frac{c_6 x_1^3}{\{k(x,t)\}^{1/2}} \right) \]

\[ \leq c_8 \exp(-c_9 x_1^2) = c_8 \exp\left( -\frac{c_9 x_1^2}{\mu k(x,t)} \right) \]

\[ \leq c_8 \exp(-c_{10} x^2) \quad \text{(since } t/k(x,t) > c_{11} > 0 \text{ when } |x| \leq t^{1/3}) \]

\[ \leq c_{12} \exp(-c_{13} |x|). \]

Formulae (4.2.27), (4.2.28) and (4.2.31) show that (4.2.2) holds in Case 4 with some \( \alpha \) and \( \nu \).

Finally, suppose that the condition of Case 5 applies, i.e., \( t^{1/3} < -x \leq (t\mu)^{1/2}/\sigma \). Denote \( m(t) = [t/\mu - \sigma t^{5/6}/\mu^{3/2}] \). Then \( k(x,t) \leq k(-t^{1/3}, t) = m(t) \).

Let us now estimate the probability in the right hand side of (4.2.27).
\[ P\{S_k(x,t) > t\} = P\{X_1 + \ldots + X_k(x,t) > t\} \]

\[ \leq P\{X_1 + \ldots + X_m(t) > t\} \quad \text{(since } X_1, X_2, \ldots \text{ are positive (a.s.))} \]

\[ = P\left\{ \frac{1}{\{m(t)\}^{1/2}} \sum_{j=1}^{m(t)} Z_j > \frac{t - m(t)\mu}{\sigma\{m(t)\}^{1/2}} \right\} \]

\[ \leq P\left\{ \frac{1}{\{m(t)\}^{1/2}} \sum_{j=1}^{m(t)} Z_j > \frac{t - (t/\mu - \sigma t^{5/6}/\mu^{3/2})\mu}{\sigma\{m(t)\}^{1/2}} \right\} \]

\[ \leq 1 - P\left\{ \frac{1}{\{m(t)\}^{1/2}} \sum_{j=1}^{m(t)} Z_j \leq x_2 \right\}, \]

where

\[ x_2 = t^{5/6}/\{\mu m(t)\}^{1/2}. \]

Since for all sufficiently large \( t \),

\[ x_2/\{m(t)\}^{1/3} = \mu^{-1/2}\{t/m(t)\}^{5/6} = O(1), \]

Theorem 2.2.4 is applicable. Therefore

\[ 1 - P\left\{ \frac{1}{\{m(t)\}^{1/2}} \sum_{j=1}^{m(t)} Z_j \leq x_2 \right\} \]

\[ = \{1 - \Phi(x_2)\} \exp\left\{ \frac{x_2^3}{\{m(t)\}^{1/2}} \lambda^{[4]}(\frac{x_2}{\{m(t)\}^{1/2}}) \right\} \left(1 + O\left(\frac{x_2 + 1}{\{m(t)\}^{1/2}}\right)\right) \]

\[ \leq c_1 \{1 - \Phi(x_2)\} \exp\left(\frac{c_2 x_2^3}{\{m(t)\}^{1/2}}\right) \leq c_3 \exp\left(-x_2^2/2 + \frac{c_2 x_2^3}{\{m(t)\}^{1/2}}\right) \]
\[
\leq c_3 \exp(-c_4 x_2^2) = c_3 \exp\left(-\frac{c_4 t^{5/3}}{\mu m(t)}\right) \leq c_5 \exp(-c_6 t^{2/3})
\]
\[
< c_5 \exp(-c_7 |x|) \quad \text{(since } |x| \leq (t\mu)^{1/2}/\sigma)\).
\]

Combining (4.2.27), (4.2.32) and (4.2.34), we show that (4.2.2) holds with some \(\alpha\) and \(\nu\) under the conditions of Case 5.

Hence (4.2.2) is satisfied (with possibly different \(\alpha\) and \(\nu\)) in each of the three applicable cases (one of which, Case 6, is trivial). As we did at the end of the proof of (4.2.1), we can choose here such \(\alpha, \nu > 0\) that (4.2.2) is satisfied for all \(x\) and all sufficiently large \(t\). It is clear therefore that there exist \(\alpha\) and \(\nu\) that satisfy both parts (i) and (ii) of the lemma.

Lemma 4.2.1 is proved. \(\square\)

4.3. APPROXIMATIONS TO THE CHARACTERISTIC AND CUMULATIVE FUNCTIONS OF \(Y_1\)

In order to approximate the distribution function of \(V(t)\), we will consider the behavior of its characteristic function. In this section we find a close approximation to \(\psi(\theta)\), the characteristic function of \(Y_1\). Later we will show that \(\psi(\theta)\), properly adjusted, under certain conditions does not differ much from the characteristic function of \(V(t)\). Together these results will be used in Section 4.6 to prove a Central Limit Theorem with the remainder term for the cumulative processes.

Lemma 4.3.1 and Lemma 4.3.2 provide the tools to approximate \(\kappa(\theta)\), the cumulative function of \(Y_1\). Lemma 4.3.3 gives the desired approximation of \(\kappa(\theta)\). The estimation of \(\psi(\theta)\) is done in Lemma 4.3.4 and Lemma 4.3.5. While it is sufficient for our purposes to assume in Lemmas 4.3.1, 4.3.2 and 4.3.3 the existence of a finite absolute moment of \(Y_1\) of the order between 2 and 3 only, we state these results in a more general way. We believe that the lemmas in this section have
possible applications beyond those in our dissertation.

**Lemma 4.3.1.**

Suppose that

\[
E|Y_1|^{k+\delta} < \infty,
\]

where \( k \geq 2 \) is an integer and \( 0 \leq \delta < 1 \). Define for \( z \geq 0 \),

\[
R(z) = \int_{|y| \geq z} |y|^k \, dG(y),
\]

where \( G(y) \) is the distribution function of \( Y_1 \). Then

\[
R(z) = o(z^{-\delta}) \quad \text{as} \quad z \to \infty.
\]

**Proof.**

Evidently,

\[
z^\delta \int_{|y| \geq z} |y|^k \, dG(y) \leq \int_{|y| \geq z} |y|^{k+\delta} \, dG(y) = o(1) \quad \text{as} \quad z \to \infty,
\]

since

\[
\int_{-\infty}^{+\infty} |y|^{k+\delta} \, dG(y) = E|Y_1|^{k+\delta} < \infty.
\]

The statement of the lemma follows directly from (4.3.4).

Lemma 4.3.1 is proved. \( \square \)

**Lemma 4.3.2.**

If (4.3.1) holds then

\[
\int_{|y| \leq z} |y|^{k+1} \, dG(y) = o(z^{1-\delta}) \quad \text{as} \quad z \to \infty.
\]
Proof.

Fix $\epsilon > 0$. Let $z_1$ be such that

\begin{equation}
\int_{|y|>z_1} |y|^{k+\delta} \ dG(y) < \epsilon/2.
\end{equation}

The existence of $z_1$ is guaranteed by assumption (4.3.1). For $z > z_1$ we can write

\begin{align*}
&\int_{|y|\leq z} |y|^{k+1} \ dG(y) \\
&\quad = \int_{|y|\leq z_1} |y|^{k+1} \ dG(y) + \int_{z_1 < |y| \leq z} |y|^{k+1} \ dG(y) \\
&\quad \leq z_1^{1-\delta} \int_{|y|\leq z_1} |y|^{k+\delta} \ dG(y) + z_1^{1-\delta} \int_{z_1 < |y| \leq z} |y|^{k+\delta} \ dG(y) \\
&\quad \leq z_1^{1-\delta} \int_{-\infty}^{+\infty} |y|^{k+\delta} \ dG(y) + \epsilon z^{1-\delta}/2 \\
&\quad < \epsilon z^{1-\delta}, \quad \text{for } z \text{ sufficiently large.}
\end{align*}

Since $\epsilon$ is chosen arbitrarily, the statement of Lemma 4.3.2 is now proved. \qed

Lemma 4.3.3.

Suppose that $EY_1 = 0$ and (4.3.1) is satisfied with some $k \geq 2$. Then

\begin{equation}
\kappa(s) = \sum_{m=2}^{k} \frac{(is)^m}{m!} \gamma_m + o(|s|^{k+\delta}) \quad \text{as } s \to 0,
\end{equation}

where $\gamma_m$ is the $m$th cumulant of $Y_1$.

Note. When $\delta = 0$, the result of Lemma 4.3.3 is well-known. See, for example, Chapter I in Petrov's book (1975).
Proof of Lemma 4.3.3.

Ibragimov (1967) showed (see Lemma 3 in his paper) that if \( E|Y_1|^k < \infty \) then

\[
(4.3.9) \quad \left| \kappa(s) - \sum_{m=2}^{k} \frac{(is)^m}{m!} \gamma_m \right| = \int_{-\infty}^{+\infty} \left( e^{isy} - \sum_{m=0}^{k} \frac{(isy)^m}{m!} \right) dG(y) + O(|s|^{k+1})
\]

as \( s \to 0 \). According to Heyde and Leslie (1972),

\[
(4.3.10) \quad \left| e^{ix} - \sum_{m=0}^{k} \frac{(ix)^m}{m!} \right| \leq \frac{|x|^{k+1}}{(k+1)!}, \quad \text{for } |x| \leq 1,
\]

\[
\left| e^{ix} - \sum_{m=0}^{k} \frac{(ix)^m}{m!} \right| \leq (1+e)|x|^k, \quad \text{for } |x| \geq 1.
\]

Hence

\[
\left| \int_{-\infty}^{+\infty} \left( e^{isy} - \sum_{m=0}^{k} \frac{(isy)^m}{m!} \right) dG(y) \right|
\]

\[
(4.3.11) \quad \leq \int_{|sy| \leq 1} |sy|^{k+1} (k+1)! dG(y) + (1+e) \int_{|sy| > 1} |sy|^k dG(y)
\]

\[
= I_1 + I_2, \text{ say.}
\]

We will estimate \( I_1 \) and \( I_2 \) separately:

\[
I_1 = \frac{|s|^{k+1}}{(k+1)!} \int_{|y| \leq |s|^{-1}} |y|^{k+1} dG(y)
\]

\[
(4.3.12) \quad = \frac{|s|^{k+1}}{(k+1)!} o(|s|^{\delta-1}) \quad \text{(by Lemma 4.3.2)}
\]

\[
= o(|s|^{k+\delta}),
\]
\[ I_2 = (1+e)|s|^k \int_{|y|>|s|^{-1}} |y|^k \, dG(y) \]

\[(4.3.13) \quad = |s|^k \mathcal{O}(|s|^\delta) \quad \text{(by Lemma 4.3.1)} \]

\[ = \mathcal{O}(|s|^{k+\delta}). \]

Putting together (4.3.9), (4.3.11), (4.3.12) and (4.3.13), we complete the proof of Lemma 4.3.3. \( \square \)

**Corollary 4.3.3.**

Suppose \(|\theta| \leq t^\beta\) for some \(\beta < \frac{1}{2}\). Then

\[(4.3.14) \quad t \kappa(\theta_t)/\mu \]

\[ = -\theta^2/2 + \sum_{m=3}^{k} \frac{(i\theta)^m \mu (m-2)/2}{\tau_m m! t^{(m-2)/2}} \gamma_m + \mathcal{O} \left( \frac{|\theta|^{k+\delta}}{t^{(k-2+\delta)/2}} \right) \quad \text{as } t \to \infty. \]

(The sum is set to 0 if \(k = 2\).)

**Proof.**

Under the conditions of this corollary, \(\theta_t \to 0\) as \(t \to \infty\). Hence Lemma 4.3.3 is applicable. Putting \(s = \theta_t\) in (4.3.8), we obtain

\[(4.3.15) \quad t \kappa(\theta_t)/\mu = \sum_{m=2}^{k} \frac{(i\theta)^m \mu (m-2)/2}{\tau_m m! t^{(m-2)/2}} \gamma_m + \mathcal{O} \left( \frac{|\theta|^{k+\delta}}{t^{(k-2+\delta)/2}} \right). \]

Since \(\gamma_2 = \tau^2\), the statement of the corollary follows immediately from (4.3.15). \( \square \)

Now we are ready to estimate the closeness of \(\{\psi(\theta_t)\}^{t/\mu}\) to \(e^{-\theta^2/2}\). We will use these estimates in Section 4.6. Two cases will be considered: \(k = 3, \delta = 0\) and
k = 2, 0 < \delta < 1. The results in these cases are slightly different, therefore we will need two separate lemmas.

Lemma 4.3.4.

Suppose $EY_1 = 0$ and $E|Y_1|^3 < \infty$. Let $\beta$ be any real number such that $0 < \beta < \frac{1}{\delta}$. Then for $|\theta| \leq t^\beta$,

\[(4.3.16) \quad \{\psi(\theta_t)\}^{t/\mu} = e^{-\theta^2/2} + O\left(\frac{e^{-\theta^2/2|\theta|^3}}{t^{1/2}}\right).\]

Proof.

Let us estimate the difference between $\{\psi(\theta_t)\}^{t/\mu}$ and $e^{-\theta^2/2}$. Conditions of the lemma guarantee that $\psi(\theta_t) \neq 0$. Thus

\[
\left|\left\{\psi(\theta_t)\right\}^{t/\mu} - e^{-\theta^2/2}\right| = \left|\exp\left\{t\kappa(\theta_t)/\mu\right\} - e^{-\theta^2/2}\right|
\]

\[(4.3.17) \quad = \left|\exp\left\{-\theta^2/2 + O(|\theta|^3t^{-1/2})\right\} - e^{-\theta^2/2}\right|
\]

(from Corollary 4.3.3 with $k = 3$ and $\delta = 0$)

\[= e^{-\theta^2/2}\left|\exp\left\{O(|\theta|^3t^{-1/2})\right\} - 1\right|.
\]

The choice of $\beta$ guarantees that $|\theta|^3t^{-1/2} < 1$ for all sufficiently large $t$. Since $e^x - 1 = O(|x|)$ for $|x| < 1$, we obtain

\[(4.3.18) \quad \left|\exp\left\{O(|\theta|^3t^{-1/2})\right\} - 1\right| = O(|\theta|^3t^{-1/2}).
\]

Together (4.3.17) and (4.3.18) show that (4.3.16) holds.

Lemma 4.3.4 is proved. \(\square\)
Lemma 4.3.5.

Suppose $EY_1 = 0$ and $E|Y_1|^{2+\delta} < \infty$ for some $\delta$ such that $0 < \delta < 1$. Let $\beta$ be any number within the range $0 < \beta < \frac{\delta}{2(2+\delta)}$. Then for $|\theta| \leq t^\beta$,

$$
(4.3.19) \quad \{\psi(\theta_t)\}_t^{1/\mu} = e^{-\theta^2/2} + o\left(\frac{e^{-\theta^2/2}|\theta|^{2+\delta}}{t^{\delta/2}}\right).
$$

Proof.

The proof is similar to that of the previous lemma. We write

$$
\left|\left\{\psi(\theta_t)\right\}_t^{1/\mu} - e^{-\theta^2/2}\right| = \left|\exp\{t\kappa(\theta_t)/\mu\} - e^{-\theta^2/2}\right|
$$

$$
(4.3.20) \quad = e^{-\theta^2/2}\left|\exp\left(o\left(\frac{|\theta|^{2+\delta}}{t^{\delta/2}}\right)\right) - 1\right| \quad \text{(by Corollary 4.3.3 with } k = 2)\]

$$
= o\left(\frac{e^{-\theta^2/2}|\theta|^{2+\delta}}{t^{\delta/2}}\right).
$$

(The choice of $\beta$ assures us that $\frac{|\theta|^{2+\delta}}{t^{\delta/2}} < 1$ for all sufficiently large $t$.)

Lemma 4.3.5 is proved.  

Note. If we allowed $\delta = 1$ in Lemma 4.3.5, then Lemma 4.3.4 would follow from Lemma 4.3.5. We chose, however, not to combine these two results together. They are used separately and also the statement of Lemma 4.3.2 does not hold with $\delta = 1$. That would make a proof of a combined result much more difficult.
4.4. AN ESTIMATE OF A CORRECTION TERM ARISING FROM AN APPROXIMATION OF N(t) WITH t/µ

In this section we prove a lemma the result of which will later be used to approximate the characteristic function of V(t).

Lemma 4.4.1.

Suppose EY₁ = 0 and (4.1.2) holds for some r > 2. Let β be such that 0 < β < 1/4. Then for all sufficiently large t and |θ| ≤ t^β,

\[
E[\exp\{κ(θ_t)(t/µ-N(t)-1)\} - 1] = O(θ^2t^{-1/2}),
\]

where θ_t is defined by (4.1.3).

The proof of Lemma 4.4.1 requires the following proposition.

Proposition 4.4.1.

For any complex number h,

\[
|e^h - 1| \leq e^{|h|} - 1.
\]

Proof of Proposition 4.4.1.

\[
|e^h - 1| = \left| \lim_{n \to \infty} \sum_{k=1}^{n} \frac{h^k}{k!} \right| = \lim_{n \to \infty} \left| \sum_{k=1}^{n} \frac{h^k}{k!} \right|
\]

\[
\leq \lim_{n \to \infty} \sum_{k=1}^{n} \frac{|h|^k}{k!} = e^{|h|} - 1.
\]

Now we can prove Lemma 4.4.1 itself.
We have

\[ E \left| \exp \left\{ \kappa(\theta_t) \left( t/\mu - N(t) - 1 \right) \right\} - 1 \right| \]

\[(4.4.3) \quad = E \left| \exp \left\{ -\kappa(\theta_t) \cdot \frac{\sigma_t^{1/2}}{\mu^{3/2}} \cdot \frac{N(t) + 1 - t/\mu}{\sigma_t^{1/2}/\mu^{3/2}} \right\} - 1 \right| \]

\[ = E|\exp\{h(\theta,t)R(t)\} - 1|, \]

where

\[(4.4.4) \quad h(\theta,t) = -\kappa(\theta_t) \sigma_t^{1/2} / \mu^{3/2} \]

and \( R(t) \) is defined by (4.1.5). Let us estimate \( h(\theta,t) \). Assumption (4.1.2) and the fact that \( \theta_t \to 0 \) as \( t \to \infty \) (which is true since \( |\theta| \leq t^{\beta} \)) allow us to apply Lemma 4.3.3 with \( k = 2 \). It implies that

\[(4.4.5) \quad \kappa(\theta_t) = -\theta_t^2 \sigma_t^2 / 2 + o(\theta_t^2) \]

\[ = -\theta^2 \mu / 2t + o(\theta^2 t^{-1}) = O(\theta^2 t^{-1}). \]

Hence for \( |\theta| \leq t^{\beta} \),

\[(4.4.6) \quad |h(\theta,t)| = O(\theta^2 t^{-1/2}). \]

Evidently,

\[ E|\exp\{h(\theta,t)R(t)\} - 1| \]

\[(4.4.7) \quad = \int_{-\infty}^{+\infty} |\exp\{h(\theta,t)x\} - 1| dF_{R(t)}(x) = \int_{0}^{\infty} + \int_{-\infty}^{0} \]

\[ = I_1 + I_2, \text{ say.} \]
According to Proposition 4.4.1,

\[ I_1 \leq \int_0^\infty \left\{ \exp\left( |h(\theta,t)|x \right) - 1 \right\} dF_R(t)(x) \]

(4.4.8) \[ = -\int_0^\infty \left\{ \exp\left( |h(\theta,t)|x \right) - 1 \right\} d\left(1 - F_R(t)(x)\right) \]

\[ = -m(\theta,t,x) + |h(\theta,t)| \int_0^\infty \exp\left( |h(\theta,t)|x \right) \left(1 - F_R(t)(x)\right) dx, \]

where

(4.4.9) \[ m(\theta,t,x) = \left\{ \exp\left( |h(\theta,t)|x - 1 \right) \right\} \left(1 - F_R(t)(x)\right). \]

We will show that

(4.4.10) \[ \int_0^\infty \exp\left( |h(\theta,t)|x \right) \left(1 - F_R(t)(x)\right) = O(1) \]

and that \( \lim_{x \to \infty} m(\theta,t,x) \) exists and is equal to 0.

According to Lemma 4.2.1, there exist \( \alpha, \nu > 0 \) such that \( 1 - F_R(t)(x) < \alpha e^{-\nu x} \) for all \( x \geq 0 \). Suppose \( t \) is large enough so that \( |h(\theta,t)| < \nu/2 \). (We know that \( h(\theta,t) \to 0 \) as \( t \to \infty \), since \( h(\theta,t) = O(\theta^2 t^{-1/2}) \) and \( |\theta| \leq t^\beta \), where \( \beta < \frac{1}{4} \)).

Then the value of the expression in the left hand side of (4.4.10) does not exceed

(4.4.11) \[ \alpha \int_0^\infty \exp\left( |h(\theta,t)|x \right) \exp(-\nu x) dx = \alpha \left( \nu - h(\theta,t) \right)^{-1} = O(1). \]

So (4.4.10) holds. Also for such \( t \),

\[ 0 \leq m(\theta,t,x) \leq \alpha \exp\left\{ (|h(\theta,t)| - \nu)x \right\} \to 0 \quad \text{as} \quad x \to \infty. \]

That proves that \( \lim_{x \to \infty} m(\theta,t,x) = 0 \) for any \( \theta \) and \( t \), such that \( t \) is sufficiently large and \( |\theta| \leq t^\beta \). Together with (4.4.8) and (4.4.10), it shows that

(4.4.12) \[ I_1 = O\left( |h(\theta,t)| \right) = O(\theta^2 t^{-1/2}). \]
Similarly,

\[ I_2 \leq \int_{-\infty}^{0} \{ \exp(|h(\theta,t)x|) - 1 \} dF_{R(t)}(x) \]

\[ = \int_{-\infty}^{0} \{ \exp(-|h(\theta,t)x|) - 1 \} dF_{R(t)}(x) \]

\[ = n(\theta,t,x)|_{x=-\infty}^{0} + |h(\theta,t)| \int_{-\infty}^{0} \exp(-|h(\theta,t)x|) F_{R(t)}(x) \, dx, \]

where

\[ n(\theta,t,x) = \{ \exp(-|h(\theta,t)x|) - 1 \} F_{R(t)}(x). \]

As in a case of $I_1$, using Lemma 4.2.1 we obtain

\[ \int_{-\infty}^{0} \exp(-|h(\theta,t)x|) F_{R(t)}(x) \, dx = O(1) \]

and

\[ \lim_{x \to -\infty} n(\theta,t,x) = 0 \]

for all $\theta$ and $t$ such that $t$ is sufficiently large and $|\theta| \leq t^{\beta}$. That along with (4.4.13) implies that

\[ I_2 = O(|h(\theta,t)|) = O(\theta^2 t^{-1/2}). \]

Lemma 4.4.1 is proved. \( \square \)

4.5. THE BEHAVIOR OF THE CHARACTERISTIC FUNCTION OF V(t)

In this section we obtain the estimates for

\[ \rho(t,\theta) \equiv E e^{i\theta V(t)}. \]
Our goal is to show that when \( \theta \) is small (as compared to \( t \)), \( \rho(t, \theta) \) can be closely approximated by a function of the characteristic function of \( Y_1 \). This will be done in Lemma 4.5.1. Next, in Lemma 4.5.2 we will demonstrate that for certain large values of \( \theta \) (again, relative to \( t \)), \( \rho(t, \theta) \) is sufficiently small as \( t \to \infty \).

The information on \( \rho(t, \theta) \) obtained in this section will be needed to approximate the distribution function of \( V(t) \).

**Lemma 4.5.1.**

If the conditions of Lemma 4.4.1 are met, then

\[
(4.5.2) \quad \rho(t, \theta) = \{\psi(\theta_t)\}^{1/\mu} + O(\theta^2 t^{-1/2}) \quad \text{as } t \to \infty.
\]

**Proof.**

In Appendix B we prove the following generalization of the Wald’s Fundamental Identity (Wald, 1944):

\[
(4.5.3) \quad E \frac{\exp(i\theta_t W(t))}{\{\psi(\theta_t)\}^{N(t)+1}} = 1.
\]

Since

\[
(4.5.4) \quad \theta_t W(t) = \theta V(t),
\]

we can write

\[
1 - E \frac{\exp(i\theta V(t))}{\{\psi(\theta_t)\}^{t/\mu}}
= \left| E \frac{\exp(i\theta V(t))}{\{\psi(\theta_t)\}^{N(t)+1}} - E \frac{\exp(i\theta V(t))}{\{\psi(\theta_t)\}^{t/\mu}} \right|
\]
\[(4.5.5) \quad \leq E \left| \frac{1}{\{\psi(\theta_t)\}^{N(t)+1}} - \frac{1}{\{\psi(\theta_t)\}^{t/\mu}} \right| \]

\[= \frac{1}{|\psi(\theta_t)|^{t/\mu}} E \left| \{\psi(\theta_t)\}^{t/\mu - N(t) - 1} - 1 \right| \]

\[= \frac{1}{|\psi(\theta_t)|^{t/\mu}} E \left| \exp\{\kappa(\theta_t)(t/\mu - N(t) - 1)\} - 1 \right|. \]

Hence

\[\left| \{\psi(\theta_t)\}^{t/\mu} - E \exp(i\theta V(t)) \right| \]

\[(4.5.6) \quad \leq E \left| \exp\{\kappa(\theta_t)(t/\mu - N(t) - 1)\} - 1 \right| \]

\[= O(\theta^2 t^{-1/2}) \quad \text{(according to Lemma 4.4.1).} \]

Lemma 4.5.1 is proved. \[\Box\]

**Lemma 4.5.2.**

Suppose \( EY_1 = 0 \) and (4.1.2) is satisfied with some \( r \) such that \( r > 2 \). Then there exists \( \eta > 0 \) that does not depend on \( \theta \) and \( t \) such that if

\[(4.5.7) \quad 5(\log t)^{1/2} \leq |\theta| \leq \eta t^{1/2} \]

then

\[(4.5.8) \quad \rho(t,\theta) = O(t^{-2}) \quad \text{as} \ t \to \infty. \]

(Obviously, there is nothing magic about a constant '5' in (4.5.7). It is large enough to satisfy the inequalities in the proof. Any constant larger than 5 would suffice too.)
Proof.

First, we may assume that $\eta$ is small enough to ensure that $\psi(\theta_t) \neq 0$, i.e., that $\kappa(\theta_t)$ exists. Indeed, if $|\theta| \leq \eta t^{1/2}$ then $|\theta_t| \leq \eta t^{1/2}/\tau$, and we can use the fact that $\psi(s) \neq 0$ when $|s|$ is sufficiently small.

Next, let

$$m_t = \frac{t}{\mu} - \beta t^{1/2} \log t$$

(4.5.9)

$$M_t = \frac{t}{\mu} + \beta t^{1/2} \log t + 1,$$

where

(4.5.10) $$\beta = \frac{2\sigma}{\nu \mu^{3/2}}$$

and $\nu$ is such that (4.2.1) and (4.2.2) are satisfied. Then

$$P\{N(t) < m_t\} = P\left\{\frac{N(t) - \frac{t}{\mu}}{\sigma t^{1/2}/\mu^{3/2}} < \frac{m_t - \frac{t}{\mu}}{\sigma t^{1/2}/\mu^{3/2}}\right\}$$

$$\leq F_R(t)\left(\frac{m_t - \frac{t}{\mu}}{\sigma t^{1/2}/\mu^{3/2}}\right) \leq F_R(t)\left(-\frac{\beta \mu^{3/2} \log t}{\sigma}\right)$$

(4.5.11)

$$= O\left(\exp\left(-\frac{\nu \beta \mu^{3/2} \log t}{\sigma}\right)\right) \quad \text{(by Lemma 4.2.1)}$$

$$= O(\exp(-2 \log t)) \quad \text{(by the choice of $\beta$)}$$

$$= O(t^{-2}).$$

Similarly,

(4.5.12) $$P\{N(t) \geq M_t\} = O(t^{-2}),$$
so

\[(4.5.13) \quad P\{N(t) \geq M_t\} + P\{N(t) < m_t\} = O(t^{-2})\]

for all sufficiently large \(t\).

Consider now a sequence of random variables \(\tilde{Y}_j, j = 1,2,\ldots\), defined as

\[(4.5.14) \quad \tilde{Y}_j = Y'_j - Y''_j,\]

where \(Y'_j, Y''_j\) are independent for all \(i,j = 1,2,\ldots\) and are such that sequences \((X_j, Y'_j)\) and \((X_j, Y''_j)\), \(j = 1,2,\ldots\), satisfy the same conditions as the original sequence \((X_j, Y_j)\). Thus, \(\tilde{Y}_1, \tilde{Y}_2,\ldots\) are independent identically distributed random variables satisfying the following conditions:

\[(4.5.15) \quad E\tilde{Y}_j = 0, \quad E\tilde{Y}_j^2 = 2r^2 \quad \text{and} \quad E|\tilde{Y}_j|^r < \infty,\]

where \(r\) is the same as in (4.1.2).

Let \(\psi_1(\theta)\) and \(\kappa_1(\theta)\) be the characteristic function and the cumulative function of \(\tilde{Y}_1\) correspondingly. From the definition of \(\tilde{Y}_1\) it follows that \(\psi_1(\theta) = |\psi(\theta)|^2\), so \(\psi_1(\theta)\) is real and non-negative.

Define the following cumulative processes:

\[W'(t) = \sum_{j=1}^{N(t)+1} Y'_j,\]

\[(4.5.16) \quad W''(t) = \sum_{j=1}^{N(t)+1} Y''_j\]

and

\[\tilde{W}(t) = \sum_{j=1}^{N(t)+1} \tilde{Y}_j.\]

Clearly, \(\tilde{W}(t) = W'(t) - W''(t)\). 

For any \( n = 0, 1, 2, \ldots \) we will define

\[
J_n(\theta; x_1, \ldots, x_{n+1})
\]

\[= \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \exp \left( i \theta \sum_{j=1}^{n+1} y_j \right) G_{x_1}(dy_1) G_{x_2}(dy_2) \cdots G_{x_{n+1}}(dy_{n+1})
\]

and

\[I_n(t; x_1, \ldots, x_{n+1}) = \begin{cases} 1, & \text{if } x_1 + \ldots + x_n \leq t < x_1 + \ldots + x_{n+1} \\ 0, & \text{otherwise.} \end{cases}\]

(The distribution function \( G_x(y) \) was defined at the beginning of this chapter.)

Then

\[
E \{ \exp(i \theta \hat{W}(t)) \mid N(t) = n \} P\{N(t) = n\}
\]

\[= \int_{N(t) = n} \exp(i \theta (\hat{Y}_1 + \ldots + \hat{Y}_{n+1})) dP
\]

\[= \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} I_n(t; x_1, \ldots, x_{n+1}) \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \exp \left( i \theta \sum_{j=1}^{n+1} (y_j' - y_j'') \right)
\]

\[\times G_{x_1}(dy_1') \cdots G_{x_{n+1}}(dy_{n+1}') G_{x_1}(dy_1'') \cdots G_{x_{n+1}}(dy_{n+1}'')
\]

\[\times F_X(dx_1) \cdots F_X(dx_{n+1})
\]

(where \( F_X \) is a distribution function of \( X_1 \))

\[= \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} I_n(t; x_1, \ldots, x_{n+1}) |J_n(\theta; x_1, \ldots, x_{n+1})|^2
\]

\[\times F_X(dx_1) \cdots F_X(dx_{n+1})
\]

—real and non-negative. Therefore we can write
\[
\{\psi_1(\theta_t)\}_t^{M_t} = E \exp(i\theta_t(\tilde{Y}_1 + \ldots + \tilde{Y}_{M_t}))
\]

\[
= \sum_{n=0}^{\infty} E\{\exp(i\theta_t(\tilde{Y}_1 + \ldots + \tilde{Y}_{M_t})) | N(t)=n\} P\{N(t)=n\}
\]

(4.5.19)

\[
\geq \sum_{n=m_t}^{M_t-1} E\{\exp(i\theta_t(\tilde{Y}_1 + \ldots + \tilde{Y}_{n+1})) | N(t)=n\} P\{N(t)=n\}
\]

(since for \(j > n+1\), \(\tilde{Y}_j\) is independent of \(\{N(t)=n\}\))

\[
\geq \{\psi_1(\theta_t)\}_t^{m_t-1} \sum_{n=m_t}^{M_t-1} E\{\exp(i\theta_t(\tilde{Y}_1 + \ldots + \tilde{Y}_{n+1})) | N(t)=n\} P\{N(t)=n\}.
\]

Hence

\[
\sum_{n=m_t}^{M_t-1} E\{\exp(i\theta_t(\tilde{Y}_1 + \ldots + \tilde{Y}_{n+1})) | N(t)=n\} P\{N(t)=n\}
\]

(4.5.20)

\[
\leq \{\psi_1(\theta_t)\}_t^{m_t+1} \leq \{\psi_1(\theta_t)\}_t^{t/2\mu} \quad \text{for all sufficiently large } t.
\]

Also

\[
\left| E\{\exp(i\theta_t W(t)) | N(t)=n\} P\{N(t)=n\}\right|^2
\]

\[
= \left| \int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty} I_n(t; x_1, \ldots, x_{n+1}) \int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty} \exp(i\theta_t(y_1 + \ldots + y_{n+1})) \right|^2
\]

(4.5.21)

\[
\times G_{x_1}(dy_1) \ldots G_{x_{n+1}}(dy_{n+1}) F_X(dx_1) \ldots F_X(dx_{n+1})
\]

\[
= \left| \int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty} I_n(t; x_1, \ldots, x_{n+1}) J_n(\theta_t; x_1, \ldots, x_{n+1}) F_X(dx_1) \ldots F_X(dx_{n+1}) \right|^2
\]
\begin{align*}
&\leq \int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty} I_n(t; x_1, \ldots, x_{n+1}) |J_n(\theta_t; x_1, \ldots, x_{n+1})|^{2} \mathcal{F}_X(dx_1) \ldots \mathcal{F}_X(dx_{n+1}) \\
&= \mathbb{E}\{\exp(i\theta_t \hat{W}(t))|N(t)=n\} \mathbb{P}\{N(t)=n\} \quad \text{(according to (4.5.18))}. \\
\end{align*}

Now we can estimate \( \mathbb{E} \exp(i\theta V(t)) \):

\begin{align*}
\left| \mathbb{E} \exp(i\theta V(t)) \right| &= \left| \mathbb{E} \exp(i\theta W(t)) \right| \\
&= \left| \sum_{n=0}^{\infty} \mathbb{E}\{\exp(i\theta_t W(t))|N(t)=n\} \mathbb{P}\{N(t)=n\} \right| \\
&\leq \sum_{n=0}^{\infty} \left| \mathbb{E}\{\exp(i\theta_t W(t))|N(t)=n\} \right| \mathbb{P}\{N(t)=n\} \\
&= \sum_{n=0}^{m_t-1} + \sum_{n=m_t}^{M_t-1} + \sum_{n=M_t}^{\infty} \\
&= S_1 + S_2 + S_3, \quad \text{say}. \\
\end{align*}

(4.5.22)

According to (4.5.13),

\begin{align*}
(4.5.23) \quad S_1 + S_3 &\leq \mathbb{P}\{N(t)<m_t\} + \mathbb{P}\{N(t)\geq M_t\} = O(t^{-2}).
\end{align*}

This estimate holds for all \( \theta \) for which \( \kappa(\theta_t) \) is defined.

In order to estimate \( S_2 \) we note that for any positive integer \( k \) and any real \( x_1, \ldots, x_k \), the following inequality holds:

\[ \sum_{i=1}^{k} x_i \leq \sqrt{k \sum_{i=1}^{k} x_i^2}. \]
Therefore,

\[ S_2 \leq \sqrt{\left( M_t - m_t \right) \sum_{n=m_t}^{M_t-1} \left| E \{ \exp(i \theta_t \tilde{W}(t)) | N(t)=n \} \right|^2} \]

\[ = \sqrt{\left( M_t - m_t \right) \sum_{n=m_t}^{M_t-1} E \{ \exp(i \theta_t \tilde{W}(t)) | N(t)=n \} P\{ N(t)=n \} \text{ (by (4.5.21))} } \]

\[ \leq \sqrt{2 \beta t^{1/2} \log t + 2 \{ \psi_1(\theta_t) \}^{t/2\mu} } \]

(by (4.5.20) and the choice of \( m_t \) and \( M_t \))

\[ < t \{ \psi_1(\theta_t) \}^{t/4\mu} \]

for all sufficiently large \( t \) and all \( \theta \) such that \( \kappa(\theta_t) \) is defined.

According to (4.5.15), for each \( j=1,2,\ldots, \tilde{Y}_j \) has mean 0, variance \( 2\tau^2 \) and a finite moment of the order \( r > 2 \). We write \( \delta = \min(r-2, \frac{1}{2}) \). Then \( \mathbb{E}|Y_1|^{2+\delta} < \infty \).

From Lemma 4.3.3, applied to random variable \( \tilde{Y}_1 \) and its cumulative function \( \kappa_1(s) \), it follows that

\[ \kappa_1(s) = -\tau^2 s^2 + o(|s|^{2+\delta}) \quad \text{as} \quad |s| \to 0. \]

(4.5.25)

Since \( \psi_1(s) \) is real-valued, so is \( \kappa_1(s) \). Clearly, \( \kappa_1(s) \leq 0 \). Let \( s_1 > 0 \) be such that \( \kappa(s) \leq -\tau^2 s^2 + |s|^{2+\delta} \) whenever \( |s| \leq s_1 \) (the existence of such \( s_1 \) follows from (4.5.25)). Take \( c_1 = \max(\tau^2/s_1^\delta, 1) \). Then \( -\tau^2 s^2 + c_1 |s|^{2+\delta} \geq 0 \) for \( |s| \geq s_1 \), so for all real \( s \) we have \( \kappa_1(s) < -\tau^2 s^2 + c_1 |s|^{2+\delta} \). Thus

\[ \psi_1(\theta_t) \leq \exp(-\tau^2 \theta_t^2 + c_1 |\theta_t|^{2+\delta}). \]

(4.5.26)
Let \( \eta \) be such that

\[
(4.5.27) \quad 0 < \eta^\delta < \frac{r^{\delta+2}}{2c_1 \mu^{\delta/2}}.
\]

If \( |\theta| \leq \eta t^{1/2} \) then

\[
(4.5.28) \quad c_1 |\theta_t|^{2+\delta} = \frac{\theta^2}{t} \cdot \frac{|\theta|^{\delta}}{t^{\delta/2}} \cdot \frac{c_1 \mu^{1+\delta/2}}{r^{2+\delta}} \leq \frac{\theta^2}{t} \cdot \eta^\delta \cdot \frac{c_1 \mu^{1+\delta/2}}{r^{2+\delta}} < \frac{\theta^2 \mu}{2t}.
\]

It follows therefore from (4.5.26) that

\[
(4.5.29) \quad \psi_1(\theta_t) \leq \exp\left(-r^2 \cdot \frac{\theta^2 \mu}{r^{2t}} + \frac{\theta^2 \mu}{2t}\right) = \exp\left(-\frac{\theta^2 \mu}{2t}\right).
\]

From (4.5.24) we conclude that

\[
S_2 < t \exp(-\theta^2/8)
\]

\[
(4.5.30) \quad = \mathcal{O}\left(t \exp\left(-\frac{25 \log t}{8}\right)\right) \quad \text{(since } |\theta| \geq 5 (\log t)^{1/2})
\]

\[
= \mathcal{O}(t^{-2}) \quad \text{for all sufficiently large } t.
\]

Together (4.5.22), (4.5.23) and (4.5.30) show that (4.5.8) holds.

Lemma 4.5.2 is proved. \( \Box \)

4.6. ESTIMATES OF THE REMAINDER TERM IN THE CENTRAL LIMIT THEOREMS FOR THE CUMULATIVE PROCESSES

We are ready now to formulate and prove the main results of this chapter. These are the Central Limit Theorems for the cumulative processes that provide the remainder terms. The order of these terms depends on the existence of finite absolute moments of \( Y_1 \). We start with Theorem 4.6.1 which requires the
imposition of the strongest assumptions on the distribution of Y's and, in return, allows the remainder term of the smallest order.

**Theorem 4.6.1.**

Let \( W(t) \) be a cumulative process defined by (1.1.2). Let \( V(t) \) be defined by (4.1.4). Suppose the distributions of random variables \( X_1 \) and \( Y_1 \) (and therefore \( X_j \) and \( Y_j \) for any \( j \)) satisfy the following requirements:

(i) the Cramér condition (4.1.1) holds for \( X_1 \),

(ii) \( EY_1 = 0 \),

(iii) \( E|Y_1|^3 < \infty \).

Then

\[
(4.6.1) \quad \| \mathbf{P}\{V(t) \leq x\} - \Phi(x)\| = O\left(\frac{\log t}{t^{1/2}}\right) \quad \text{as } t \to \infty.
\]

**Proof.**

Let \( \phi(\theta) = e^{-\theta^2/2} \) denote the characteristic function of the standard normal random variable and let \( \rho(t,\theta) \) be the characteristic function of \( V(t) \) as defined by (4.5.1). From a version of the Berry-Esseen inequality (see, for example, Petrov (1975), Chapter V, Theorem 2), it follows that there exists an absolute constant \( c \) such that for any \( T > 0 \) we have

\[
(4.6.2) \quad \sup_x |\mathbf{P}\{V(t) \leq x\} - \Phi(x)| \leq \frac{1}{\pi} \int_{-T}^{T} \left| \frac{\rho(t,\theta) - \phi(\theta)}{\theta} \right| d\theta + cT^{-1}.
\]

Take \( T = \eta t^{1/2} \), where \( \eta \) is chosen as in Lemma 4.5.2. Then \( cT^{-1} = O(t^{-1/2}) \). Hence we only need to estimate the integral term in (4.6.2). We have
\[ \int_{-T}^{T} \left| \frac{\rho(t,\theta) - \phi(\theta)}{\theta} \right| d\theta \]

\[ \leq \int_{|\theta| \leq 5(\log t)^{1/2}} \left| \frac{\rho(t,\theta) - \phi(\theta)}{|\theta|} \right| d\theta + \int_{5(\log t)^{1/2} < |\theta| \leq \eta t^{1/2}} \left| \frac{\rho(t,\theta)}{|\theta|} \right| d\theta + \int_{|\theta| > 5(\log t)^{1/2}} \frac{\phi(\theta)}{|\theta|} d\theta = I_1 + I_2 + I_3, \text{ say.} \]

Let us estimate \( I_1, I_2 \) and \( I_3 \) separately. First,

\[ I_1 \leq \int_{|\theta| \leq 5(\log t)^{1/2}} \left| \frac{\rho(t,\theta) - \{\psi(\theta_t)\}^{t/\mu}}{|\theta|} \right| d\theta \]

\[ + \int_{|\theta| \leq 5(\log t)^{1/2}} \left| \frac{\{\psi(\theta_t)\}^{t/\mu} - \phi(\theta)}{|\theta|} \right| d\theta \]

\[ = J_1 + J_2, \text{ say.} \]

According to Lemma 4.5.1,

\[ J_1 \leq \int_{|\theta| \leq 5(\log t)^{1/2}} \frac{c\theta^2}{|\theta|^{1/2} t^{1/2}} d\theta \]

\[ = O\left(\frac{\log t}{t^{1/2}}\right). \]
Further,  

\[
J_2 \leq \int_{|\theta| \leq 5(log t)^{1/2}} \frac{e^{-\theta^2/2} |\theta|^2}{t^{1/2}} \, d\theta \quad \text{(by Lemma 4.3.4)}
\]

(4.6.6)  

\[
\leq t^{-1/2} \int_{-\infty}^{+\infty} e^{-\theta^2/2} |\theta|^2 d\theta = O(t^{-1/2}).
\]

Hence  

(4.6.7)  

\[
I_1 = O\left(\frac{\log t}{t^{1/2}}\right).
\]

Also, according to Lemma 4.5.2,  

\[
I_2 \leq \int_{5(log t)^{1/2} < |\theta| \leq \eta t^{1/2}} \frac{c_1 t^{-2}}{|\theta|} \, d\theta
\]

(4.6.8)  

\[
= 2c_1 t^{-2} \left\{ \log(\eta t^{1/2}) - \log(5(log t)^{1/2}) \right\}
\]

\[
= O(t^{-1/2}).
\]

Finally,  

\[
I_3 = \int_{|\theta| > 5(log t)^{1/2}} e^{-\theta^2/2 \frac{1}{|\theta|}} \, d\theta < \frac{1}{(2\pi)^{1/2}} \int_{|\theta| > 5(log t)^{1/2}} e^{-\theta^2/2} \, d\theta
\]

(4.6.9)  

\[
= 2\left\{ 1 - \Phi \left(5(log t)^{1/2}\right) \right\} < 2(2\pi)^{1/2} \exp\left(-\frac{25}{2} \log t\right) = O(t^{-1/2}).
\]
Estimates for $I_1$, $I_2$ and $I_3$ show that the expression in the left hand side of (4.6.3) is $O\left(\frac{\log t}{t^{1/2}}\right)$. That together with (4.6.2) completes the proof of the theorem.

Theorem 4.6.1 is proved.

We will now consider a situation when $Y_1$ has a finite absolute moment of the order of $(2 + \epsilon)$, where $0 < \epsilon < 1$. It turns out that in this case we can obtain a remainder term of the order of $o(t^{-\delta/2})$, which is what we would expect from the finiteness of $\mathbb{E}|Y_1|^{2+\delta}$. (A remainder term of a similar order was obtained by Niculescu (1984) in the (non-uniform in x) Central Limit Theorem for the renewal process based on the i.i.d. X’s.) The reason that we can not get a similar result for $\delta = 1$ is that there is a correction term caused by the randomness of $N(t)$ (we dealt with this term in Lemma 4.4.1). The order of this term can not be made less than $O\left(\frac{\log t}{t^{1/2}}\right)$. While such error can be ignored if $\epsilon < 1$, i.e., when the desired accuracy is $O(t^{-1/2+\epsilon})$ for some $\epsilon > 0$, it becomes the largest component of the remainder term for $\delta = 1$.

The following theorem covers the case of the $Y$’s with a finite absolute moment of the order of greater than 2 but less than 3.

**Theorem 4.6.2.**

Suppose conditions (i) and (ii) of Theorem 4.6.1 are satisfied and also $E|Y_1|^{2+\delta} < \infty$ for some $\delta$ such that $0 < \delta < 1$. Then

\[(4.6.10)\quad \|P\{V(t) \leq x\} - \Phi(x)\| = o(t^{-\delta/2})\quad \text{as } t \to \infty.\]
Proof.

As in the proof of Theorem 4.6.1, using the Berry-Esseen inequality (see (4.6.2)) with $T = \eta t^{1/2}$, where $\eta$ is chosen as in Lemma 4.5.2, we can write

\[
(4.6.11) \quad \| P \{ V(t) \leq x \} - \Phi(x) \| \leq \frac{1}{\pi} \int_{|\theta| \leq \eta t^{1/2}} \frac{|\rho(t, \theta) - \phi(\theta)|}{|\theta|} \, d\theta + O(t^{-1/2}).
\]

Let $\beta$ be any real number such that $0 < \beta < \min\left(\frac{1-\delta}{4}, \frac{\delta}{2(2+\delta)}\right)$. Then

\[
\int_{|\theta| \leq \eta t^{1/2}} \frac{|\rho(t, \theta) - \phi(\theta)|}{|\theta|} \, d\theta \leq \int_{|\theta| \leq t^{\beta}} \frac{|\rho(t, \theta) - \{\psi(\theta)\}^{t/\mu}|}{|\theta|} \, d\theta
\]

\[
(4.6.12) \quad + \int_{|\theta| \leq t^{\beta}} \frac{|\{\psi(\theta)\}^{t/\mu} - \phi(\theta)|}{|\theta|} \, d\theta + \int_{t^{\beta} < |\theta| \leq \eta t^{1/2}} \frac{|\rho(t, \theta)|}{|\theta|} \, d\theta + \int_{|\theta| > t^{\beta}} \frac{\phi(\theta)}{|\theta|} \, d\theta
\]

\[
= A_1 + A_2 + A_3 + A_4, \text{ say.}
\]

From Lemma 4.5.1 it follows that

\[
A_1 \leq \int_{|\theta| > t^{\beta}} \frac{c_0^2}{|\theta| t^{1/2}} \, d\theta = O(t^{-1/2 + 2\beta})
\]

\[
(4.6.13) \quad = O(t^{-\delta/2}), \quad \text{since } \beta < (1-\delta)/4.
\]
Also,

\[ A_2 \leq \int_{|\theta| \leq t^{1/2}} e^{\frac{-\theta^2/2}{|\theta|^{1+\delta}}} \frac{t^{\delta/2}}{\delta/2} d\theta \]

(by Lemma 4.3.5, which is applicable since \( \beta < \frac{\delta}{2(2+\delta)} \))

\[ = O(t^{-\delta/2}), \]

\[ A_3 \leq \int_{t^{1/2} < |\theta| \leq \eta t^{1/2}} \frac{c_1 t^{-2}}{|\theta|} d\theta \quad \text{(according to Lemma 4.5.2)} \]

\[ = O(t^{-1/2}) \quad \text{as } t \to \infty \]

and

\[ A_4 = O(t^{-1/2}), \]

as it was shown in the course of the proof of Theorem 4.6.1 (see (4.6.9)).

The above estimates for \( A_1, A_2, A_3 \) and \( A_4 \) combined with (4.6.11) and (4.6.12) complete the proof of the theorem.

Theorem 4.6.2 is proved. \( \square \)

4.7. A CASE WHEN Y’s HAVE A NON-ZERO MEAN

In this section we drop the assumption \( EY_1 = 0 \) and prove two general Central Limit Theorems with the remainder terms. The first of these two theorems follows almost immediately from the results of Section 4.6, while the proof of the second one is much less obvious. We believe, however, that that second result (Theorem 4.7.2) represents the correct approach to the Central Limit Theorem,
because, unlike Theorem 4.7.1, its statement does not involve any random variables besides the one, namely \( W(t) \), whose distribution is of interest to us.

In the remainder of Chapter IV we will denote

\[
\lambda = EY_1,
\]

(4.7.1)

\[
\gamma^2 = \text{Var}(Y_1 - \frac{1}{\lambda}X_1).
\]

We will assume that \( \gamma > 0 \) and \( \lambda \) can be any real number.

We start with the following result.

**Theorem 4.7.1.**

Let \( W(t) \) be a cumulative process defined by (1.1.2) and suppose that the Cramér condition (4.1.1) holds for \( X_1 \). Then

(i) if \( E|Y_1|^3 < \infty \) then

\[
(4.7.2) \quad \left\| P \left\{ \frac{W(t) - \lambda (N(t) + 1)}{\tau(t/\mu)^{1/2}} \leq x \right\} - \Phi(x) \right\| = O\left(\frac{\log t}{t^{1/2}}\right) \quad \text{as } t \to \infty,
\]

(ii) if \( E|Y_1|^{2+\delta} < \infty \) for some \( \delta \) such that \( 0 < \delta < 1 \), then

\[
(4.7.3) \quad \left\| P \left\{ \frac{W(t) - \lambda (N(t) + 1)}{\tau(t/\mu)^{1/2}} \leq x \right\} - \Phi(x) \right\| = o\left(t^{-\delta/2}\right) \quad \text{as } t \to \infty.
\]

**Proof.**

The proof follows immediately from Theorem 4.6.1 and Theorem 4.6.2 applied to the cumulative processes based on random variables \( X_j \) and \( Y'_j \), \( j = 1,2,\ldots \), where \( Y'_j = Y_j - \lambda \), so that \( EY'_j = 0 \). □

The proof of the next theorem will require two additional lemmas.
Lemma 4.7.1.

Let $q_t$ be any function of $t$. Then

\[(4.7.4) \quad P\{X_{N(t)+1} > q_t\} = O(te^{-gq_t}),\]

where $g$ is the constant in the Cramér condition (4.1.1) that applies to $X_n$'s.

Proof.

Let us introduce function $H(t) \equiv EN(t)$, a renewal function that corresponds to the renewal process defined by the $X_n$'s. From (2.3.3) it follows that $H(t) \leq ct$ for some fixed $c$ and all sufficiently large $t$. Here and later $c, c_1, c_2, \ldots$ will be some appropriate constants that do not depend on $t$. Also, the Cramér condition (4.1.1) implies that for a fixed $t$ and any given $n$,

\[(4.7.5) \quad P\{X_n > q_t\} \leq c_1 e^{-gq_t}.\]

We cannot use (4.7.5) directly to prove the lemma because in (4.7.4) the index of $X$ depends on $t$. The proof requires a few more steps. We write

\[
P\{X_{N(t)+1} > q_t\}
\]

\[
= \sum_{n=0}^{\infty} P\{N(t) = n, X_{N(t)+1} > q_t\}
\]

\[
= \sum_{n=0}^{\infty} P\{N(t) = n, X_{n+1} > q_t\}
\]

\[
\leq \sum_{n=0}^{\infty} P\{N(t) \geq n, X_{n+1} > q_t\}
\]

\[
(4.7.6) \quad = \sum_{n=0}^{\infty} P\{S_n \leq t, X_{n+1} > q_t\} \quad \text{(because } \{N(t) \geq n\} = \{S_n \leq t\})
\]
\[ \sum_{n=0}^{\infty} P\{S_n \leq t\} P\{X_{n+1} > q_t\} \quad \text{(since } X_{n+1} \text{ and } S_n \text{ are independent)} \]

\[ \leq c_1 e^{-gqt} \sum_{n=0}^{\infty} P\{S_n \leq t\} \quad \text{(by (4.7.5))} \]

\[ = c_1 e^{-gqt} \sum_{n=0}^{\infty} P\{N(t) \geq n\} = c_1 e^{-gqt}(1 + H(t)) \]

\[ = O(te^{-gqt}) \quad \text{(since, as we showed above, } H(t) = O(t)) \]

Lemma 4.7.1 is proved. \[ \square \]

For any \( t > 1 \) set

\[ q_t = \frac{3}{2g} \log t. \quad (4.7.7) \]

According to Lemma 4.7.1,

\[ P\{X_{N(t)+1} > q_t\} = O(te^{-gqt}) \]

\[ = O(t \exp(-\log t^{3/2})) = O(t^{-1/2}). \quad (4.7.8) \]

Let us now introduce a new cumulative process \( W_1(t) \) based on the i.i.d. pairs \( (X_n, Y_n^{(1)}) \), where \( Y_n^{(1)} = Y_n - \frac{1}{\mu}X_n, \quad n = 1, 2, \ldots \). We have \( W_1(t) = \sum_{j=1}^{N(t)+1} Y_j^{(1)} \). Clearly, \( EY_n^{(1)} = 0 \). If condition (4.1.2) is satisfied for \( Y_1 \) and some real \( r \) then (because of (4.1.1)) it is also satisfied for \( Y_1^{(1)} \).

The following lemma allows us to convert a Central Limit Theorem for \( W(t) \) into one for \( W_1(t) \).

**Lemma 4.7.2.**

If \( X_1 \) satisfies the Cramér condition then for all sufficiently large \( t \),
\[
\|P\left\{ \frac{W(t) - \frac{1}{\gamma(t/\mu)^{1/2}}}{\gamma(t/\mu)^{1/2}} \leq x\right\} - \Phi(x) \| \leq 2 \left\| P\left\{ \frac{W_1(t)}{\gamma(t/\mu)^{1/2}} \leq x\right\} - \Phi(x) \| + \frac{c \log t}{t^{1/2}}
\]

(4.7.9)

where a constant \( c \) depends only on the joint distribution of \((X_1, Y_1)\).

Proof.

If \( \lambda = EY_1 = 0 \) then the result is trivial since \( W(t) = W_1(t) \). Suppose now that \( \lambda > 0 \). Obviously,

\[
P\left\{ \frac{W(t) - \frac{1}{\gamma(t/\mu)^{1/2}}}{\gamma(t/\mu)^{1/2}} \leq x\right\} = P\left\{ \frac{W(t) - \frac{1}{\gamma(t/\mu)^{1/2}}(X_1 + \ldots + X_{N(t)} + 1)}{\gamma(t/\mu)^{1/2}} + \frac{\lambda}{\gamma(t/\mu)^{1/2}}(X_1 + \ldots + X_{N(t)} + 1 - t) \leq x\right\}
\]

(4.7.10)

\[
= P\left\{ \frac{W_1(t)}{\gamma(t/\mu)^{1/2}} \leq x - \frac{\lambda(X_1 + \ldots + X_{N(t)} + 1 - t)}{\gamma(\mu)^{1/2}} \right\}.
\]

Since \( \lambda > 0 \) and \( X_1 + \ldots + X_{N(t)} + 1 > t \), formula (4.7.10) implies that

\[
P\left\{ \frac{W(t) - \frac{1}{\gamma(t/\mu)^{1/2}}}{\gamma(t/\mu)^{1/2}} \leq x\right\} \leq P\left\{ \frac{W_1(t)}{\gamma(t/\mu)^{1/2}} \leq x \right\}
\]

(4.7.11)

for any real \( x \).

Let us now find a lower bound for the probability in (4.7.10). Denote \( x_1 = x - \frac{\lambda q_t}{\gamma(\mu)^{1/2}} \), where \( q_t \) is defined by (4.7.7). We have
\[
P\left\{ \frac{W_1(t)}{\gamma(t/\mu)^{1/2}} \leq x - \frac{\lambda (X_1 + \ldots + X_{N(t)} + 1 - t)}{\gamma(\mu t)^{1/2}} \right\}
\geq \ P\left\{ \frac{W_1(t)}{\gamma(t/\mu)^{1/2}} \leq x - \frac{\lambda X_{N(t)} + 1}{\gamma(\mu t)^{1/2}} \right\} \quad \text{(since } X_1 + \ldots + X_{N(t)} \leq t)\]

\[
\geq \ P\left\{ X_{N(t)} + 1 \leq q_t, \quad \frac{W_1(t)}{\gamma(t/\mu)^{1/2}} \leq x - \frac{\lambda q_t}{\gamma(\mu t)^{1/2}} \right\}
\]

\[
(4.7.12) \quad \geq \ P\left\{ X_{N(t)} + 1 \leq q_t, \quad \frac{W_1(t)}{\gamma(t/\mu)^{1/2}} \leq x \right\} \quad \text{(by the definition of } x_1)\]

\[
= \ P\left\{ X_{N(t)} + 1 \leq q_t, \quad \frac{W_1(t)}{\gamma(t/\mu)^{1/2}} \leq x_1 \right\} - \ P\left\{ X_{N(t)} + 1 > q_t, \quad \frac{W_1(t)}{\gamma(t/\mu)^{1/2}} \leq x_1 \right\}
\]

\[
\geq \ P\left\{ \frac{W_1(t)}{\gamma(t/\mu)^{1/2}} \leq x_1 \right\} - \ P\left\{ X_{N(t)} + 1 > q_t \right\}
\]

\[
\geq \ P\left\{ \frac{W_1(t)}{\gamma(t/\mu)^{1/2}} \leq x_1 \right\} - c_1 t^{-1/2},
\]

for some \(c_1\), according to (4.7.8).

From (4.7.11) it follows that

\[
(4.7.13) \quad \ P\left\{ \frac{W(t) - \frac{1}{\beta} t}{\gamma(t/\mu)^{1/2}} \leq x \right\} - \Phi(x) \leq \ P\left\{ \frac{W_1(t)}{\gamma(t/\mu)^{1/2}} \leq x \right\} - \Phi(x).
\]

From (4.7.10) and (4.7.12) we obtain that
\[
P\left\{ \frac{W(t) - \frac{1}{2} t}{\gamma(t/\mu)^{1/2}} \leq x \right\} - \Phi(x)
\]

\[
(4.7.14) \quad \geq P\left\{ \frac{W_1(t)}{\gamma(t/\mu)^{1/2}} \leq x_1 \right\} - \Phi(x_1) + \left\{ \Phi(x_1) - \Phi(x) \right\} - c_1 t^{-1/2}
\]

\[
\geq P\left\{ \frac{W_1(t)}{\gamma(t/\mu)^{1/2}} \leq x_1 \right\} - \Phi(x_1) - c_2 t^{-1/2}\log t
\]

(since \( |\Phi(y_1) - \Phi(y_2)| \leq |y_1 - y_2| \) for all real \( y_1, y_2 \),

for some \( c_2 \) that does not depend on \( x \) or \( t \).

It is easy to verify that for any real numbers \( y_1, y_2 \) and \( y_3 \) the following property holds: if \( y_1 \leq y_2 \leq y_3 \) then \( |y_2| \leq |y_1| + |y_3| \). Applying it to (4.7.13) and (4.7.14) with

\[
y_1 = P\left\{ \frac{W_1(t)}{\gamma(t/\mu)^{1/2}} \leq x_1 \right\} - \Phi(x_1) - c_2 t^{-1/2}\log t,
\]

\[
y_2 = P\left\{ \frac{W(t) - \frac{1}{2} t}{\gamma(t/\mu)^{1/2}} \leq x \right\} - \Phi(x)
\]

and

\[
y_3 = P\left\{ \frac{W_1(t)}{\gamma(t/\mu)^{1/2}} \leq x \right\} - \Phi(x),
\]

we obtain

\[
\left| P\left\{ \frac{W(t) - \frac{1}{2} t}{\gamma(t/\mu)^{1/2}} \leq x \right\} - \Phi(x) \right|
\]

\[
(4.7.15) \quad \leq \left| P\left\{ \frac{W_1(t)}{\gamma(t/\mu)^{1/2}} \leq x \right\} - \Phi(x) \right| + \left| P\left\{ \frac{W_1(t)}{\gamma(t/\mu)^{1/2}} \leq x_1 \right\} - \Phi(x_1) \right|
\]

\[+ c_2 t^{-1/2}\log t.\]
Hence

\[
\left\| P \left\{ \frac{W(t) - \bar{W}}{\gamma(t/\mu)^{1/2}} \leq x \right\} - \Phi(x) \right\| \leq \left\| P \left\{ \frac{W_1(t)}{\gamma(t/\mu)^{1/2}} \leq x \right\} - \Phi(x) \right\| 
+ \left\| P \left\{ \frac{W_1(t)}{\gamma(t/\mu)^{1/2}} \leq x_1 \right\} - \Phi(x_1) \right\| + c_2 t^{-1/2} \log t,
\]

(4.7.16)

since for any functions \( \alpha(x) \) and \( \beta(x) \) it is true that \( \| \alpha(x) + \beta(x) \| \leq \| \alpha(x) \| + \| \beta(x) \| \).

Consider now the second term in the right hand side of (4.7.16). For any fixed \( t \), it is equal to

\[
\sup_{-\infty < x < \infty} \left| P \left\{ \frac{W_1(t)}{\gamma(t/\mu)^{1/2}} \leq x - \frac{\lambda t}{\gamma(\mu t)^{1/2}} \right\} - \Phi \left( x - \frac{\lambda t}{\gamma(\mu t)^{1/2}} \right) \right|
\]

(4.7.17)

\[
= \sup_{-\infty < x < \infty} \left| P \left\{ \frac{W_1(t)}{\gamma(t/\mu)^{1/2}} \leq x \right\} - \Phi(x) \right|.
\]

That, together with (4.7.16), proves formula (4.7.9).

Similarly, (4.7.9) holds when \( \lambda < 0 \).

Lemma 4.7.2 is proved. \( \Box \)

We are now ready to state the most general Central Limit Theorem for the cumulative processes. It does not involve any random variables other than \( W(t) \) and allows \( Y_1 \) to have a non-zero mean.
Theorem 4.7.2.

If conditions of Theorem 4.7.1 are satisfied then

(i) if $E[Y_1^3] < \infty$ then

$$
(4.7.18) \quad \left\| P \left\{ \frac{W(t) - \frac{1}{2} t}{\gamma(t/\mu)^{1/2}} \leq x \right\} - \Phi(x) \right\| = O\left(\frac{\log t}{t^{1/2}}\right) \quad \text{as } t \to \infty,
$$

(ii) if $E[Y_1^{2+\delta}] < \infty$ for some $\delta$ such that $0 < \delta < 1$, then

$$
(4.7.19) \quad \left\| P \left\{ \frac{W(t) - \frac{1}{2} t}{\gamma(t/\mu)^{1/2}} \leq x \right\} - \Phi(x) \right\| = o\left(t^{-\delta/2}\right) \quad \text{as } t \to \infty.
$$

Proof.

The proof follows from Lemma 4.7.2 used together with Theorem 4.6.1 (that will demonstrate that (i) holds) and Theorem 4.6.2 (that will prove (ii)).

4.8. ESTIMATES OF AN AVERAGE ERROR

Results proved in the previous section give us the remainder terms in the Central Limit Theorems for the cumulative processes. As it was done for the sums of independent random variables, it is interesting to see if more can be said of these remainder terms when they are averaged over all possible values of $t$. For example, if conditions of Theorem 4.7.2(ii), are satisfied then it follows immediately from (4.7.19) that for any $\epsilon > 0$

$$
(4.8.1) \quad \int_1^{\infty} t^{-1+\delta/2-\epsilon} \left\| P \left\{ \frac{W(t) - \frac{1}{2} t}{\gamma(t/\mu)^{1/2}} \leq x \right\} - \Phi(x) \right\| dt < \infty.
$$

However, it does not follow from Theorem 4.7.2 that (4.8.1) holds for $\epsilon = 0$. In this section we will be concerned with advancing Theorem 4.7.2 by obtaining a
formula similar to (4.8.1) with \( \epsilon = 0 \). This corresponds to how Theorem 2.2.3 improves on Theorem 2.2.1.

Here is the main result of this section.

**Theorem 4.8.1.**

Let \( W(t) \) be defined by (1.1.2). If the Cramér condition (4.1.1) holds for \( X_1 \) and \( E|Y_1|^{2+\delta} < \infty \) for some \( \delta \in (0,1) \) then

\[
(4.8.2) \quad \int_0^\infty t^{-1+\delta/2} \left\| P \left\{ \frac{W(t) - \frac{1}{\sqrt{t}}}{\gamma(t/\mu)^{1/2}} \leq x \right\} - \Phi(x) \right\| dt < \infty.
\]

(Note that the lower limit of integration in (4.8.2) is 0, not 1 as in (4.8.1)).

Before we begin the proof of Theorem 4.8.1, we will state two auxiliary lemmas.

**Lemma 4.8.1.**

Suppose \( EY_1 = 0 \) and \( E|Y_1|^{2+\delta} < \infty \) for some \( \delta \) such that \( 0 < \delta < 1 \). Let function \( a(u) \) be such that

\[
(4.8.3) \quad \kappa(u) = -\frac{\tau^2 u^2}{2} + u^2 a(u),
\]

where \( \kappa(u) \) is a cumulative function of \( Y_1 \) and \( \tau^2 = \text{Var} Y_1 \). Let \( \Delta > 0 \) be such that \( \psi(u) \neq 0 \) (therefore, \( \kappa(u) \) is defined) for \( |u| \leq \Delta \), where \( \psi \) is the characteristic function of \( Y_1 \). Then

\[
(4.8.4) \quad \int_{-\Delta}^{\Delta} \frac{|a(u)|}{|u|^{1+\delta}} du < \infty.
\]

**Proof.**

It was shown by Heyde and Leslie (1972, Lemma 2) that if \( EY_1 = 0 \) and
\[ E|Y_1|^{2+\delta} < \infty \text{ for some } \delta \in (0,1), \text{ then} \]

\[(4.8.5) \quad \int_0^A \frac{|a(u)|}{u^{1+\delta}} \, du < \infty \]

for any \( A > 0 \) such that \( 0 < A \leq \Delta \). In the same paper (see page 262) Heyde and Leslie noted that \(|a(u)|\) is symmetric in \( u \), so the statement of the lemma follows from (4.8.5).

Lemma 4.8.1 is proved. \( \square \)

The next result is somewhat similar to Lemma 4.3.5. The difference is that now we want a remainder term to depend on function \( a(u) \) defined by (4.8.3).

Lemma 4.8.2.

Suppose \( EY_1 = 0 \) and \( E|Y_1|^{2+\delta} < \infty \), where \( \delta \in (0,1) \). Let \( \beta \) be any positive real number such that \( \beta < \frac{\delta}{2(2+\delta)} \). Then for \( |\theta| \leq t^\beta \),

\[(4.8.6) \quad \{\psi(\theta_t)\}^{t/\mu} = e^{-\theta^2/2} + O(e^{-\theta^2/2} \theta^2 |a(\theta_t)|). \]

(Note that the conditions of this lemma are identical to those of Lemma 4.3.5.)

Proof.

We have

\[ \left| \{\psi(\theta_t)\}^{t/\mu} - e^{-\theta^2/2} \right| \]

\[ = \left| \exp(t \kappa(\theta_t)) \mu - e^{-\theta^2/2} \right| \]

\[(4.8.7) \quad = \left| \exp\left(-\frac{t \theta_t^2}{2\mu} + \frac{t \theta_t^2 a(\theta_t)}{\mu}\right) - e^{-\theta^2/2} \right| \text{ (by the definition of } a(u) \text{)} \]
\[
\begin{align*}
&= \left| \exp\left(-\theta^2/2 + \frac{\theta^2 a(\theta_t)}{\tau^2}\right) - e^{-\theta^2/2}\right| \\
&= e^{-\theta^2/2} \left| \exp\left(\frac{\theta^2 a(\theta_t)}{\tau^2}\right) - 1 \right|.
\end{align*}
\]

From Lemma 4.3.3 it follows that \(|a(u)| = o(u^\delta)\) as \(u \to 0\). Since \(\theta_t \to 0\), as \(t \to \infty\) and \(|\theta| \leq t^\beta\), we obtain

\[
(4.8.8) \quad \frac{\theta^2 a(\theta_t)}{\tau^2} = \theta^2 o(\theta^\delta_t) = \theta^2 o\left(\frac{|\theta|\delta}{t^{\delta/2}}\right) = o\left(\frac{|\theta|^{2+\delta}}{t^{\delta/2}}\right) = o(1)
\]
as \(t \to \infty\), because of our choice of \(\beta\). Since \(e^x - 1 = O(|x|)\) as \(x \to 0\), the right hand side of (4.8.7) can be written as \(O\left(e^{-\theta^2/2} \theta^2 |a(\theta_t)|\right)\).

Lemma 4.8.2 is proved. \(\square\)

Now we have all the necessary tools to prove the main result of this section.

**Proof of Theorem 4.8.1.**

First, we notice that if we prove that for some \(y\) that does not depend on \(t\),

\[
(4.8.9) \quad \int_y^\infty t^{-1+\delta/2} \left\| P\left\{ \frac{W(t) - \frac{\lambda t}{\gamma(t/\mu)^{1/2}}} \leq x \right\} - \Phi(x) \right\| \, dt < \infty,
\]
then the theorem will be proved, since

\[
\int_0^y t^{-1+\delta/2} \left\| P\left\{ \frac{W(t) - \frac{\lambda t}{\gamma(t/\mu)^{1/2}}} \leq x \right\} - \Phi(x) \right\| \, dt \leq \int_0^y t^{-1+\delta/2} \, dt < \infty.
\]

We will choose an appropriate \(y\) later.

Second, let us show that it is possible to assume that the \(Y\)'s have mean 0. Indeed, if \(EY_1 = \lambda\) then it follows from Lemma 4.7.2 that the value of the integral
expression in (4.8.9) does not exceed

\[ 2 \int_y^\infty t^{-1+\delta/2} \left\| P \left\{ \frac{W_4(t)}{\gamma(t/\mu)^{1/2}} \leq x \right\} - \Phi(x) \right\| \, dt + c \int_y^\infty t^{-3-\delta/2} \log t \, dt. \]

The second integral in (4.8.10) is finite since \( \delta < 1 \). A cumulative process \( W_4(t) \) is based on \( Y \)'s having zero means. Therefore, from now on we can assume that \( EY_1 = 0 \). Thus we only need to show that for some \( y \) that does not depend on \( t \),

\[ \int_y^\infty t^{-1+\delta/2} \left\| P\{V(t) \leq x\} - \Phi(x) \right\| \, dt < \infty, \]

where \( V(t) \) is defined by (4.1.4).

Let \( \beta \) and \( \eta \) be defined as in the proof of Theorem 4.6.2. From formula (4.6.11) it follows that it is sufficient to show that

\[ \int_y^\infty t^{-1+\delta/2} \int_{|\theta| \leq \eta t^{1/2}} \frac{|\rho(t,\theta) - \phi(\theta)|}{|\theta|} \, d\theta \, dt < \infty, \]

where \( \phi(\theta) = e^{-\theta^2/2} \) is the characteristic function of a standard normal random variable. The inner integral does not exceed \( A_1 + A_2 + A_3 + A_4 \), where the \( A_i \)'s, \( i = 1,2,3,4 \), are defined by formula (4.6.12). We saw in (4.6.13), (4.6.15) and (4.6.16) that \( A_1 + A_3 + A_4 \leq c_1 t^{-1/2} + 2 \beta \) for all sufficiently large \( t \), so

\[ \int_y^\infty t^{-1+\delta/2} (A_1 + A_2 + A_3) \, dt \]

is finite for any \( y > 0 \).

We now have to take care of the integral that involves \( A_2 \), i.e., prove that

\[ \int_y^\infty t^{-1+\delta/2} \int_{|\theta| \leq t^\beta} \frac{|(\psi(t))^{t/\mu} - e^{-\theta^2/2}|}{|\theta|} \, d\theta \, dt < \infty. \]
Lemma 4.8.2 tells us that it would be sufficient to prove the finiteness of

\[
(4.8.14) \quad \int_y^\infty t^{-1+\delta/2} \int_{|\theta|\leq t^\beta} - e^{-\theta^2/2} |\theta| |a(\theta)| \, d\theta \, dt.
\]

By changing the variable of integration in the inner integral from \( \theta \) to \( u = \theta = \frac{\theta \mu^{1/2}}{\tau t^{1/2}} \), we see that the value of the expression in (4.8.14) is equal to

\[
(4.8.15) \quad \int_y^\infty t^{-1+\delta/2} \int_{|u| \leq \frac{\mu^{1/2}}{\tau t^{1/2-\beta}}} \frac{\tau^2 t}{\mu} \exp\left( - \frac{u^2 \tau^2 t}{2\mu} \right) |u| |a(u)| \, du \, dt.
\]

We can assume that \( y \) is large enough so that \( \frac{\mu^{1/2}}{\tau t^{1/2-\beta}} \leq \Delta \) whenever \( t > y \). (We defined \( \Delta \) in Lemma 4.8.1.) Then the quantity in (4.8.15) does not exceed

\[
\frac{\tau^2 t}{\mu} \int_y^\infty t^{\delta/2} \int_{|u| \leq \Delta} \exp\left( - \frac{u^2 \tau^2 t}{2\mu} \right) |u| |a(u)| \, du \, dt
\]

\[
= \frac{\tau^2 t}{\mu} \int_{-\Delta}^{\Delta} |u| |a(u)| \int_y^\infty t^{\delta/2} \exp\left( - \frac{u^2 \tau^2 t}{2\mu} \right) \, dt \, du
\]

\[
\leq \frac{\tau^2 t}{\mu} \int_{-\Delta}^{\Delta} |u| |a(u)| \int_0^\infty t^{\delta/2} \exp\left( - \frac{u^2 \tau^2 t}{2\mu} \right) \, dt \, du
\]

\[
(4.8.16) \quad = \frac{\tau^2 t}{\mu} \int_{-\Delta}^{\Delta} |u| |a(u)| \int_0^\infty \left( \frac{2\mu v}{2\tau^2} \right)^{\delta/2} e^{-v} \frac{2\mu}{u^2 \tau^2} \, dv \, du
\]

(substituting \( v = \frac{u^2 \tau^2 t}{2\mu} \) for \( t \))
\[ c_2 \int_{-\Delta}^{\Delta} \frac{|a(u)|}{|u|^{1+\delta}} \, du \cdot \int_{0}^{\infty} v^{\delta/2} e^{-v} \, dv \quad \text{(for some constant } c_2) \]

\[ < \infty, \quad \text{according to Lemma 4.8.1}. \]

That completes the proof of (4.8.13).

Theorem 4.8.1 is proved. \( \square \)

4.9. AREAS OF FURTHER RESEARCH

In conclusion, we mention several other interesting problems that we have not yet investigated and indeed may never investigate. One can attempt to improve the error term in (4.7.18) and (4.7.19) when \(|x| \to \infty\). For the sequences of random variables such results were obtained by Bikelis (1966), Osipov (1972) and Ahmad and Lin (1977). Also, it would be great to get rid of the Cramér condition (4.1.1) on \( X_1 \) and consider instead a cumulative process based on \( X \)'s and \( Y \)'s such that the \( k^{th} \) moment of \( X_1 \) and the \( m^{th} \) moment of \( Y_1 \) are finite. So far, we could not achieve any improvement in the estimate of the remainder term by assuming that \( Y_1 \) has a finite moment of the order greater than 3. It is also of some interest to look into a case when \( X_n \)'s and \( Y_n \)'s are not identically distributed.
APPENDIX A

Let $X$ be a random variable with the characteristic function $\phi$, i.e., $\phi(\theta) = E e^{i\theta X}$. If $\phi(\theta) \neq 0$ for a given $\theta$, we can define a cumulative function of $X$ as

$$
(A.1) \quad \kappa(\theta) = \log \phi(\theta).
$$

If $X$ has a moment $\alpha_k$ of some integer order $k$, then a cumulant of order $k$ is defined by the equality

$$
(A.2) \quad \gamma_k = \left. \frac{d^k}{d\theta^k} \kappa(\theta) \right|_{\theta=0}.
$$

Write $\sigma^2 = \text{Var} X$ and let $H_m$ be the Hermite polynomial of order $m$, that is,

$$
(A.3) \quad H_m(x) = (-1)^m e^{x^2/2} \frac{d^m}{dx^m} e^{-x^2/2}.
$$

We are now ready to introduce functions $Q_\nu(x)$ that are referred to in the text. Define

$$
(A.4) \quad Q_\nu(x) = -\frac{1}{\sqrt{2\pi}} e^{-x^2/2} \sum H_{\nu+2s-1}(x) \prod_{m=1}^s \frac{1}{k_m!} \left( \frac{\gamma_{m+2}}{(m+2)! \sigma^2} \right)^{k_m},
$$

where the summation is taken over all integers $k_1, \ldots, k_\nu \geq 0$, such that $k_1 + 2k_2 + \ldots + \nu k_\nu = \nu$ and $s = k_1 + \ldots + k_\nu$. Functions $Q_\nu(x)$ are needed to obtain the Edgeworth expansion in the Central Limit Theorem.
APPENDIX B

Here we prove a generalization of the Wald's Fundamental Identity (see Wald, 1944). This proof has been developed by W.L. Smith. Professor Smith presented it in his lectures given at the University of North Carolina. The method of the proof is similar to that used by A. Wald.

Let $W(t)$ be a cumulative process based on the i.i.d. bivariate random variables $\{(X_n, Y_n)\}$, $n = 1, 2, ..., $ as defined in Chapter I. Let $N(t)$ be a renewal process based on the $X_n$'s and let $\psi(\theta)$ be the characteristic function of the $Y_n$'s. We assume that $t$ and $\theta$ are fixed and $\theta$ is such that $\psi(\theta) \neq 0$. The main result of this Appendix is the following theorem.

**Theorem B.**

\[(B.1) \quad E \frac{e^{i\theta W(t)}}{\{\psi(\theta)\}^{N(t)+1}} = 1.\]

First we need to prove an auxiliary lemma.

**Lemma B.1.**

For any $q > 0$ there exists a positive constant $A = A(q)$ such that for all $n \geq 1$,

\[(B.2) \quad P\{N(t) \geq n\} \leq Aq^n.\]
Proof.

Since $X_1 > 0$, there exists $\nu > 0$ such that

(B.3) $E e^{-\nu X_1} < q$.

We have

$E e^{-\nu (S_n - t)} \geq P\{S_n \leq t\} E\{e^{-\nu (S_n - t)} | S_n \leq t\}$

$\geq P\{S_n \leq t\} = P\{N(t) \geq n\}$.

Hence

$P\{N(t) \geq n\} \leq e^{\nu t} E e^{-\nu S_n} = e^{\nu t} \left(E e^{-\nu X_1}\right)^n$

$\leq e^{\nu t} q^n$.

So (B.2) holds with $A = e^{\nu t}$.

Lemma B.1 is proved. $\square$

We denote $a = |\psi(\theta)|$ and will use Lemma B.1 with $q = a/2$ only.

Proof of Theorem B.

Fix any integer $M \geq 1$.

(B.4) $E \frac{e^{i\theta W(t)}}{\{\psi(\theta)\}^{N(t)+1}} = \sum_{n=0}^{\infty} P\{N(t) = n\} E\left\{\frac{e^{i\theta W(t)}}{\{\psi(\theta)\}^{N(t)+1}} | N(t) = n\right\}$

$= \sum_{n=0}^{M-1} + \sum_{n=M}^{\infty} = P_M + R_M$, say.
For $n \leq M-1$, we obtain

$$E\left\{ e^{i\theta W(t)} \left| \{\psi(\theta)\}^{n+1} \right. \frac{N(t) = n}{\{\psi(\theta)\}^{n+1}} \right\} = E\left\{ e^{i\theta (Y_1 + \ldots + Y_{n+1})} \left| \{\psi(\theta)\}^{n+1} \right. \frac{N(t) = n}{\{\psi(\theta)\}^{n+1}} \right\}$$

(by the definition of $W(t)$)

$$= E\left\{ e^{i\theta (Y_1 + \ldots + Y_M)} \left| \{\psi(\theta)\}^M \right. \frac{N(t) = n}{\{\psi(\theta)\}^M} \right\}$$

(since $Y_{n+2}, \ldots, Y_M$ are independent of $\{N(t) = n\}$).

Therefore

$$P_M = \sum_{n=0}^{M-1} P\{N(t) = n\} E\left\{ e^{i\theta (Y_1 + \ldots + Y_M)} \left| \{\psi(\theta)\}^M \right. \frac{N(t) = n}{\{\psi(\theta)\}^M} \right\}$$

$$= 1 - T_M,$$

where

$$T_M = \sum_{n=M}^{\infty} P\{N(t) = n\} E\left\{ e^{i\theta (Y_1 + \ldots + Y_M)} \left| \{\psi(\theta)\}^M \right. \frac{N(t) = n}{\{\psi(\theta)\}^M} \right\}.$$

Evidently,

$$|T_M| \leq \sum_{n=M}^{\infty} P\{N(t) = n\} \cdot \frac{1}{|\psi(\theta)|^M}$$

$$= \frac{P\{N(t) \geq M\}}{A(q)q^M} \leq \frac{A(q)q^M}{A(q)} = A(q) \cdot \frac{1}{2^M} \rightarrow 0 \text{ as } M \rightarrow \infty.$$
Also,

\[ |R_M| \leq \sum_{n=M}^{\infty} P\{N(t) = n\} \cdot \frac{1}{|\psi(\theta)|^{n+1}} \leq \sum_{n=M}^{\infty} \frac{P\{N(t) \geq n\}}{|\psi(\theta)|^{n+1}} \]

\[ \leq \sum_{n=M}^{\infty} \frac{A(q)q^n}{a^{n+1}} \quad \text{(by Lemma B.1)} \]

\[ = \frac{A(q)}{a} \sum_{n=M}^{\infty} \frac{1}{2^n} \to 0 \quad \text{as } M \to \infty. \]

Our estimates of \( R_M \) and \( T_M \) show that the expression in the left hand side of (B.4) tends to 1 as \( M \to \infty \). Since that expression does not depend on \( M \), it is equal to 1.

Theorem B is proved. \( \square \)
REFERENCES


