

DYNAMICAL SYSTEMS, FRACTIONAL BROWNIAN MOTION  
AND LIMIT THEOREMS

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ABSTRACT

Let  $(X, \mu, T)$  be a dynamical system, and for  $f \in L^2(X)$  set

$$S_n f = f + f \circ T + \dots + f \circ T^{n-1}, \quad \sigma_n^2 = \int_X (S_n f)^2 d\mu$$

and for  $0 \leq t \leq 1$

$$S_n f(t) = \begin{cases} S_k f & t = k/n, \quad k = 0, 1, 2, \dots, n \\ \text{linear inbetween} & . \end{cases}$$

Let  $(X, \mu, T)$  be aperiodic, and fix  $0 < H < 2$ . Let  $B_H(t)$ ,  $0 \leq t \leq 1$  be the fractional Brownian motion of index  $H$ . (See [MvN]) Then there is an  $f \in L^2(X)$  so that  $\sigma_n^{-1} S_n f(\cdot)$  converges weakly to  $B_H(\cdot)$  in  $C[0,1]$ . The corresponding result for the law of the iterated logarithm also holds. Both results, by way of a new construction, extend a central limit theorem due to Burton and Denker [BD].

## 1. Introduction

In an interesting paper, Burton and Denker [BD] established the central limit theorem of probability theory in some dynamical systems. To state the result, let  $(X, \mu, T)$  be a dynamical system with  $\mu(X) = 1$ . That is,  $T$  is a measurable,  $\mu$ -measure preserving map on  $X$ , and  $(X, \mu)$  is a Lebesgue space. Let  $f \in L^2(\mu)$  be centered,  $\int_X f \, d\mu = 0$ , set  $S_n f = f + f \circ T + \dots + f \circ T^{n-1}$ , and  $\sigma_n^2 = \int_X (S_n f)^2$ . Then

Theorem 1.1. If  $(X, \mu, T)$  is aperiodic, then there is an  $f \in L^2(X)$  so that for all  $-\infty < u < +\infty$ ,

$$(1.2) \quad \mu \left\{ x : \frac{S_n f(x)}{\sigma_n} \leq u \right\} \rightarrow (2\pi)^{-1/2} \int_{-\infty}^u \exp(-v^2/2) dv, \text{ as } n \rightarrow +\infty.$$

(1.2) is a central limit theorem, briefly, CLT. Most CLT's are proved under some restriction on the dependence structure of  $f, f \circ T, \dots$ . Thus, the result above is most striking in the zero entropy case, where one expects the most dependence. We refer the reader to [BD] for more background on these points, and note that the papers [Ma], [GS] and [CS] are related to Theorem 1.1. For specific dynamical systems they find general sufficient conditions for (1.2) to hold.

The CLT continues to be one of the most studied results in probability theory, and has been the subject of many generalizations. Theorem 1.1 clearly raises the possibility that some of these generalizations can hold in the context of dynamical systems; it is this point which we address in this paper. Two fruitful generalizations in the case of independent, identically

distributed summands are due to Donsker [D] and Strassen [St]. These results are, respectively, the functional central limit theorem and law of the iterated logarithm. Both results treat the limiting behavior of the following continuous functions on  $t \in [0,1]$ . For  $x \in X$ , and integers  $n$ , let

$$(1.3) \quad S_n f(t,x) = \begin{cases} S_k f & t = k/n, k = 0,1,2,\dots,m \\ \text{linear inbetween} & \end{cases}$$

Note that  $S_n f(\cdot,x)$  is a continuous function on  $[0,1]$ . Thus  $\sigma_n^{-1} S_n f(\cdot,x)$  induces a probability measure on  $C[0,1]$ , the space of continuous functions on  $[0,1]$ , with the usual sup-norm. The FCLT states that these measures converge weakly towards a limit. More specifically, let  $Y(\cdot)$ ,  $t \in [0,1]$  be a stochastic process with  $P(Y(\cdot) \in C[0,1]) = 1$ . Write  $f \in CLT(Y)$  if for all bounded, continuous  $h : C[0,1] \rightarrow \mathbb{R}$ , and all  $-\infty < u < +\infty$ ,

$$(1.4) \quad \mu(x \in X : h(\sigma_n^{-1} S_n f(\cdot,x)) < u) \rightarrow P(h(Y(\cdot))) < u), n \rightarrow +\infty.$$

This is nothing more than the definition of weak convergence of probability measures on the separable metric space  $C[0,1]$ . (See e.g. [B]). We also note the FCLT is a statement about the statistical regularity of the evolution of  $S_n f$ .

Donsker's theorem can be summarized as follows. Let  $B(t)$  be a standard Brownian motion. (See [B].) For independent identically distributed random variables, with  $\int f=0$  and  $\int f^2=1$ ,  $f \in CLT(B)$ .

The class of processes we consider include the Brownian motion as a special case. They are the fractional Brownian motions. ([MvN]). For  $0 < H < 2$ , let  $B_H(t)$ ,  $t > 0$  be a Gaussian process which satisfies

$$B_H(0) = 0, E B_H(t) = 0, t > 0;$$

and for  $s, t > 0$ ,

$$(1.5) \quad E(B_H(s+t) - B_H(t))^2 = s^H.$$

For a construction of these processes, in particular, a proof that  $B_H$  admits a version with a.s. continuous paths, we refer the reader to [Ka, Chapter 18].

We further comment that  $B_H$  has a self-similarity property: For all  $a > 0$

$$(1.6) \quad \{a^{-H/2} B_H(at) : t > 0\} \stackrel{d}{=} \{B_H(t) : t > 0\}$$

( $\stackrel{d}{=}$  is equality in finite dimensional distributions.) Finally, observe that for  $H=2$ ,  $B_2(s)$  can be realized as  $s \cdot g$  where  $g$  is a standard Gaussian random variable (mean 0, variance 1). That is  $B_2$  is a trivial process.

Theorem 1.7. Fix  $0 < H < 2$ . For all aperiodic dynamical systems  $(X, \mu, T)$ , there is an  $f \in L^2(X)$  for which  $f \in CLT(B_H)$ .

By a theorem of Lamperti, [L, Theorem 2, p. 64], [M, Theorem 1], if  $f \in CLT(Y)$  for some stochastic process  $Y(t)$ , then  $Y(t)$  is self-similar. That is,  $Y(t)$  satisfies (1.5) for some  $H > 0$ . Moreover, if  $Y(t)$  is square-integrable, then  $\sigma_n^2 = E Y^2(1) \cdot n^H L(n)$ , where  $L(n)$  is slowly varying. Further, by the stationarity of  $f \circ T^k$ ,  $k \geq 0$ ,  $Y(t)$  must have stationary increments, so that (1.4) is satisfied. Finally, if  $Y(t)$  is Gaussian, it must be a version of  $B_H$ . Thus, the fractional Brownian motions,  $B_H$  for  $0 < H < 2$ , are the largest class of non-trivial Gaussian processes which can arise in Theorem 1.7.

The LIL can be thought of as an almost everywhere version of the CLT. To state the FLIL, set  $a_n^2 = 2 \sigma_n^2 \log^+(1 + \log^+ \sigma_n^2)$ . For a compact subset  $K$  of  $C[0,1]$ , write  $f \in FLIL(K)$  if there is a set  $N \subset X$  with  $\mu(N) = 0$  so that for all  $x \in X \setminus N$ , the following two conditions hold. For  $\mathcal{F}(x) = \{a_n^{-1} S_n f(\cdot, x) : n \geq 1\} \subset C[0,1]$ ,

$$(1.8.a) \quad \mathcal{F}(x) \text{ is relatively compact in } C[0,1];$$

and

$$(1.8.b) \quad \text{the set of cluster points of } \mathcal{F}(x) \text{ is } K.$$

For each  $0 < H < 2$  there is a compact  $K_H \subset C[0,1]$  which is the unit ball of the reproducing kernel Hilbert space associated with  $B_H$ . (See [JM].) For  $H=1$ ,  $K_1$  was identified by Strassen, [S].

$$K_1 = \{F \in C[0,1] : F(t) = \int_0^t f(s)ds, \text{ where } \int_0^1 f^2(s)ds \leq 1\}.$$

For  $H \neq 1$ , there is a description of  $K_H$  in terms of fractional integration, [TC1], indeed Mandelbrot first defined  $B_H$  by fractional integrals, hence the name.

Theorem 1.9. Fix  $0 < H < 2$ . For all aperiodic dynamical systems  $(X, \mu, T)$ , there is an  $f \in L^2(X)$  with  $f \in \text{FLIL}(K_H)$ .

Consider the projection  $g \rightarrow g(1)$  on  $C[0,1]$ . Then as a corollary to this theorem, we get

$$\limsup_n a_n^{-1} S_n f = \sup_{g \in K_H} g(1) = 1 \quad \text{a.e. } (X)$$

This is the usual law of the iterated logarithm.

The set of  $f$  in  $L^2(X)$  satisfying any of these theorems are dense in  $L_0^2(X) = \{g \in L^2(X) : \int_X g = 0\}$ . This is easiest to see from the fact that  $\{g - g \circ T : g \in L^2(X)\}$  is dense in  $L_0^2(X)$ , and  $\sigma_n = O(n^H)$ . (See [B, proposition 2.2, p. 191] and [GL Remark 1, p. 393].) The classes of  $f$  which satisfy Theorems 1.7 and 1.9, for which the rate of convergence in the pointwise ergodic theorem is very regular, contrast with the classes constructed by Kakutani and Petersen, [P, Theorem 2.3, p. 94], for which the rate is bad. Other questions about the nature of these classes, such as first or second category, are open.

Also, there is a broad class of processes  $Y(t)$  satisfying (1.5) for some  $H > 0$ . Besides stable processes which are not square-integrable, there are non-Gaussian self-similar processes which are square integrable. It would be of interest to extend theorems 1.6 and 1.8 to any of these processes. (See

[TC2], [M] for surveys of limit theory and self-similar processes. Also see the bibliography [T].)

We now comment on the proof of theorems 1.7 and 1.9, and the organization of this paper. The constructions of the functions  $f$  which satisfy these Theorems are very different from those in [BD]. (This seems to be necessary.) In section 2, we treat the case of irrational rotation. For fixed  $0 < H < 2$ , the function  $f$  is constructed in terms of its Fourier expansion. Then classical facts about lacunary trigonometric series are used to give a complete proof of Theorem 1.7. We do this to avoid carrying out very similar calculations in the technically more complicated section 3, in which the general case is taken up. Moreover, many aspects of section 2 are closely mimicked in section 3, thus section 2 should be read before section 3. The full proof of Theorem 1.7 is contained in section 3. The extra steps needed to prove Theorem 1.9 are described in section 4.

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2. Irrational Rotation. Let  $X = \mathbb{R}/2\pi$ , and let  $T x = x + \alpha \pmod{2\pi}$  where  $\alpha$  is irrational with respect to  $\pi$ . Write  $f(x) = \sum b_j e^{ijx}$ , where  $\sum b_j^2 < +\infty$ , and

$$(2.1) \quad K_n(\lambda) = 1 + e^{i\lambda} + \dots + e^{i(n-1)\lambda} = \frac{1 - e^{in\lambda}}{1 - e^{i\lambda}}.$$

Then observe that

$$S_n f(x) = f(x) + f \circ T(x) + \dots + f \circ T^{n-1}(x) = \sum_j b_j K_n(j\alpha) e^{ijx}.$$

Set  $\sigma_n^2 = \|S_n(f)\|_2^2 = \sum_j b_j^2 |K_n(j\alpha)|^2$ . We will choose the Fourier coefficients to satisfy in addition,  $b_0 = 0$  and  $b_j = b_{-j}$ . Thus  $f(x) = 2 \sum_1^{+\infty} b_j \cos jx$ . In fact,  $f$  will be a sum of lacunary trigonometric functions. Thus, the proof will use well-known properties of such functions, e.g. the Salem-Zygmund CLT and exponential-squared integrability.

The next lemma gives the construction of  $f$ :

Lemma 2.2. Fix  $0 < H < 2$ . There is a set  $J = \{j_1, j_2, \dots\} \subset \mathbb{N}$  and constants  $b_j$  so that

$$(2.2.a) \quad j_{r+1} > 4 j_r \quad r = 1, 2, 3, \dots;$$

$$(2.2.b) \quad b_j = 0 \quad \text{if } |j| \notin J;$$

$$(2.2.c) \quad \sum b_j^2 < +\infty;$$

$$(2.2.d) \quad \begin{aligned} \sigma_n^2 &= \sum b_j^2 |K_n(j\alpha)|^2 \\ &= n^H + O(\log n) \quad \text{as } n \rightarrow +\infty; \end{aligned}$$

and

$$(2.2.e) \quad \max_j \sigma_n^{-1} b_j |K_n(j\alpha)| = O(n^{H/2}) \quad \text{as } n \rightarrow +\infty.$$

Conditions (a) and (b) guarantee lacunarity; (c) implies that  $f \in L^2(X)$ ; (d) shows that the  $L^2$ -norm of  $S_n f$  has the correct behavior; and concerning (e), note that  $S_n(f)$  is the row-wise sum of the array of functions

$$\{2 b_r \operatorname{Re}(K_n(j_r \alpha)) \cos j_r x\}_{r, n \geq 1}.$$

Then (e) implies that this array is row-wise "infinitesimal" ( $S_n f$  is the sum of many small pieces). Such conditions are frequently imposed to obtain a CLT.



In particular, the CLT  $\sigma_n^{-1} S_n f \xrightarrow{d} N(0,1)$  is immediate from (the array version of) the Salen-Zygmund CLT. ([Z, Vol. II, Theorem 5.4, p. 264]).

The next lemma completes the proof of Theorem 1.6 in this special case. (2.3.a) below shows that the finite dimensional distributions of  $\sigma_n^{-1} S_n f(t)$ ,  $0 \leq t \leq 1$  converge to those of  $B_H(t)$ . (2.3.c) below shows "tightness" of the functions  $\sigma_n^{-1} S_n f(t)$ . That is, the finite dimensional distributions determine the distribution of  $\sigma_n^{-1} S_n f(t)$ . This and the a.s. continuity of  $B_H(t)$  imply Theorem 1.6. For a proof of this, see [B, Theorem 8.1, p. 54].

Lemma 2.3. The following two results hold.

(2.3.a) For all  $0 \leq t_1 < t_2 < \dots < t_r \leq 1$ , as  $n \rightarrow +\infty$ ,

$$\{\sigma_n^{-1} S_n f(t_u)\}_{1 \leq u \leq v} \xrightarrow{d} \{B_H(t_u)\}_{1 \leq u \leq v};$$

and for all  $\delta > 0$ ,  $n > 2/\delta$  and  $\lambda^2 > C \delta^H$ ,

$$(2.3.b) \quad \left| \{x \in \mathbb{R}/2\pi : \sup_{|s-t| < \delta} |S_n f(s,x) - S_n f(t,x)| > \lambda \sigma_n\} \right| \leq C \delta^{-1} \exp\left(-\frac{\lambda^2}{C \delta^H}\right).$$

In particular, for all  $\lambda > 0$

$$(2.3.c) \quad \lim_{\delta \downarrow 0} \left| \{x \in \mathbb{R}/2\pi : \sup_{|s-t| < \delta} |S_n f(s) - S_n f(t)| > \lambda \sigma_n\} \right| = 0.$$

We now turn to the proof of two lemmas above. Lemma 2.2 proceeds by approximating  $\sigma_n^2$  by an appropriate integral, which we introduce now.

Lemma 2.4. Fix  $0 < H < 2$ . There is a real-valued, symmetric, integrable function  $w(x)$  on  $[-\pi, \pi]$  so that

$$(2.4.a) \quad \int_{-\pi}^{\pi} w(x) |K_n(x)|^2 dx = m^H, \quad m \geq 1;$$

and

$$(2.4.b) \quad w(x) = O(|x|^{1-H}), \quad |x| \downarrow 0.$$

Proof. For  $H=1$ , we can take  $w \equiv 1$ . Take  $H \neq 1$ ; assume for the moment that  $w(x)$  exists. Then by symmetry,  $\hat{w}(n) = \hat{w}(-n)$ . Moreover,

$$\begin{aligned} (n+1)^H - n^H &= \int_{-\pi}^{\pi} (|K_{n+1}(x)|^2 - |K_n(x)|^2) w(x) dx \\ &= \int_{-\pi}^{\pi} (e^{-jnx} + \dots + e^{jnx}) w(x) dx \\ &= \hat{w}(0) + 2\hat{w}(1) + \dots + 2\hat{w}(n). \end{aligned}$$

Setting  $K_0 = 0$ , we then see that  $\hat{w}(0) = 1$ . For  $n \geq 1$ ,

$$\begin{aligned} (2.5) \quad \hat{w}(n) &= \frac{1}{2}((n+1)^H - 2n^H + (n-1)^H) \\ &= \frac{1}{2} H(H-1)n^{H-2} + O(n^{H-3}), \quad n \rightarrow +\infty. \end{aligned}$$

The remainder of the proof consists of showing that a function with Fourier coefficients as in (2.5) is integrable and satisfies (2.4.b).

Consider the case  $1 < H < 2$ . Then

$$\begin{aligned} g(x) &= \sum_{n>1} n^{H-2} \cos nx \\ &\simeq x^{1-H} \Gamma(3-H) \sin(\pi(H-2)/2) \quad \text{as } x \downarrow 0, \end{aligned}$$

and the sum converges uniformly on  $|x| > \epsilon$  for all  $\epsilon > 0$ . (See [7, Vol. I, p. 186].) Thus  $g$  is integrable, and satisfies (2.4.b). To conclude the proof, observe that by (2.5),  $w(x) - \frac{1}{2} H(H-1)g(x)$  is a continuous, bounded function on  $[-\pi, \pi]$ .

For the case  $0 < H < 1$ , note that the Fourier expansion for  $w(x)$  converges uniformly on  $[-\pi, \pi)$  to a continuous function, hence  $w(x)$  is integrable.

Concerning (2.4.b), note that  $w(0) = 0$ , but  $w(x)$  is non-negative on an neighborhood of 0. (This follows from continuity of  $w$ , (2.4.a) and (2.7) below.) Nonnegativity and integrability are the only facts used in the proof of lemma 2.2. Nevertheless, to establish (2.4.b), from [Z, Vol. I, p. 186],

$$h(x) = \sum_{n>0} n^{H-1} \sin nx$$

$$\simeq x^{H-2} \Gamma(3-H) \cos \pi(H/2)/2$$

Now,

$$w'(x) = -2 \sum_{n>0} n \hat{w}(n) \sin nx.$$

Thus, by (2.5),  $w'(x) - H(H-1)h(x)$  is a bounded, continuous function. This proves (2.4.b). ■

Proof of lemma 2.1. Fix  $0 < H < 2$ , and  $w(x)$  as in lemma 2.4. Then  $w(x)$  is non-negative on a non-empty interval  $[0, a)$ . Write this interval as a disjoint union of decreasing intervals  $I_r$  with midpoints  $i_r$ , for  $r \in \mathbb{N}$ . Further, require that for all  $r$

$$(2.6) \quad |I_r| \leq \exp(-|i_r|^{-1}).$$

Inductively choose  $J = \{j_1, j_2, \dots\} \subset \mathbb{N}$  as follows. Pick  $j_1$  so that  $j_1 \alpha \in I_1$ . (Recall that  $Tx = x + \alpha$ ). Having chosen  $j_r \in J$ ,  $r \geq 1$ , pick  $j_{r+1} > 4j_r$  so that  $j_{r+1} \alpha \in I_{r+1}$ . Then set  $b_j = 0$  for  $j \notin J$ ,  $b_j = b_{-j}$  and

$$b_{j_r}^2 = \int_{I_r} w(x) dx, \quad r \geq 1.$$

Thus (a) and (b) hold, and (c) is satisfied as  $w(x)$  is integrable.

To check the last two conditions of the lemma, we use the following two properties of  $K_n$ . For all  $n$ ,

$$(2.7) \quad |K_n(x)| \leq C|x|^{-1}, \quad 0 < |x| < \pi$$

and

$$(2.8) \quad |K_n(x) - K_n(y)| \leq n^2|x-y|.$$

Then

$$\begin{aligned} & |n^H - \sum_j b_j^2 |K_n(j\alpha)|^2| \\ &= | \int_{-\pi}^{\pi} |K_n(x)|^2 w(x) dx - \sum_j b_j^2 |K_n(j\alpha)|^2 | \\ &= \int_{|x|>a} |K_n(x)|^2 |w(x)| dx \\ &+ 2C \sup\{|K_n(x)|^2 : |x| > (8 \log n)^{-1}\} \\ &+ C \sup_r \{ \sup_{x,y \in I_r} |K_n(x) - K_n(y)|^2 : I_r < (8 \log n)^{-1} \} \\ &\leq O(1) + O(\log n) + O(1), \quad n \rightarrow +\infty. \end{aligned}$$

Here, the first and second estimates follow from the integrability of  $w(x)$  and (2.7). The third follows from (2.6) and (2.8). For use later, we note that we have shown a little more than (c) above: For all  $n$ ,

$$(2.9) \quad |n^H - \sum_{r>\log n} \sum_{j_r} 2 b_{j_r}^2 |K_n(j_r \alpha)|^2| < C \log n.$$

(Recall that the intervals  $I_r$  are decreasing.)

It remains to prove (d). By (2.4.b), (2.6) and (2.7)

$$\begin{aligned} & \sup_r \sup_n |b_r K_n(j_r \alpha)|^2 \\ & \leq C \sup_{0 < x < a} x^{-1-H} \exp(-x^{-1}) < +\infty. \end{aligned}$$

Proof of lemma 2.3. We apply the Salem-Zygmund CLT [Z, Vol. II, Theorem 5.4, p. 264], to prove (a). Fix  $0 \leq t_1 < t_2 < \dots < t_r \leq 1$ . It is enough to show that for all  $a_1, \dots, a_v$

$$(2.10) \quad G_n = \sigma_n^{-1} \sum_1^v a_u S_n f(t_u) \xrightarrow{d} \sum_1^v B_H(t_u), \text{ as } n \rightarrow +\infty.$$

By (2.2.d),

$$\int_0^{2\pi} G^2(x) dx \rightarrow \tau^2 = E\left(\sum_1^v B_H(t_u)\right)^2, \text{ as } n \rightarrow +\infty.$$

Moreover,  $G(x) = \sum_r g_{rn} \cos(j_r k)$ , with  $\max_r |g_{rn}| \rightarrow 0$ , as  $n \rightarrow +\infty$ . Therefore (2.10) holds.

Clearly (c) follows from (b), and (b) is in turn a consequence of well-known techniques from the theory of empirical processes, see e.g. [B]. Lacunarity is again used: For all constants  $a_r$ ,  $r \geq 1$  and  $\lambda > 0$ ,

$$(2.11) \quad \begin{aligned} & \left| \{ |x| < \pi : \left| \sum_r a_r \cos(j_r x) \right| > \lambda \} \right| \\ & \leq C \exp\left(-\frac{\lambda^2}{C \sum_r a_r^2}\right). \end{aligned}$$

We shall prove (c) with  $n = 2^v$  and  $\delta = 2^{-d}$  for integers  $v$  and  $d$ , with  $v > d$ . By (2.2.d) and the definition (1.3), this yields (c) as stated, with a slightly worse constant. Again by (1.3), for each  $n$  observe that  $S_n f(t)$  has local extrema at the nodes  $t = k/n$ ,  $k = 0, 1, 2, \dots, n$ . Using this observation, and the dyadic decomposition of the integers one has

$$\sup_{|s-t| < \delta} |S_n f(\delta) - S_n f(t)|$$

$$< \sum_{k=d-1}^v \sup_{\ell < 2^k} |S_{\ell 2^{v-k}} f - S_{(\ell+1)2^{v-k}} f|$$

Also observe that

$$\sum_{k=d-1}^v 2^{H(d-k)/4} < C,$$

for all  $v > d$ . Thus it is enough to bound the expression below. To do so we will use stationary and (2.11):

$$\begin{aligned} \mu(x : \sum_{k=d-1}^v \sup_{\ell < 2^k} |S_{\ell 2^{v-k}} f(x) - S_{(\ell+1)2^{v-k}} f(x)| \\ > C \lambda \sigma_n \sum_{k=d-1}^v 2^{H(d-k)/4}) \\ \leq \sum_{k=d-1}^v 2^k \mu(x : |S_{2^{v-k}} f(x)| > C \lambda \sigma_n 2^{H(d-k)-4}) \\ \leq C_1 \sum_{k=d-1}^k 2^k \exp(-C_2 \lambda^2 2^{H(d+k)/2}) \\ \leq C \delta^{-1} \exp(-C_2 \lambda^2 \delta^{-H}). \end{aligned}$$

3. Towers. In this section we construct the functions  $f$  used to prove Theorems 1.7 and 1.9. Their relevant properties are collected in lemma 3.9; indeed, Theorem 1.7 is an immediate corollary to this lemma, see the comments following lemma 3.9. The construction uses Kakutani-Rochlin towers, which we define now. Fix an aperiodic dynamical system  $(X, \mu, T)$ .

Definition. For a positive integer  $N$  and  $\epsilon > 0$ , call a measurable set  $F \subset X$  an

$(N, \epsilon)$  tower if  $F, TF, \dots, T^{N-1}F$  are disjoint, and  $\mu(R) < \epsilon$  where  $R = X \setminus \bigcup_0^{N-1} T^\ell F$  is the residual set associated with  $F$ .

The Kakutani-Rochlin lemma states that a tower exists for each pair  $(N, \epsilon)$ , see e.g. [P, lemma 4.7, p. 48]. We also note that towers have an additional property: Set  $k(x) = \inf\{k : T^k x \in F\}$ , for  $x \in R$ , then  $T^{k(x)} x \in F$ . This implies that the orbit of any point in  $X$  can only enter the tower at the bottom level.

Definition. Let  $L$  and  $N$  be positive integers,  $\epsilon > 0$  and  $F$  a  $(LN, \epsilon)$  tower with residual set  $R$ . Fix  $0 < \lambda < \pi$  so that  $e^{i\lambda}$  is an  $N$ th root of unity:  $e^{iN\lambda} = 1$ . Call a function  $g : X \rightarrow \mathbb{C}$  special for  $(L, N, \epsilon, F, \lambda)$  if the following conditions hold.

$$(3.1.a) \quad g|_F \text{ is equidistributed on } \{1, e^{i\lambda}, \dots, e^{i(N-1)\lambda}\};$$

$$(3.1.b) \quad g(T^\ell x) = e^{i\ell \lambda} g(x), \quad x \in F, \quad 0 \leq \ell < NL;$$

and

$$(3.1.c) \quad g(x) = 0, \quad x \in R.$$

Recall the definition of  $K_n$  in (2.1) above.

Lemma 3.2. Let  $g$  be special for  $(L, N, \epsilon, F, \lambda)$ . Then

$$(3.2.a) \quad |S_n(g)| \leq C/|\lambda|;$$

$$(3.2.b) \quad S_n(g)(x) = g(x) K_n(\lambda), \quad \text{for } x \in \bigcup_0^{(L-1)N} T^\ell F, \quad 0 \leq n \leq N;$$

and

$$(3.2.c) \quad \int_{T^\ell F} g \, d\mu = 0, \quad 0 \leq \ell < LN.$$

Proof. (b) and (c) follow immediately from the definitions of  $g$  and  $K_n$ .

To see (a), observe that for  $x \in F$  and  $0 < n \leq LN$ ,

$$|S_n g(x)| = |K_n(g(x))| \leq C/|\lambda|,$$

by (2.7). Moreover,  $S_{LN} g(x) = 0$ . Using this, and the fact that orbits only enter tower at the bottom level, (a) follows. ■

Our function will be a sum of special functions, defined appropriately.

To do this we need the next lemma and the notation which follows.

Lemma 3.3. Let  $F$  be an  $(LN, \epsilon)$  tower, and let  $\pi$  be any finite partition of  $X$ . Then there is a finite partition  $\beta$  of  $F$  so that  $\{T^\ell \beta : 0 \leq \ell < LN\}$ , a partition of  $X \setminus R$ , refines  $\pi$  on  $X \setminus R$ .

Proof. Consider the partitions of  $F$

$$\beta_\ell = T^{-\ell}(\pi|_{T^\ell F}), \quad 0 \leq \ell < LN.$$

Then let  $\beta$  be a common refinement of  $\beta_1, \dots, \beta_{LN-1}$ . This concludes the proof. ■

Fix  $0 < H < 2$ , and let  $w(x)$  be as in lemma 2.4. Then  $w(x)$  is positive on an interval  $[0, a)$ , with  $a > 0$ . Write this interval as a disjoint union of decreasing intervals  $\Lambda_r$ ,  $r \geq 1$ , with rational midpoints  $\lambda_r$  so that

$$(3.4.a) \quad \text{measure}(\Lambda_r) \leq \exp(-\lambda_r^{-1}).$$

Set

$$(3.4.b) \quad b_r^2 = \int_{\Lambda_r} w(x) dx.$$

Choose an increasing sequence of integers  $N_r$  so that

$$(3.4.c) \quad N_r > e^r \quad \text{and} \quad \exp(i N_r \lambda_r) = 1.$$

Define  $r_0 = r_0(n) = \inf\{r : N_r \geq n\}$ ; then for all  $n$ ,



$$(3.4.d) \quad r_0 < \log n.$$

Choose  $L_r$  to be an increasing sequence of integers, and  $\epsilon_r > 0$  so that

$$(3.4.e) \quad \sum_{r > r_0} \lambda_r^{-1} (L_r^{-1} + \epsilon_r)^{1/2} < n^{-8}.$$

For  $r \geq 1$ , let  $F_r$  be an  $(L_r, N_r, \epsilon_r)$  tower with residual set  $R_r$ .

Inductively define special functions  $g_r$  as follows. Let  $g_1$  be special for  $(L_1, N_1, \epsilon_1, F_1, \lambda_1)$ . Given  $g_{r-1}$ , let  $\pi_{r-1}$  be the finite partition of  $X$  generated by

$$(3.5.a) \quad g_{r-1} \circ T^\ell, T^\ell \{R_s; T^k F_s, 0 \leq k < L_s N_s\},$$

$$s = r-1, r; 0 \leq \ell \leq N_{r-1}.$$

Let  $\mathcal{F}_{r-1}$  be the sigmafield generated by  $\pi_{r-1}$ . Apply lemma 3.3 with  $F = F_r$ , and  $\pi = \pi_{r-1}$ ; let  $\beta_r$  be a partition of  $F_r$  given by the conclusion of the lemma. Then, let  $g_r$  be special for  $(L_r, N_r, \epsilon_r, \lambda_r)$  so that for all  $A \in \beta_r$ ,

$$(3.5.b) \quad g_r|_A \text{ is equidistributed on } \{e^{i\ell \lambda_r}; 0 \leq \ell < N_r\}.$$

Let  $g_{-r} = \bar{g}_r$ ,  $b_{-r} = b_r$ ,  $b_0 = 0$ ,  $g_0 \equiv 0$ , and

$$(3.5.c) \quad f = \sum_{-\infty}^{\infty} b_r g_r.$$

This is the function which proves the Theorems. For purposes of the proof, we further define for all  $n$ ,

$$(3.5.d) \quad f_n = \sum_{|r| > r_0(n)} b_r g_r.$$

Observe that for

$$(3.5.e) \quad S_n f_n = \sum_{r>r_0(n)} b_r S_n(g_r + g_{-r})$$

$$= \sum_{r>r_0(n)} e_r,$$

and we have that  $e_r$  is  $\mathcal{F}_r$  measurable.

The next lemma contains the relevant properties of the broken-line functions  $S_n f(t)$ . Of it's four parts, the last is a CLT with rate for the finite dimensional distributions of  $\sigma_n^{-1} S_n f(\cdot)$ . (More remarks on the lemma are below.) To state it succiently, we will use the following notation. Fix an integer  $v$ . For  $\underline{a} = (a_1, \dots, a_v) \in \mathbb{R}^u$ , with  $\sum |a_u| \leq 1$ , and  $\underline{k} = (k_1, \dots, k_v) \in \mathbb{N}^u$  with  $|k_u| \leq n$ , define

$$\varphi(y; \underline{a}, \underline{k}) = \mu(x \in X : \sigma_n^{-1} \sum_1^v a_u S_{k_u} f_n \leq y)$$

and

$$p(y; \underline{a}, \underline{k}) = P(\sum_1^v a_u B_H(k_u/n) \leq y).$$

Also let  $\tau^2$  be the variance of the last sum, which is explicitly given in (3.14) below. By way of explanation, note that if  $\delta(t)$  denotes a point mass at  $0 \leq t \leq 1$ , then  $h = \sum a_k \delta(k_u/n)$  is a linear functional on  $C[0,1]$  of norm less than 1. Thus  $\varphi(\cdot)$  above is the distribution of  $h(\sigma_n^{-1} S_n f_n(\cdot))$  and similarly for  $p(\cdot)$ . (d) below gives an estimate for the rate at which  $\varphi(\cdot)$  converges to  $p(\cdot)$ .

Lemma 3.6. The following properties hold for all  $n$ .

$$(3.6.a) \quad \sup_{k \leq n} |S_k(f-f_n)(x)| \leq c \log n, \text{ all } x \in X;$$

$$(3.6.b) \quad |\sigma_n^2 - n^H| \leq c(\log n)^2;$$

$$(3.6.c) \quad \text{for all } \delta > 2/n, \text{ and } \lambda^2 > C \delta^H,$$

$$\begin{aligned} \mu\{x : \sup_{|s-t| < \delta} |S_n f_n(s,x) - S_n f_n(t,x)| > \sigma_n \lambda\} \\ \leq 2 n^{-8} + C_1 \delta^{-1} \exp(-C_2 \delta^{-H} \lambda^2); \end{aligned}$$

and

$$(3.6.d) \quad \text{for all } \underline{a} \text{ and } \underline{k}, \text{ (which depend on } v)$$

$$\begin{aligned} D_n &= \sup_y |\varphi(y; \underline{a}, \underline{k}) - p(y; \underline{a}, \underline{k})| \\ &\leq C_v \tau^{-4/5} (\log n)^{2/5} n^{-2H/5}. \end{aligned}$$

Observe that by (a) and (b) imply that

$$\lim_{n \rightarrow \infty} \sup_t \sigma_n^{-1} |S_n(f-f_n)(t)| = 0 \quad \text{a.e.}(X).$$

Thus, one only needs to consider  $S_n f_n$  in proving Theorems 1.7 and 1.9. (c) shows tightness of the probability measures of  $\sigma_n^{-1} S_n f_n$  on  $C[0,1]$ , and (d) shows weak convergence to  $B_H$  for a dense set of linear functionals on  $C[0,1]$ . Together, they imply the functional central limit theorem. For a proof of this, see e.g. [B, Theorem 8.1, p. 54].

Proof. Fix an integer  $n$ . To see (a) observe that for all  $k \leq n$ ,

$$\begin{aligned} |S_k(f-f_n)| &\leq \sum_{|r| \leq r_0} |b_r S_k(g_r)| \\ &\leq c (\log n) \max_r |b_r \lambda_r^{-1}| \\ &\leq c \log n. \end{aligned}$$

Here, we have used (3.2.a), (3.4.a) and (3.4.d).

For the remainder of the proof, martingales play a crucial role. (For background on martingales, see [D].) Recall the construction of the sigmafields  $\mathcal{F}_r$ , (3.5.a) above. We see that they are increasing in  $r$ . By  $E(\cdot | \mathcal{F}_r)$  denote conditional expectation (on  $X$ !) with respect to  $\mathcal{F}_r$ . Fix  $0 \leq j < k \leq n$ . Then, mimicking (3.5.e), write

$$(3.7) \quad \begin{aligned} S_k(f_n) - S_j(f_n) &= \sum_{r>r_0} b_r (S_k(g_r + g_{-r}) - S_j(g_r + g_{-r})) \\ &= \sum_{r>r_0} e_r(j, k, n) = \sum_{r>r_0} e_r. \end{aligned}$$

Then  $e_r$  is  $\mathcal{F}_r$  measurable, and setting  $d_r = e_r - E(e_r | \mathcal{F}_{r-1})$ , we see that  $(d_r, \mathcal{F}_r, r > r_0)$  is a martingale difference sequence (quickly, mds).

Let

$$(3.8) \quad G_r = \bigcup_0^{(L_r-1)N_r} T^{\ell} F, \quad \text{and } B_r = X \setminus G_r.$$

Thus  $\mu(B_r) \leq (L_r^{-1} + \epsilon_r)$ . (Recall (3.4.e), which controls the size of  $B_r$ .)

The following conditions hold.

$$(3.9.a) \quad E(e_r | \mathcal{F}_{r-1})(x) = 0 \quad x \in G_r, \quad 0 \leq j < k \leq n;$$

$$(3.9.b) \quad \sup_x \sup_{j < k \leq n} \sup_{r > r_0} |d_r| \leq 2 \sup \sup \sup |e_r| = O(1), \quad n \rightarrow +\infty;$$

and

$$(3.9.c) \quad \mu\{x : \max_{0 \leq j < k \leq n} |(k-j)^H - \sum_{r>r_0} E(d_r^2 | \mathcal{F}_{r-1})| > c \log n\} < n^{-8}.$$

We now show that (3.9) implies (b) and (c) of the lemma. To see (b), note that by (3.6.a)

$$\begin{aligned}
 \sigma_n^2 &= \int_X (S_n f)^2 \\
 &= O((\log n)^2) + \int_X (S_n f_n)^2 d\mu \\
 &= O((\log n)^2) + \int_X (\sum_r E(e_r | \mathcal{F}_{r-1}))^2 d\mu \\
 &\quad + 2 \int_X (\sum_r E(e_r | \mathcal{F}_{r-1})) (\sum_r d_r) \\
 &\quad + \int_X (\sum_r d_r)^2.
 \end{aligned}$$

Thus (b) will follow from the next two estimates. First, by (3.2.a), (3.4.e) and (3.9.a)

$$\begin{aligned}
 &(\int_X (\sum_r E(e_r | \mathcal{F}_{r-1}))^2 d\mu)^{1/2} \\
 (3.10) \quad &\leq C \sum_{r \geq r_0} \lambda_r^{-1} \mu(B_r)^{1/2} \\
 &\leq C n^{-8}.
 \end{aligned}$$

Second, by (3.9.c) and orthogonality of martingale differences, observe that,

$$\begin{aligned}
 \int_X (\sum_r d_r^2) d\mu &= \int_X \sum_r d_r^2 d\mu \\
 &= \int_X \sum_r E(d_r^2 | \mathcal{F}_{r-1}) d\mu \\
 &= n^H + O((n^{-6} \sum_{r \geq r_0} b_r^2)) + O((\log n)^2) \\
 &= n^H + O((\log n)^2).
 \end{aligned}$$

Here, we've used  $\sum b_r^2 \leq \int_{-\pi}^{\pi} |w(x)| dx < +\infty$ . Then use Schwartz's inequality to

control the middle term above.

Part (c) of the lemma follows from (3.9.b) and (c) and an argument very similar to that used to prove (2.3.b) above. The only missing ingredient is an exponential squared distribution inequality, like (2.11) above. For this, one can use Bernstein's inequality for martingale mds. Use [S, Corollary 5.4.1, p. 299]. (For comparison, Bernstein's inequality for independent r.v. can also be found in [S].)

To prove (3.9.a), observe that by (3.5.a),  $G_r$  is  $\mathcal{F}_{r-1}$  measurable. Hence, if  $A \subset G_r$  is any  $\mathcal{F}_{r-1}$  atom, then  $A \subset T^\ell F_r$  for some  $0 \leq \ell < (L_r - 1)N_r$ . That is  $A$  is completely contained in a level of the  $r$ th tower. Moreover, as  $r > r_0(m)$ , we have  $N_r > m$ . Therefore for  $0 \leq k \leq m$ , we have  $T^k A \subset T^{\ell+k} F_r$ , which is a level of the tower. Then the definition of  $g_r$  and  $\beta_r$  ((3.5.b) above) show that

$$\int_A g \circ T^k d\mu = 0, \quad 0 \leq k \leq m.$$

Thus  $E(e_r | \mathcal{F}_{r-1})(x) \equiv 0$  for  $x \in G_r$ .

To see (3.9.b), by (3.2.a),

$$\begin{aligned} & \sup_x \sup_{0 \leq j < k \leq n} \sup_{r > r_0} |e_r(x)| \\ & \leq C \sup_{r \geq r_0} b_r \lambda_r^{-1} = o(1), \quad n \rightarrow +\infty. \end{aligned}$$

In proving (3.9.c) we treat the case  $j = 0$ , the case  $0 < j < k \leq n$  being similar. We claim that

$$(3.11) \quad \sup_{k \leq n} |k^H - E(e_r | \mathcal{F}_{r-1})(x)| < C \log n, \quad \text{for } x \in \bigcap_{r > r_0} G_r.$$

This proves (3.9.c): use (3.9.a) and (3.4.e) to see that  $\mu(\bigcap_{r > r_0} G_r) > 1 - m^{-8}$ .

To prove (3.11), for  $x \in G_r$  we have

$$\begin{aligned} e_r(x) &= b_r S_k(g_r + g_{-r})(x) \\ &= b_r (g_r(x) K_k(\lambda_r) + \overline{g_r(x) K_k(\lambda_r)}). \end{aligned}$$

Then for an atom  $A$  of  $\mathcal{F}_{r-1}$ , with  $A \subset G_r$ ,  $\int_A g_r^2(x) d\mu = 0$ . Thus

$$\int_A e_r^2(x) = 2 b_r^2 |K_k(\lambda_r)|^2.$$

Hence, for  $x \in \bigcap_{r>r_0} G_r$ ,

$$\sum_{r \geq r_0} E(e_r^2 | \mathcal{F}_{r-1})(x) = 2 \sum_{r \geq r_0} b_r^2 |K_k(\lambda_r)|^2.$$

Therefore, (3.11) follows from (2.9), (3.4.a), and (3.4.b).

It remains to prove part (d) of the lemma. Fix  $\underline{a}$  and  $\underline{k}$  as in (d). Then write

$$\begin{aligned} &\sigma_n^{-1} \sum_{u=1}^v a_u S_{k_u}(f_n) \\ &= \sigma_n^{-1} \sum_{r>r_0} b_r \sum_{u=1}^v a_u S_{k_u}(g_r + g_{-r}) \\ &= \sum_{r>r_0} e_r. \end{aligned}$$

And by a further abuse of notation, set  $d_r = e_r - E(e_r | \mathcal{F}_{r-1})$ . Then,  $d_r$  is a linear combination of mds, all with respect to  $\mathcal{F}_r$ , hence it is a mds. This observation allows us to use a martingale CLT with a rate to prove (d). Set

$$\tau^2 = E\left(\sum_{u=1}^v a_u B_H(k_u/n)\right)^2 \leq C$$

as  $\sum |a_u| \leq 1$ . Let

$$L_n = L_n(\underline{a}, \underline{k}) = \sum_{r \geq r_0} \int_X d_r^4 d\mu$$

and

$$N_n = N_n(\underline{a}, \underline{k}) = \int_X |\tau^2 - \sum_{r \geq r_0} E(d_r^2 | \mathcal{F}_{r-1})|^2 d\mu.$$

As a consequence of the main result of [HB], (also see [H], Theorem 1) applied with  $\delta = 1$  and to  $\tau^{-1} e_r$ , we see that the l.h.s. of (d) is dominated by

$$(3.12) \quad C \tau^{-4/5} (L_n + N_n)^{1/5}.$$

Our task is then to estimate this last quantity.

Observe that by (3.4.c),

$$\begin{aligned} L_n &\leq 4 \sigma_n^{-4(v+1)} \sum_{r \geq r_0} \sum_{u=1}^v a_u^4 E |S_{k_u}(g_r)|^4 \\ &\leq C \sigma_n^{-4(v+1)} \left( \sum_{r \geq r_0} b_r^4 |K_{k_u}(\lambda_r)|^4 \right. \\ (3.13) \quad &\left. + n^4 \sum_{r \geq r_0} b_r^4 \mu(B_r) \right) \\ &\leq C_v \sigma_n^{-2} \end{aligned}$$

Secondly, to bound  $N_n$ , write

$$\begin{aligned} \tau^2 &= \sum_{t=1}^v \sum_{u=1}^v E a_t a_u B_H(k_t/n) B_H(k_u/n) \\ (3.14) \quad &= \frac{1}{2} \sum_{t=1}^v \sum_{u=1}^v a_t a_u E(-B_H^2(k_t/n) - B_H^2(k_u/n) \\ &\quad + (B_H(k_t/n) - B_H(k_u/n))^2) \end{aligned}$$



and likewise for  $E(e_r^2 | \mathcal{F}_{r-1})$ . Then by (3.9.c) we have

$$|\tau^2 - \sum_{r > r_0} E(e_r^2 | \mathcal{F}_{r-1})| < C \sigma_n^{-1} \log n$$

on a set of measure at least  $1 - n^{-8}$ . Off of this set we have

$$\begin{aligned} & |\tau^2 - \sum_{r \geq r_0} E(e_r^2 | \mathcal{F}_{r-1})| \\ & \leq \tau^2 + 2 n^2 \sum_{r \geq r_0} b_r^2 \leq C n^2. \end{aligned}$$

Thus,  $N_n \leq C \sigma_n^{-2} (\log n)^2$ . This, (3.13) and (3.14) finish the proof. ■

4. FLIL. For the proof of Theorem 1.9, we will use the following notation.

Let  $Z(n, t, x) = \sigma_n^{-1} S_n f_n(t, x)$ ;  $\varphi(n) = (2 \log \log \sigma_n^2)^{1/2}$ ; for  $\alpha > 1$ ,  $k \geq 1$ , let  $\alpha_k = [\alpha^k]$ . (Recall that  $a_n = \sigma_n \varphi_n$ .) We need to prove both parts of (1.8). (For a general discussion of our approach to this theorem, in particular our reduction to (4.2) below, we refer the reader to [Ph, section 2].)

(1.8.a) follows from the Arzela-Ascoli theorem, and (3.6.c) as we now show. Let

$$\begin{aligned} (4.1) \quad \alpha_k(\epsilon) = \{x : & \sup_{|s-t| \leq \alpha^{-1}} |Z(\alpha_k, s, x) - Z(\alpha_k, t, x)| \\ & > \epsilon \varphi(\alpha_k)\} \end{aligned}$$

To see that  $\mu(x : \mathcal{F}(x) \text{ is bounded in } C[0,1]) = 1$  ( $\mathcal{F}(x)$  is defined in (1.8.a).), observe that

$$\begin{aligned} & \sup_{2^{k-1} \leq n < 2^k} \sup_t |Z(n, t)| \\ & \leq C(k \sigma_{2^k}^{-1} + \sup_t |Z(2^k, t)|). \end{aligned}$$

Moreover, setting  $\alpha = 2$  in (4.1), we have

$$\begin{aligned} & \sum_k \mu(x : \sup_t |Z(2^k, t, x)| > \epsilon \varphi(2^k)) \\ & \leq \sum_k \mu(2_k(\epsilon)) < +\infty \end{aligned}$$

for  $\epsilon$  sufficiently large, by (3.6.c). The Borel-Cantelli lemma then implies that  $\mu(x : \mathcal{F}(x) \text{ is bounded in } C[0,1]) = 1$ .

A very similar argument will show that  $\mu(x : \mathcal{F}(x) \text{ is equicontinuous}) = 1$ .

Observe that

$$\begin{aligned} & \sup_{\alpha_{k-1} \leq n \leq \alpha_k} \sup_{|s-t| \leq \alpha^{-1}} |Z(n, s) - Z(n, t)| \\ & \leq C_\alpha \sup_{|s-t| \leq \alpha^{-1}} |Z(\alpha_k, s) - Z(\alpha_k, t)| \end{aligned}$$

where  $C_\alpha \rightarrow 1$  as  $\alpha \downarrow 1$ . Further, for all  $\epsilon > 0$ ,  $\sum \mu(\alpha_k(\epsilon)) < +\infty$  for  $\alpha$  sufficiently close to 1, by (3.6.c). The proof is then completed by the Borel-Cantelli lemma.

We turn to the proof of (1.8.b), with  $K = K_H$ , the unit ball of the reproducing kernel Hilbert space of  $B_H$ . (See [JM] for background on these spaces.) By applying [Kb theorem 3.1, and lemma 2.1 (iv)], we have the following sufficient condition for (1.7.b): For all  $g \in C[0,1]^*$ ,

$$(4.2) \quad \limsup_n \varphi_n^{-1} g(Z(n)) = (E g(B_H)^2)^{1/2} \quad \text{a.e.}(\mu).$$

(Kuelbs' result has been used by several other authors in the context of FLIL associated with fractional Brownian motion. See the survey [TC2].) Moreover, by relative compactness, we can, and do, take  $g$  in (4.2) to be a finite linear combination of point masses on the interval  $[0,1]$ . That is,  $g = \sum_1^v a_u \delta_{t_u}$ , with  $\sum |a_u| = 1$  and  $\tau^2 = E g(B_H)^2$ . This reduction will allow us to use (3.6.d) in the argument below.

The upper half of (4.2) follows from relative compactness and

$$\limsup_k \varphi^{-1}(\alpha_k) g(Z(\alpha_k)) \leq \tau \quad \text{a.e.}(\mu)$$

for all  $\alpha > 1$ . To establish this limit, fix  $\epsilon > 0$ . As  $g$  is fixed, (3.6.d) implies that

$$\begin{aligned} \mu(x : g(Z(\alpha_k)) < (1+\epsilon) \tau \varphi(\alpha_k)) \\ \leq C \tau^{-1} k^{2/5} \alpha_k^{-2H/5} + P(g(B_H) < (1+\epsilon)\tau \varphi(\alpha_k)) \\ \leq C \tau^{-1} k^{2/5} \alpha_k^{-2H/5} + \exp(-(1+\epsilon)^2 \varphi^2(\alpha_k)), \end{aligned}$$

which is summable in  $k$ , for all  $\epsilon > 0$ . This concludes the proof of the upper half.

To prove the lower half of (4.2), we will show that

$$\limsup_k \varphi(2^k)^{-1} g(Z(2^k)) \geq \tau \quad \text{a.e.}(\mu).$$

Fix  $\epsilon > 0$  and set  $A_k = (x : g(Z(2^k, x)) \geq (1-\epsilon)^{1/2} \tau \varphi(2^k))$ . It is enough to show that  $\mu(A_k \text{ i.o.}) = 1$ . By a result in [ER], two sufficient conditions for this are:  $\sum \mu(A_k) = +\infty$  and

$$(4.4) \quad \limsup_n \frac{\sum_{1 \leq j, k \leq n} \mu(A_j A_k)}{(\sum_{1 \leq k \leq n} \mu(A_k))^2} \leq 1.$$

The first condition follows from an argument very close to (4.3). In fact, letting  $\psi(u) = (2\pi)^{-1/2} \int_u^{+\infty} e^{-v^2/2} dv$ , observe that

$$(4.5) \quad \lim_n \frac{\sum_{k=1}^n \mu(A_k)}{\sum_{k=1}^n \psi((1-\epsilon)^{1/2} \varphi(2^k))} = 1,$$

and

$$(4.6) \quad C_1 n^\epsilon \leq \sum_{k=1}^n \psi((1-\epsilon)^{1/2} \varphi(2^k)) \leq C_2 n^\epsilon.$$

Verifying (4.4) will be more delicate. Recall that  $g = \sum_{u=1}^v a_u \delta(t_u)$  is a finite linear combination of point masses, with norm  $\sum |a_u| = 1$ . Let  $\tilde{g}$  be any other norm 1 linear functional which is a combination of at most  $v$  point masses. We have in mind  $\tilde{g} = g_k = \sum a_u \delta(2^{-k} t_u)$ . As the constant on the r.h.s. of (3.6.d) depends only on  $v$  and  $E g(B_H)^2$ , we have the following two-dimensional CLT, with a rate: Set  $\tilde{\tau}^2 = E \tilde{g}(B_H)^2$ . Then

$$(4.8) \quad \begin{aligned} & \sup_{y, z} |\mu(x : g(Z(m)) \geq y; \tilde{g}(Z(m)) \geq z) \\ & - P(g(B_H) \geq y; \tilde{g}(B_H) \geq z)| \\ & \leq C_v (\tau \wedge \tilde{\tau})^{-4/5} (\log m)^{2/5} m^{-2H/5}. \end{aligned}$$

Set  $\tau_k^2 = E g_k(B_H)^2$ , and observe that by self-similarity, (1.6),  $\tau_k^2 = 2^{-kH} \tau^2$ . Then for  $j, k > 0$

$$\begin{aligned}
 & \mu(A_j A_{j+k}) \\
 (4.9) \quad & \leq m(x : g(Z(2^{j+k})) > (1-\epsilon)^{1/2} \tau \varphi(2^{j+k}); \\
 & \quad g_k(Z(2^{j+k})) > (1-\epsilon)^{1/2} \tau_k \varphi(2^j) - \delta(j,k))
 \end{aligned}$$

where

$$\begin{aligned}
 (4.10) \quad \delta(j,k) &= 2^{-kH/2} \sup_t |S_{2^{k+j}} f_{2^{k+j}} - S_{2^{k+j}} f_{2^j}| \\
 &+ \tau \left| \frac{\sigma_{2^j}}{\sigma_{2^{j+k}}} - 2^{-kH/2} \right| \varphi(2^j) \\
 &\leq C((\log k) 2^{-kH/2} + |j+k| 2^{-(k+j)H/2}).
 \end{aligned}$$

Here, we've used (3.6.a) and (3.6.b).

We will also need the following lemma on  $B_H$ .

Lemma 4.11. For all intergers  $j$  and  $k$ ,

$$\begin{aligned}
 \Delta_1(j,k) &= |P(g(B_H) > (1-\epsilon)^{1/2} \tau \varphi(2^{j+k}); g_k(B_H) > (1-\epsilon)^{1/2} \tau_k \varphi(2^j)) \\
 &- P(g(B_H) > (1-\epsilon)^{1/2} \tau \varphi(2^{j+k})) P(g_k(B_H) > (1-\epsilon)^{1/2} \tau_k \varphi(2^j))| \\
 &\leq C 2^{-kG} \exp\left(-\frac{(1-\epsilon)(\varphi(2^{j+k})^2 + \varphi(2^j)^2)}{2(1+C 2^{-kG})}\right) \\
 &\leq C 2^{-kG}.
 \end{aligned}$$

where  $G = \min(H/2, (2-H)/2)$ .

Proof. Let  $Y = \tau^{-1} g(B_H)$  and  $Y_k = \tau_k^{-1} g_k(B_H)$ . Then  $Y$  and  $Y_k$  are standard normal random variables, with covariance

$$\begin{aligned}
 |E Y Y_k| &\leq \tau^{-2} 2^{+kH/2} \left| \sum_{1 \leq u, w \leq u} E B(t_u) B(2^{-k} t_w) \right| \\
 &\leq \frac{1}{2} \tau^{-2} 2^{+kH/2} \sum_{1 \leq u, w \leq v} |t_u|^H + |2^{-k} t_w|^H - |t_u 2^{-k} t_w|^H \\
 &\leq C 2^{kH/2} (2^{-kH} + 2^{-k}) \\
 &\leq C (2^{-kH/2} + 2^{-(1-H/2)k}) \leq C 2^{-kG}.
 \end{aligned}$$

With this observation, the necessary estimate can be carried out in a straightforward way. See e.g. the Normal Comparison lemma, [LLR, Theorem 4.2.1, p. 81].

We now have all the estimates necessary to prove (4.4). We need to show that

$$(4.12) \quad \limsup_n \frac{\sum_{j=1}^n \sum_{1 \leq k \leq n-j} \mu(A_j A_{j+k})}{\left( \sum_{1 \leq j \leq n} \mu(A_j) \right)^2} \leq \frac{1}{2}.$$

Here, we've used (4.6) to eliminate the diagonal for (4.4). Let

$$R_0 = \{(j,k) : 1 \leq j < n, 1 \leq k \leq (n-j)\}$$

$$R_1 = \{(j,k) : 1 \leq j < n, 1 \leq k \leq (n-j) \wedge n^{\epsilon/2}\}$$

and

$$R_2 = \{(j,k) : 1 \leq j < n, 1 \leq k < j^{2/\epsilon} \wedge (n-j)\}.$$

Notice that  $R_0$  is the set of indices over which we sum;  $R_1$  is "close" to the diagonal of the square; and  $R_1$  and  $R_2$  overlap.

(4.12) will follow from the calculations below, which can be routinely verified. By (4.5) and (4.6),

$$(4.13) \quad \sum \sum_{R_1} \mu(A_j A_{j+k}) \leq n^{\epsilon/2} \sum_1^n \mu(A_j) = O(n^3 \epsilon/2)$$

and likewise for

$$(4.14) \quad \sum \sum_{R_0 \setminus R_2} \mu(A_j A_{j+k}) = O(n^3 \epsilon/2).$$

Thus we can restrict our attention to sums over the set  $R_2 \setminus R_1$ .

Next, we reduce to the case of the fractional Brownian motion. Observe that by (4.8),

$$\begin{aligned} \Delta_2(j,k) &= |\mu(A_j A_{j+k}) - P(g(B_H) > (1-\epsilon)^{1/2} \tau \varphi(2^{j+k}))| \\ &= |g_k(B_H) > (1-\epsilon)^{1/2} \tau_k \varphi(2^j)| < C(k+j)^{2/5} 2^{-2jH/5}. \end{aligned}$$

Thus

$$(4.15) \quad \sum \sum_{R_2} \Delta_2(j,k) \leq C \sum_{j=1}^n 2^{-2jH/5} \sum_{k \leq j}^{2/\epsilon} (k+j)^{2/5} = O(1).$$

On  $R_2 \setminus R_1$ , we can reduce to the case of independent random variables. By (4.11),

$$(4.16) \quad \sum \sum_{R_2 \setminus R_1} \Delta_1(j,k) \leq C \sum_{j=1}^n \sum_{k \geq n^{\epsilon/2}} 2^{-kG} = O(1).$$

And for independent events, (4.12) is obvious.

We note that this proof has not taken (4.9) and (4.10) into account. It can be trivially modified to do so.

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