

A Note on the almost everywhere central limit theorem

Michael T. Lacey and Walter Philipp

Department of Statistics, UNC, Chapel Hill

1. Introduction. The theorem in the title refers to a recent result in [2] and [11], (also see [3]), which is as follows: Let S_k be the k -th partial sum of i.i.d. real-valued random variables (r.v.s) X_i with mean 0, variance 1 and finite $(2+\delta)$ -th moments ($\delta > 0$ in [2] and $\delta = 1$ in [11]). Let the r.v.s be defined on a probability space (Ω, \mathcal{F}, P) . Then there is a P -null set $N \subset \Omega$ such that for all $\omega \in N^c$,

$$(1) \quad (\log n)^{-1} \sum_{k \leq n} k^{-1} 1_A(k^{-1/2} S_k(\omega)) \rightarrow (2\pi)^{-1/2} \int_A e^{-1/2 u^2} du$$

for all Borel sets $A \subset \mathbb{R}$ with $\lambda(\partial A) = 0$. If $\delta(x)$ denotes the point mass at $x \in \mathbb{R}$, then (1) can be restated in this way. For all $\omega \in N^c$

$$(2) \quad (\log n)^{-1} \sum_{k \leq n} k^{-1} \delta(k^{-1/2} S_k(\omega)) \xrightarrow{d} N(0,1).$$

The functional version of (2), proved in [2], is as follows. Define the usual "broken line process" on $[0,1]$,

$$(3) \quad s_n(t, \omega) = \begin{cases} n^{-1/2} S_k & \text{if } t = k/n, k = 0, 1, \dots, n \\ \text{linear in between} & \end{cases}$$

and denote by $\delta(x)$ the point mass at $x \in C[0,1]$. Then

$$(4) \quad (\log n)^{-1} \sum_{k \leq n} k^{-1} \delta(s_k(\cdot, \omega)) \xrightarrow{d} W \quad P\text{- a.s.}$$

Here W is standard Brownian motion on $[0,1]$.

In this note we prove the following results.

Theorem 1. Let $\{X_j, j \geq 1\}$ be any sequence of real-valued r.v.s defined on (Ω, \mathcal{F}, P) . Suppose there exists a sequence $\{Y_j, j \geq 1\}$ of i.i.d. $N(0,1)$ r.v.s such that with probability 1

$$(5) \quad \sum_{k \leq n} X_k - \sum_{k \leq n} Y_k = o(n^{1/2}).$$

Then (4) holds.

Theorem 2. Let $\{X_j, j \geq 1\}$ be a sequence of i.i.d. r.v.s with mean 0 and variance 1. Then (4) holds.

The difference between the proof of Theorem 1 and that in [2] is two-fold. First, we assume an almost sure invariance principle (5) (ASIP) and present a simple reduction of the theorem to the Gaussian case. Second, the latter case is done in a straightforward, probabilistic way, with no appeal to ergodic theory, as in [2]. The proof of Theorem 2 is a corollary to the proof of Theorem 1. Further remarks on the theorems and some extensions are at the end of (this) note.

2. Proofs. a. Denote by $BL = BL(C[0,1], \|\cdot\|_\infty)$ the class of functions $f : C[0,1] \rightarrow \mathbb{R}$ with $\|f\|_{BL} := \|f\|_L + \|f\|_\infty < \infty$. Here

$$\|f\|_L := \sup\{|f(x) - f(y)| / \|x-y\|_\infty : x, y \in C[0,1], x \neq y\}.$$

We first prove that (4) is equivalent to the following statement: For each $f \in BL$

$$(6) \quad (\log n)^{-1} \sum_{k \leq n} k^{-1} f((s_k(\cdot, \omega)) \rightarrow Ef(W(\cdot)) \quad P\text{- a.s.}$$

Although the left-hand side of (4) does not define a probability measure, it is so close to one that the theory of weak convergence still applies. But (6) does not yet imply (4) since, in general, the exceptional P-null set may depend on f. However, it is clear from the proof of [5, Theorem 8.3, II => III] that if we have (6) for a certain countable subset of BL, which is not difficult to write down explicitly, then we have only a countable union N of P-null sets to contend with, and, outside N, the convergence in (6) is uniform in $f \in BL$. Thus (4) and (6) are equivalent.

b. Next, by (3) and (5), there is a P-null set, call it also N, such that for all $\omega \in N^c$

$$|f(s_k(\cdot, \omega)) - f(t_k(\cdot, \omega))| \leq \|f\|_{BL} \|s_k(\cdot, \omega) - t_k(\cdot, \omega)\|_\infty \\ \leq \|f\|_{BL} k^{-1} \max_{j \leq k} |s_{1j}(X_1 - Y_1)| = o(1).$$

Here we denoted by t_n the broken line process, defined in (3) with X_j replaced by Y_j . Thus (4) is equivalent to

$$(7) \quad (\log n)^{-1} \sum_{k \leq n} k^{-1} f(t_k(\cdot, \omega)) \xrightarrow{P} E f(W(\cdot)) \quad P\text{- a.s.}$$

for each $f \in BL$. Hence we have reduced the proof to the case of Gaussian r.v.s.

c. For the proof of (7) we note that $t_k \xrightarrow{d} W$ by Donsker's theorem applied to $\{Y_j, j \geq 1\}$, i.e. $E f(t_k) \rightarrow E f(W)$. Hence it is enough to prove that

$$(8) \quad (\log n)^{-1} \sum_{k \leq n} k^{-1} \xi_k \rightarrow 0 \quad P\text{- a.s.}$$

where

$$(9) \quad \xi_k := f(t_k(\cdot)) - E f(t_k(\cdot)).$$

We need the following estimate.

e. The use of the logarithmic mean looks perhaps peculiar at first glance, but it appears that it is essentially the only summation method that works. The Cesaro mean certainly does not, since

$$\frac{1}{n} \#\{k \leq n; S_k > 0\} \not\rightarrow \arcsin.$$

A similar argument shows that the weights $k^{-\alpha}$, $0 < \alpha < 1$ don't work either. Moreover, the covariance estimate (10) is sharp, and the present method will not allow the use of any summation method which is not close to the logarithmic mean.

f. The rates of convergence in these theorems cannot be $o(1/\log n)$ as can be seen for instance by setting $A = R$ in (1).

REFERENCES

- [1] Billingsley, Patrick, Convergence of probability measures, J. Wiley, N.Y., 1968.
- [2] Brosamler, Gunnar A., An almost everywhere central limit theorem, *Math. Proc. Camb. Phil. Soc.* 104, 561-574 (1988).
- [3] Brosamler, Gunnar A., Un théorème limite presque sur pour les mesures d'occupation du mouvement brownien sur une variété riemannienne compacte, *C.R. Acad. Sci.* 307, Ser. I, 919-922 (1988).
- [4] Chung, K.L., A course in probability theory, Second edition, Academic Press, New York, 1974.
- [5] Dudley, R.M., Probabilities and Metrics, Lecture Note Series 45, Aarhus Universitet, 1976.
- [6] Dudley, R.M. and Walter Philipp, Invariance principles for Banach space valued random elements and empirical processes, *Z. Wahrscheinlichkeitstheorie verw. Geb.* 62, 509-552 (1983).
- [7] Einmahl, Uwe, Strong invariance principles for partial sums of independent random vectors, *Annals Prob.* 15, 1419-1440 (1987).

- [8] Philipp, Walter. Invariance principles for sums of mixing random elements and the multivariate empirical process, *Coll. Math. Soc. Janos Bolyai* 36 (Limit Theorems in Probability and Statistics), 843-873, Veszprém (Hungary), 1982.
- [9] Philipp, Walter. Invariance principles for independent and weakly dependent random variables. In Dependence in Probability and Statistics, E. Eberlein and M.S. Taqqu, ed. Birkhauser, Boston 1986.
- [10] Philipp, Walter and William Stout, Almost sure invariance principles for partial sums of weakly dependent random variables, *Memoirs AMS* 161 (1975).
- [11] Schatte, Peter. On strong versions of the central limit theorem, *Math. Nachr.* 137, 249-256 (1988).

Michael T. Lacey

Walter Philipp
Departments of Mathematics
and Statistics
University of Illinois
Urbana, IL 61801

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