Optimally Stopping the Sample Mean of a Wiener Process
with an Unknown Drift

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ABSTRACT

It is well–known that optimally stopping the sample mean $W(t)/t$ of a standard Wiener process is associated with a square root boundary. It is shown that when $W(t)$ is replaced by $X(t)=W(t)+\theta t$ with $\theta$ normally distributed $N(\mu,\sigma^2)$ and independently of the Wiener process, the optimal stopping problem is equivalent to the time–truncated version of the original problem. It is also shown that the problem of optimally stopping $(b+X(t))/(a+t)$, with constants $a>0$ and $b$, is equivalent to the time–truncated version of the original problem or the one–arm bandit problem depending on whether $\sigma^2\leq a^{-1}$ or $\sigma^2>a^{-1}$. Furthermore, the optimal stopping region changes drastically as the prior parameters $(\mu,\sigma^2)$ are slightly perturbed in a neighborhood of $(b/a,1/a)$.

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1. **Introduction.** Taylor (1968), Shepp (1969), and Walker (1969) have independently shown that, starting from a positive time $T$, it is optimal to stop the sample mean $W(t)/t$ of a Wiener process $W = \{W(t), t \geq 0\}$ at the first time $t \geq T$ for which $W(t) \geq \alpha t^{\frac{1}{2}}$, where $\alpha = 0.83992\ldots$ is the unique root of the equation $\alpha \phi(\alpha) = (1-\alpha^2)\Phi(\alpha)$, and where $\phi$ and $\Phi$ are the standard normal probability density function and distribution function, respectively. An optimal rule, in this setting, is a stopping rule which maximizes the expected stopped sample mean $E\{W(\tau)/\tau\}$ among all stopping rules $\tau$ that continue until at least time $T$.

An equivalent formulation of their result is to say that, in seeking to maximize the ratio $\frac{b+tW(t)}{a+t}$ by stopping at an optimal time $t \geq 0$, it is optimal to stop immediately if and only if $b \geq \alpha a^{\frac{1}{2}}$ ($a > 0$). This type of equivalence does not remain when an independent drift rate $\theta$ is imposed on $W$ which is normally distributed with mean $\mu$ and variance $\sigma^2$, i.e., when $W$ is replaced by $X = \{X(t) = W(t) + \theta t, t \geq 0\}$.

We have two objectives for this note: to show that, with this added drift, (i) the first formulation of the "Taylor–Shepp–Walker problem" is equivalent to a time–truncated version of the Taylor–Shepp–Walker problem, without the drift, and (ii) the second formulation is unstable with respect to the (prior) parameters assigned to the distribution of $\theta$. Below a horizontal "line of degeneracy" (in the $(\mu,\sigma^2)$–half–plane), the solution of the optimal stopping problem, for the second formulation, is determined by the solution of the time–truncated version of the Taylor–Shepp–Walker problem referred to in (i); above the line, the solution is determined by the solution to the one–arm bandit problem considered by Chernoff and Ray (1965); and on the line of degeneracy, the optimal stopping problem "degenerates" in three possible ways: on this line, the ratio $\frac{b+X(t)}{a+t}$, $t \geq 0$, is a martingale at a specific "point of instability"; it is a supermartingale to the left of this point; and it is a submartingale to the right of this point. As one approaches the point of instability, the shape of the optimal stopping boundary converges to a limiting shape that can be simply described analytically. However, the precise limit depends upon how, and from which side, the point is approached.
One encounters an added surprise in the region where the one-arm bandit problem is relevant. It is optimal to continue when the observation $X(t)$ is sufficiently large, not sufficiently small.

One must also take into account the possible non-finiteness of the optimal stopping rule.

The first and second formulations are discussed in Sections 2 and 4, respectively. Section 3 provides a very brief description of the one-arm bandit problem considered by Chernoff and Ray (1965), which is needed in Section 4.

2. First formulation. Let $X = \{X(t) = W(t) + \theta t, t \geq 0\}$, where $W = \{W(t), t \geq 0\}$ is a standard Wiener process, and where the random variable $\theta$ is independent of $W$ and normally distributed, $N(\mu, \sigma^2)$. We are concerned here with maximizing the expectation $E\{X(\tau)/\tau\}$ among stopping times for which $\tau \geq T$, where $T$ is an arbitrary positive value. Because $X$ is a Markov process, the task is one of specifying which points $(x, t)$ are "optimal continuation points" and which are "optimal stopping points", $t > 0$. (These are independent of $T$.) Then the optimal rule can be expressed in the form "stop the first time $t \geq T$ for which the point $(X(t), t)$ is an optimal stopping point." (A point $(x, t)$ is an optimal continuation point if given $X(t) = x$, there is a stopping rule $\tau \geq t$ which improves on the ratio $x/t$ in the sense that $E\{X(\tau)/\tau | X(t) = x\} > x/t$. Otherwise, it is an optimal stopping point.) This rule might fail to stop, and, in fact, it does fail in the present setting with positive probability: if $(X(t), t)$ is an optimal continuation point there is positive probability that no future optimal stopping point is encountered. When any stopping rule fails to stop, it is appropriate to view the stopping time as infinite and the stopping reward as the limiting value of the reward: $X(t)/t \to \theta$ as $t \to \infty$. With this convention, the stopping rule described above is optimal. (Cf., Chow–Robbins–Siegmund (1971), pages 77–84.)
Now, consider the process $Z = \{Z(u), 0 \leq u \leq 1\}$, where

$$Z(u) = \sigma \cdot (X(t) - t \cdot E\{\theta | X(t)\}), \quad u = t/(t + \sigma^2), \quad 0 \leq t < \omega.$$  \hspace{1cm} (1)

By direct calculations, it is found that $Z$ is a standard (driftless) Wiener process in the variable $u$, and that

$$\frac{X(t)}{t} = \frac{\sigma Z(u)}{u} + \mu.$$  \hspace{1cm} (2)

When $t = \omega$, continuity considerations dictate the values $u = 1$ and $Z(u) = (\theta - \mu)/\sigma$. It follows that, starting from an arbitrary positive time $T$, the problem of optimally stopping $X(t)/t$ is mathematically equivalent to the problem of optimally stopping $Z(u)/u$ in the time interval $[U, 1]$, where $U = \sigma^2 T/(1 + \sigma^2 T)$, which is a time-truncated version of the Taylor–Shepp–Walker problem. If one can describe the optimal stopping points $(z, u)$ for the latter problem, then one can easily describe the optimal stopping points $(x, t)$ for the former. Clearly, the optimal stopping points $(z, u)$ can be described by an inequality of the form $z \geq \bar{z}(u)$ for some "boundary function" $\bar{z} = \{\bar{z}(u), 0 < u < 1\}$. Then, defining the function $\bar{x} = \{\bar{x}(t), t > 0\}$ by means of the equation

$$\frac{\bar{x}(t)}{t} = \frac{\sigma \bar{z}(u)}{u} + \mu, \quad (t < \omega, u < 1),$$

one obtains the former problem's optimal stopping points as $\{(x, t): x \geq \bar{x}(t)\}$. Chernoff and Petkau (1984) have numerically evaluated the function $\bar{z}$ to three decimal places.

Clearly, $0 < \bar{z}(u) < \alpha u^{1\over 2}$, $0 < u < 1$, because of the Taylor–Shepp–Walker result for $W$, cited in Section 1. It seems intuitively clear that the quality of the upper bound must be excellent when $u$ is small. Indeed, we have:
THEOREM 1. As $u \to 0$, $\tilde{z}(u) = \alpha u^{1/2} - o(u^{1/2})$. As $u \to 1$, $\tilde{z}(u) = \beta(1-u)^{1/2} + O((1-u)^{3/2})$, where $\beta = 0.63883 \cdots$ is the root of the equation $(1-\beta^2)\phi(\beta) = \beta^3\Phi(\beta)$.

This theorem is stated in Chernoff and Petkau (1984) without proof. The first part can be proved using the Taylor–Shepp–Walker result along with the invariance property that $c^{-4}W(t/c)$ remains standard Brownian motion for any $c > 0$. The second part can be established using methods found in the work of Chernoff and Ray (1965). It can actually be shown that as $u \to 1$, $\tilde{z}(u)$ has an asymptotic expansion of the form $(1-u)^{3/4} \sum_{i=0}^{\infty} c_i (1-u)^i$ for some constants $c_i$.

Transformations based on the difference $X'(t) = X(t) - t \cdot \mathbb{E}\{\theta|X(t)\}$, such as in (1), seem not to have been directly explored before. We have chosen to use one here since it so quickly leads to the revealing equation (2), and because it seems interesting. It is possible to obtain equation (2) by a more traditional route popularized by H. Chernoff, one based on the reverse-time Brownian motion process discussed in the next section. Of some interest, we believe, is G. Kallianpur’s (1987) observation that the traditional normality assumption for $\theta$ is unnecessary: if $\theta$ is independent of $W$ and has a finite second moment, then $X'(t) = X(t) - t \cdot \mathbb{E}\{\theta|X(t)\}$ is a continuous, mean-zero martingale in time $t$, and it can be viewed as standard Brownian motion in time $\nu = \int_0^t (1 - w \sigma_w^2)^2 \, dw$, where $\sigma_w^2 = \text{Var}\{\theta|X(w)\}$. More precisely, $X'(\tau_{\nu})$ is standard Brownian motion in $\nu$, where $\tau_{\nu} = \inf\{t: \int_0^t (1 - w \sigma_w^2)^2 \, dw > \nu\}$. (When $\theta \sim N(\mu, \sigma^2)$, $\sigma_w^2 = (\sigma^{-2} + w)^{-1}$, $\tau_{\nu} = \nu/(1 - \sigma^2 \nu)$ $(\nu < \sigma^2, \sigma > 0)$, and $X'(\tau_{\nu}) = \sigma^{-1}Z(\sigma^2\nu)$, implying that $Z(u)$ is standard Brownian motion. If $\theta$ is a constant, $\tau_{\nu} = \nu$ and $X'(\tau_{\nu}) = X(\nu) - \nu \cdot \mathbb{E}\{\theta|X(\nu)\} = W(\nu)$.)

If one needs to optimally stop the process $Z(c,d) = \{\frac{d+Z(u)}{c+u}, 0 \leq u \leq 1\}$ ($c > 0$), one can rescale time and exploit the Markovian property of the process $Z$: the process $Z^* = \{Z^*(u^*) = (c+1)^{-1/4}[d + Z(u)], (1+c^{-1})^{-1} \leq u^* \leq 1\}$, with $u^* = (c + u)/(c + 1)$, is a Wiener process beginning with the value $(c+1)^{-1/4}d$ at time $u^* = (1+c^{-1})^{-1}$; and one obtains $\frac{d+Z(u)}{c+u} = (c+1)^{-1/4}\{Z^*(u^*)/u^*\}$. It follows that $(z, u)$ is an optimal stopping point for $Z(c,d)$ if and only if $z \geq (c+1)^{1/4}\frac{c+u}{c+1} - d$. This will be used in Section 4.
3. Chernoff and Ray's one-arm bandit. Chernoff and Ray's (1965) one-arm bandit problem can be formulated as the problem of optimally stopping the ratio \( Y(s)/s \), in a time interval \( S \geq s \geq 1 \), where \( Y = \{Y(s)\} \) is reverse-time Brownian motion starting at \((y,s) = (Y(S),S) \) \((E\{dY(s)\} = 0, \text{Var}\{dY(s)\} = -ds)\). Stopping is mandated when \( s \) reaches 1. Here, a point \((y,s)\) is an optimal stopping point if given \( Y(s) = y \), there is no reverse stopping time \( \tau, s \geq \tau \geq 1 \), which improves on the ratio \( y/s \) in the sense that \( E\{Y(\tau)/\tau|Y(s) = y\} > y/s \). (This notion is independent of the starting point \( Y(S),S) \).

According to Chernoff and Petkau (1986), the optimal stopping points can be described by an inequality of the form \( y \leq \tilde{y}(s) \) for some decreasing (as \( s \) increases) boundary function function \( \tilde{y} = \{\tilde{y}(s), 1 < s < \infty\} \). They have numerically evaluated \( \tilde{y} \) to four decimal places. As \( s \to 1 \), \( \tilde{y}(s) = -\beta(s-1)^{1/2} + O((s-1)^{3/2}) \), where \( \beta = 0.63883 \cdots \) is the same constant appearing in Theorem 1.

We remark that "Chernoff and Petkau (1986)" is an abbreviated version of the unpublished technical report "Chernoff and Petkau (1984)." The former discusses the one-arm bandit problem but not the time-truncated version of the Taylor–Shepp–Walker problem.

A well-known way for the process \( Y \) to arise is through the transformation \( Y(s) = \frac{X(t) + \mu/\sigma^2}{t + \sigma^2}, s = \frac{1}{t + \sigma^2} \). It starts at the point \( (Y(S),S) = (\mu,\sigma^2) \), when \( t = 0 \). Since \( X(t) = Y(s)/s - \mu/\sigma^2 \), it is apparent that the problem of optimally stopping the ratio \( Y(s)/s \) is equivalent to the problem of optimally stopping \( X(t) \) over a finite interval \([0,T]\). (This gets us closer to Chernoff and Ray's (1965) original discrete-time problem.) We shall see below that it is equivalent, as well, to the problem of optimally stopping the ratio \( \frac{b+X(t)}{a+t} \), \( 0 \leq t < \infty \), when \( a\sigma^2 > 1 \).

4. Second formulation. Here, we are concerned with the more general problem of optimally stopping the process \( X(a,b) = \{\frac{b+X(t)}{a+t}, t > 0\} \) \((a > 0)\), i.e., of finding a stopping time \( \tau \) which maximizes the expectation \( E\{\frac{b+X(\tau)}{a+\tau}\} \).
4.1. On the line of degeneracy. The two–dimensional parameter \((\mu, \sigma^2)\), which controls the drift rate \(\theta\), is confined to the open upper half plane. The "line of degeneracy", referred to in the introduction, is the horizontal line \(\sigma^2 = a^{-1}\) in this half plane. Suppose for now that \(\sigma^2 = a^{-1}\). Then \(\frac{b+X(t)}{a+t} = \sigma Z(u) + u(\mu - ba^{-1}) + ba^{-1}\), a consequence of equation (2) and the relationship \(u = (t + \sigma^{-2})^{-1}t\). Since \(Z\) is a Wiener process, and hence a martingale, the optimal stopping rule for \(X(a,b)\) depends solely on the sign of the coefficient of \(u\). If \(\mu < ba^{-1}\), then \(X(a,b)\) is a supermartingale, the optimal stopping rule is \(\tau \equiv 0\), and the expected reward is \(ba^{-1}\). If \(\mu > ba^{-1}\), then \(X(a,b)\) is a submartingale, the optimal stopping rule is \(\tau \equiv \infty\), and the expected reward is \(\mu\). Finally, if \(\mu = ba^{-1}\), then \(X(a,b)\) is a martingale, every stopping rule is optimal, and the expected reward is \(ba^{-1}\).

The point \((\mu, \sigma^2) = (b/a, 1/a)\), where \(X(a,b)\) is a martingale, is the "point of instability" referred to in the introduction. It is discussed in Subsection 4.4 below.

4.2. Below the line of degeneracy. Suppose now that \(\sigma^2 < a^{-1}\). Then

\[
\frac{b+X(t)}{a+t} = (1-a\sigma^2)^{-1}\left\{\sigma \cdot \frac{d+Z(u)}{c+u} + \mu - ba^{-1}\right\},
\]

where \(c = \frac{a\sigma^2}{1-a\sigma^2}\) and \(d = \frac{\sigma(b-a\mu)}{1-a\sigma^2}\). Since the point \((z, u)\) is an optimal stopping point for the process \(Z(c, d)\), defined in Section 2, if and only if \(z \geq (c+1)^{\frac{1}{2}}\frac{z}{c+1} - d\), it follows from (2) that \((x, t)\) is an optimal stopping point for the process \(X(a,b)\) if and only if

\[
x \geq (\sigma^{-2} - a)^{-\frac{1}{2}}(\sigma^{-2} + t) \cdot \frac{a+t}{\sigma^{-2} + t} + \frac{\mu - ba^{-1}}{1-a\sigma^2} + \frac{a\mu - b}{1-a\sigma^2}.
\]

Stopping occurs at time \(t = 0\) if \(\tilde{z}(a\sigma^2) \leq (b-\mu)(\sigma^{-2} - a)^{-\frac{1}{2}}\). Otherwise, "stopping" at \(t = \infty\) is a possibility.

4.3. Above the line of degeneracy. Suppose now that \(\sigma^2 > a^{-1}\). As noted in Section 3, the transformation \(Y(s) = \frac{X(t) + \mu/\sigma^2}{t + \sigma^2}\), \(s = \frac{1}{t + \sigma^2}\) produces reverse–time Brownian motion.
The linear shift \( s^* = 1 + (a-\sigma^2)s \) sends the \( t \)-time interval \( 0 \leq t < \infty \) into the \( s^* \)-time interval \( a\sigma^2 \geq s^* > 1 \). Then \( Y^*(s^*) = (a-\sigma^2)^{-\frac{1}{2}}Y(s) - (b-\mu\sigma^2)(a-\sigma^2)^{-1} \) is reverse-time Brownian motion starting from \( Y^*(a\sigma^2) = (a\mu-b)(a-\sigma^2)^{-1} \). Moreover,

\[
\frac{b+X(t)}{a+t} = (a-\sigma^2)^{-\frac{1}{2}}Y^*(s^*)/s^* + (b-\mu\sigma^2)(a-\sigma^2)^{-1}.
\]

So this case reduces to the the one-arm bandit problem discussed in Section 3. By straightforward algebra, one finds that \( (x,t) \) is an optimal stopping point for the process \( X(a,b) \) if and only if

\[
x \leq (a-\sigma^2)^{-\frac{1}{2}}(\sigma^2+t) \cdot \tilde{y}(\frac{x+t}{\sigma^2+t}) + \frac{\mu-b\sigma^2}{1-\sigma^2}t + \frac{a\mu-b}{1-a\sigma^2}.
\]

(4)

Stopping occurs at time \( t = 0 \) if \( \tilde{y}(a\sigma^2) \geq - (b-\mu)(a-\sigma^2)^{-\frac{1}{2}} \). Otherwise, "stopping" is possible at \( t = \infty \).

4.4 Approaching the point of instability. By Theorem 1, \( \tilde{z}(s) = \beta(1-s)^{1/2} + O((1-s)^{3/2}) \) as \( s \to 1^- \). Thus at a point \( (\mu,\sigma^2) \) near the point of instability, with \( \sigma^2 < a^{-1} \), the boundary function for \( X(a,b) \) assumes the approximate shape

\[
\phi(a+t)^{\frac{1}{2}} + \frac{bt}{a} + \frac{\mu-b/a}{1-a\sigma^2}(b+a)
\]

with an error of size \( (a+t)^{-\frac{1}{2}}O(a\sigma^2-1) \) as \( a\sigma^2 \to 1^- \). On the line \( \mu = b/a \), the approximate shape simplifies to \( \phi(a+t)^{\frac{1}{2}} + \frac{bt}{a} \).

Likewise, \( \tilde{y}(s) = -\beta(s-1)^{1/2} + O((s-1)^{3/2}) \) as \( s \to 1^+ \). Thus at a point \( (\mu,\sigma^2) \) near the point of instability, with \( \sigma^2 > a^{-1} \), the boundary function for \( X(a,b) \) assumes the approximate shape

\[
-\beta(a+t)^{\frac{1}{2}} + \frac{bt}{a} + \frac{\mu-b/a}{1-a\sigma^2}(b+a)
\]
with an error of size \((a+t)^{-\frac{1}{2}}O(a\sigma^2-1)\) as \(a\sigma^2 \to 1^+\). On the line \(\mu = b/a\), the approximate shape simplifies to \(-\beta(a+t)^{\frac{1}{2}} + \frac{b}{a}t\). The obvious difference between these two expansions is the change of sign in the coefficient of the term \((a+t)^{\frac{1}{2}}\). Also, note that the two inequalities (3) and (4) are in opposite directions.

It is easily checked that \((b/a, 1/a)\) is the only unstable point in the \((\mu, \sigma^2)\)-half-plane, i.e., the optimal stopping region changes continuously with \((\mu, \sigma^2)\) at all other points, including all other points on the line of degeneracy.

**References**


