THE INSTITUTE OF STATISTICS

THE CONSOLIDATED UNIVERSITY OF NORTH CAROLINA

ON LIMIT DISTRIBUTIONS ARISING FROM ITERATED RANDOM SUBDIVISIONS OF AN INTERVAL

by

Norman L. Johnson
and
Samuel Kotz

October 1993

Mimeo Series #2308

DEPARTMENT OF STATISTICS
Chapel Hill, North Carolina

On Limit Distributions Arising from
Iterated Random Subdivisions of an Interval

Norman L. Johnson
University of North Carolina
at Chapel Hill

Samuel Kotz
University of Maryland
College Park

ABSTRACT

A method of calculating moments of distributions of limit points of certain procedures for iterated random subdivision of finite intervals is applied to some specific examples. Since the ranges of these distributions are finite, the moments, in principle determine the limit distributions. Several specific applications are included.

Key words: Beta distributions; Limit distributions; Moments; Recurrence relations; Uniform distributions

AMS 1991 SUBJECT CLASSIFICATION: Primary 62E17, Secondary 62E15

1. INTRODUCTION

In this paper we present a simple method for calculating moments of random variables representing the limit position resulting from certain procedures involving iterated subdivision of a finite interval. The first procedure we consider was described by Chen et al. (1984).

This procedure commences with observation of the value of a random variable, X_1, uniformly distributed over the interval (0,1). One of the two resulting subintervals $(0,X_1)$ $(X_1,1)$ is chosen, with probabilities p, 1−p of choosing the longer or shorter interval, respectively. Denoting the chosen interval by (L_1,U_1), the value of a random variable X_2, uniformly distributed over (L_1,U_1) and one of the two subintervals (L_1,X_2), (X_2,U_1) chosen according to the scheme just described. This interval is denoted (L_2,U_2). Continuing in this way, the length (U_n-L_n) of the chosen interval tends to zero with probability one as $n \to \infty$, so there is a limiting value, Y_p, to which both L_n and U_n tend.

Chen et al. (1981) showed that the distribution of $Y_{\frac{1}{2}}$ is beta $(\frac{1}{2},\frac{1}{2})$ (the 'arc sine' law), and Chen et al. (1984) showed that the distribution of Y_1 is beta (2,2). [In this paper the 'beta (α,β)' distribution means the distribution with probability density function

$$p_X(x) = \{B(\alpha,\beta)\}^{-1} x^{\alpha-1}(1-x)^{\beta-1} \quad (0 < x < 1).]$$

It is natural to conjecture that, for all p, Y_p has (at least) an approximate beta distribution. O'Connor et al. (1985) present results of a sampling experiment strongly supporting this conjecture. The values of moments of Y_p, presented in this paper, provide further evidence supporting this conjecture, and also a remarkably accurate conjecture of O'Connor et al. (1985) on the parameter value (α) for the approximating beta (α,α) distributions.

Variants of this procedure, in which the value of p may not remain constant for all iterations, are also considered.

The method is also applied to a procedure studied by Kennedy (1988). In this procedure, values of k mutually independent random variables, $Z_{n1},\ldots,Z_{nk}$ uniformly distributed over (L_n,U_n) are observed and (L_{n+1},U_{n+1}) chosen as the subinterval

(1) $(\min(Z_{n1},\ldots,Z_{nk}),\ U_n)$, $(L_n,\ \max(Z_{n1},\ldots,Z_{nk}))$, $(\min(Z_{n1},\ldots,Z_{nk}),$

$\max(Z_{n1},\ldots,Z_{nk}))$

with probabilities p, q, r respectively ($p+q+r = 1$). Kennedy showed that the *exact* distribution of $\lim_{n\to\infty} L_n$ ($= \lim_{n\to\infty} U_n$ with probability 1) is beta $(k(p+r),\ k(q+r))$, by a different method.

In all cases, the initial interval is (0,1). Results for a general initial interval (L_0,U_0) say, are easily established by a linear transformation.

[In an earlier application of the method employed in the present paper, the authors (Johnson and Kotz (1990)) attacked a problem introduced and studied by Van Assche (1986, 1987). This concerned the distribution of a random variable, Z^*, "uniformly distributed between two random variables, X_1, X_2", which means that the conditional distribution of Z^*, given X_1 and X_2, is uniformly distributed over the range of values between X_1 and X_2.]

2. MOMENTS OF Y_p

We use an approach of a type suggested to Chen et al. (1984) by R. Pyke, although it was not used by them to calculate moments. It is based on noticing that, once the values of L_1 and U_1 are determined, the conditional distribution of Y_p (taken as the limit point of L_n in the present context) is the same as that of

$$L_1 + (U_1-L_1)Y_p.$$

We further note that:

(i) Y_p has a symmetric distribution over (0,1). This is a very useful feature in reducing the needed algebraic manipulation because (using the notation $E[Y_p^r] = \mu_r'$,

$E[(Y_p - \mu_1')^r] = \mu_r$)

(2.1) $$\mu_1' = \frac{1}{2}.$$

(2.2) since $\mu_3 = 0$, $\mu_3' = \frac{3}{2}\mu_2' - \frac{1}{4}$, and hence

(2.3) $$\mu_4 = \mu_4' - \frac{3}{2}\mu_2' + \frac{5}{16}.$$

(ii) $\Pr[(L_1,U_1) \equiv (0,X_1) | X_1 < \frac{1}{2}] = \Pr[(L_1,U_1) \equiv (X_1,U_1) | X_1 > \frac{1}{2}] = 1-p$,

and $\Pr[(L_1,U_1) \equiv (X_1,U_1) | X_1 < \frac{1}{2}] = \Pr[(L_1,U_1) \equiv (0,X_1) | X_1 > \frac{1}{2}] = p$.

(iii) $\Pr[X_1 < \frac{1}{2}] = \Pr[X_1 > \frac{1}{2}] = \frac{1}{2}$, and the conditional distribution of X_1, given $X_1 < \frac{1}{2}$ is uniform over $(0,\frac{1}{2})$; given $X_1 > \frac{1}{2}$, it is uniform over $(\frac{1}{2},1)$.

Hence Y_p has the same distribution as a mixture of the distributions of four random variables

WY_p, $W + (1-W)Y_p$, $(1-W)Y_p$ and $1-W+WY_p$ with probabilities

$\frac{1}{2}(1-p)$, $\frac{1}{2}p$, $\frac{1}{2}p$ and $\frac{1}{2}(1-p)$ respectively,

where W and Y_p are mutually independent, and W has a uniform distribution over $(0,\frac{1}{2})$ [and so $(1-W)$ has a uniform distribution over $(\frac{1}{2},1)$].

From the resulting equations (for all positive integers r)

(3) $$\mu_r' = \tfrac{1}{2}(1-p)\, E[(WY_p)^r] + \tfrac{1}{2}\, pE[\{W + (1-W)Y_p\}^r] + \tfrac{1}{2}\, p\, E[\{(1-W)Y_p\}^r] + \tfrac{1}{2}(1-p)\, E[(1-W+WY_p)^r]$$

$$= \tfrac{1}{2}\{(1-p)M_{r,0} + p\, M_{0,r}\} + \tfrac{1}{2} \sum_{j=0}^{r} \binom{r}{j} (pM_{r-j,j} + (1-p)M_{j,r-j})\mu_j'$$

(where $M_{a,b} = E[W^a(1-W)^b]$) we derive

(4) $$\mu_r' = \frac{1}{2}\,\frac{pM_{r,0} + (1-p)M_{0,r} + \sum_{j=1}^{r-1} \binom{r}{j} \{pM_{r-j,j} + (1-p)M_{j,r-j}\}\mu_j'}{1 - (1-p)\, M_{r,0} - pM_{0,r}}.$$

Values of $M_{a,b}$ are shown in Table 1, for $a + b \leq 4$.

TABLE 1 VALUES OF $M_{a,b} = E[W^a(1-W)^b]$

b \ a	0	1	2	3	4
0	1	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{1}{32}$	$\frac{1}{80}$
1	$\frac{3}{4}$	$\frac{1}{6}$	$\frac{5}{96}$	$\frac{3}{160}$	
2	$\frac{7}{12}$	$\frac{11}{96}$	$\frac{1}{30}$		
3	$\frac{15}{32}$	$\frac{13}{160}$			
4	$\frac{31}{80}$				

Using equations (2), Table 1 and equation (4) with r = 2,4 we obtain

(5.1)
$$\mathrm{var}(Y_p) = \frac{7-6p}{4(11-6p)}$$

and

(5.2)
$$\beta_2(Y_p) = \frac{3(11-6p)(151-204p + 60p2)}{(7-6p)^2(79-30p)}.$$

[$\beta_2(= \mu_4/\mu_2^2)$ is the kurtosis index.]

From (5.1), we see that $\mathrm{var}(Y_p)$ decreases (from $\frac{7}{44}$ to $\frac{1}{20}$) as p increases from 0 to 1.

If Y_p had a beta (α,α) distribution, the value of α giving the correct value for $\mathrm{var}(Y_p)$ would be $2(7-6p)^{-1}$. It is remarkable that O'Connor et al. (1985) suggested this formula for α on the basis of a sampling experiment. This value of α would give the following 'nominal' value for β_2.

(6)
$$\text{'nominal'}\ \beta_2 = 3(11-6p)(25-18p)^{-1}.$$

Table 2 compares the actual values from (4.2) with the 'nominal' values of β_2 from (6), for selected values of p.

TABLE 2 ACTUAL AND 'NOMINAL' VALUES OR β_2

	p	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
Actual	β_2	1.287	1.315	1.348	1.388	1.438	1.500	1.580	1.687	1.831	2.019	2.14
'Nominal'	β_2	1.320	1.345	1.375	1.408	1.449	1.500	1.563	1.645	1.754	1.909	2.14

The reasonably close agreement between actual and 'nominal' value supports the conjecture that beta $(2(7-6p)^{-1}, 2(7-6p)^{-1})$ would be a good approximation to the distribution of Y_p. As must be the case, there is exact agreement between actual and 'nominal' values when $p = \frac{1}{2}$, corresponding to beta$(\frac{1}{2},\frac{1}{2})$, and when $p = 1$, corresponding to beta (2,2). (See Introduction.)

The approximating beta distributions are U–shaped if 'nominal' $\beta_2 < 1.8$. If $\beta_2 = 1.8$ (corresponding to $p = 5/6$) the approximating distribution is uniform. However, for $p = 5/6$, the *actual* β_2 is $17/9 = 1.889$, so $Y_{5/6}$ does *not* have a uniform distribution, as O'Connor et al. (1985) correctly concluded. Heuristically, it can be argued that, when p is small, it is very likely that the shorter subinterval (L_1,U_1) will be chosen at the first stage, and then all subsequent intervals (L_n,U_n) will be close to 0 or 1. Also, since the limiting values (Y_p) tend to be more spread out (closer to 0 or 1) as p decreases, one would expect their variance to increase, as already noted.

3. EXTENSIONS

The method described above can be applied in many circumstances. Here we describe applications to slightly more complicated modifications of the procedure analyzed in Section 2, and also to the problem considered by Kennedy (1988), and described in the Introduction.

First, suppose that in the procedure described in Section 2, the probability of choosing the longer interval alternates between p_1 (for odd–numbered stages, 1st, 3rd, ...) and p_2 (for even–numbered stages, 2nd, 4th,...). To deal with this, we introduce *two* random variables:

$Y_{\underset{\sim}{p}}$, representing the limit point for this procedure

and $Y^*_{\underset{\sim}{p}}$, representing the limit point for the complementary procedure, with probabilities p_2 for odd–numbered, and p_1 for even–numbered stages. The same line of argument as that used in Section 2 leads to the conclusion that the distribution of $\underset{\sim}{Y}_p$ is that of a mixture of distributions of

$$WY^*_{\underset{\sim}{p}},\ W + (1-W)Y^*_{\underset{\sim}{p}},\ (1-W)Y^*_p \text{ and } 1-W+WY^*_p$$

with probabilities $\frac{1}{4}(1-p_1)$, $\frac{1}{4}\,p_1$, $\frac{1}{4}\,p_1$ and $\frac{1}{4}(1-p_1)$ respectively, while for $\underset{\sim}{Y}^*_p$ we have a mixture of

$$WY_{\underset{\sim}{p}},\ W + (1-W)Y_{\underset{\sim}{p}},\ (1-W)Y_{\underset{\sim}{p}} \text{ and } 1-W + WY_{\underset{\sim}{p}}$$

with probabilities $\frac{1}{4}(1-p_2)$, $\frac{1}{4}p_2$, $\frac{1}{4}p_2$ and $\frac{1}{4}(1-p_2)$ respectively.

We then get two sets of equations analogous to (4), namely

$$(7.1)\quad \mu'_r = \tfrac{1}{4}\{p_1\, M_{r,0} + (1-p_1)\, M_{gr}\} + \tfrac{1}{4} \sum_{j=1}^{r-1} \binom{r}{j} \{p_1 M_{r-j,j} + (1-p)M_{j,r-j}\}\, \mu'^*_j + \{(1-p_1)\, M_{r,0} + p_1\, M_{0,r}\}\mu'^*_r$$

and

$$(7.2)\quad \mu'^*_r = \tfrac{1}{4}\{p_2\, M_{r,0} + (1-p_2)\, M_{0,r}\} + \tfrac{1}{4} \sum_{j=1}^{r-1} \binom{r}{j} \{p_2 M_{r-j,j} + (1-p_2)M_{j,r-j}\}\, \mu'_j + \{(1-p_2)\, M_{r,0} + p_2\, M_{0,r}\}\mu'_r$$

where $\mu'_j = E[Y^j_{\underset{\sim}{p}}]$ and $\mu'^*_j = E[Y^{*j}_{\underset{\sim}{p}}]$. The distributions of $Y_{\underset{\sim}{p}}$ and $Y^*_{\underset{\sim}{p}}$ are each symmetrical over (0,1), so formulas (1) apply.

Working though the case r = 2 in detail we obtain

$$(8.1)\quad \mu'_2 = \frac{1}{24}(7-6p_1) + \frac{1}{12} + \frac{1}{12}(1+6p_1)\mu'^*_2 = \frac{3-2p_1}{8} + \frac{1+6p_1}{12}\mu'^*_2$$

and

(8.2) $$\mu_2'^* = \frac{3-2p_2}{8} + \frac{1+6p_2}{12}\mu_2'$$

whence

(9.1) $$\mu_2' = \frac{3}{2}\,\frac{39-6p_1-2p_2-12p_1p_2}{144-(1+6p_1)(1+6p_2)}$$

and

(9.2) $$\mu_2'^* = \frac{3}{2}\,\frac{39-2p_1-6p_2-12p_1p_2}{144-(1+6p_1)(1+6p_2)}\,.$$

The variances are

(10.1) $$\mathrm{var}(Y_{\underset{\sim}{p}}) = \frac{3}{4}\,\frac{91-30p_1-6p_2-36p_1p_2}{144-(1+6p_1)(1+6p_2)}$$

and (10.2) $$\mathrm{var}(Y_p^*) = \frac{3}{4}\,\frac{91-6p_1-30p_2-36p_1p_2}{144-(1+6p_1)(1+6p_2)}\,.$$

Note that

$$\mathrm{var}(Y_{\underset{\sim}{p}}) - \mathrm{var}(Y_{\underset{\sim}{p}}^*) = \frac{6(p_2-p_1)}{144-(1+6p_1)(1+6p_2)}\,,$$

so that

$$\mathrm{var}(Y_{\underset{\sim}{p}}) \gtrless \mathrm{var}(Y_{\underset{\sim}{p}}^*) \text{ according as } p_1 \lessgtr p_2.$$

Values of μ_4' and $\mu_4'^*$ can be obtained by solving the simultaneous equations

(11.1) $$\mu_4' - \frac{1}{80}(1+30p_1)\mu_4'^* = \frac{1}{64}(17-18p_1) + \frac{1}{32}(5+6p_1)\mu_2'^*$$

(11.2) $$\mu_4'^* - \frac{1}{80}(1+30p_2)\mu_4' = \frac{1}{64}(17-18p_2) + \frac{1}{32}(5+6p_2)\mu_2'\,.$$

As a second example we consider extension to a Markov–style sequence, in which the conditional probability of choosing the longer subinterval depends on the choice at the immediately preceding stage. Suppose that the probability of choosing the longer (shorter) subinterval is θ if the longer (shorter) subinterval was chosen at the immediately preceding stage – i.e. the probability of repeating the same type of choice (whether longer or shorter) is θ.

We now introduce two random variables, Y_L and Y_S, representing the position of the limit when we start (with interval (0,1)) *as if* there had been a preceding choice resulting in selection of the longer or shorter subinterval, respectively. If the longer subinterval had been chosen, we would have Y_L, distributed as a mixture of distributions of

$$WY_S,\ W + (1-W)Y_L,\ (1-W)Y_L \text{ and } 1-W+WY_S$$

with probabilities $\frac{1}{2}(1-\theta)$, $\frac{1}{2}\theta, \frac{1}{2}\theta$ and $\frac{1}{2}(1-\theta)$ respectively. The distribution of Y_S is of the same form, with θ replaced by $1-\theta$. Hence

$$(11)\quad \mu_r'(Y_L) = \tfrac{1}{2}(1-\theta)M_{r,0}\,\mu_r'(Y_S) + \tfrac{1}{2}\theta\,M_{0,r}\,\mu_r'(Y_L)$$

$$+ \tfrac{1}{2}\sum_{j=0}^{r} \binom{r}{j}\{\theta\,M_{r-j,j}\,\mu_r'(Y)\,(1-\theta)M_{j,r-j}\mu_r'(Y_S)\}$$

and $\mu_r'(Y_S)$ is equal to a formula similar to the right–hand side of (11), with θ replaced by $1-\theta$.

Rearranging the formulas, we obtain

$$(12.1)\quad (1-\theta M_{0,r})\mu_r'(Y_L) - (1-\theta)M_{r,0}\,\mu_r'(Y_S)$$

$$= \tfrac{1}{2}\{\theta M_{r,0} + (1-\theta)M_{0,r}\} + \tfrac{1}{2}\sum_{j=1}^{r-1}\binom{r}{j}\{\theta M_{r-j,j}\,\mu_j'(Y_L) + (1-\theta)M_{j,r-j}\,\mu_j'(Y_S)\}$$

and

$$(12.2)\quad (1-\theta M_{r,0})\mu_r'(Y_S) - (1-\theta)M_{0,r}\,\mu_r'(Y_L)$$

$$= \tfrac{1}{2}\{(1-\theta)M_{r,0} + \theta M_{0,r}\} + \tfrac{1}{2}\sum_{j=1}^{r-1}\binom{r}{j}\{(1-\theta)M_{r-j,j}\,\mu_j'(Y_L) + \theta M_{j,r-j}\,\mu_j'(Y_S)\}.$$

As in earlier cases, the distributions of Y_L and Y_S are symmetrical over the range 0 to 1, so formulas (1) apply.

Taking $r = 2$ in (12.1) and (12.2) we find

$$(13.1)\qquad (1 - \tfrac{7}{12}\theta)\mu_2'(Y_L) - \tfrac{1}{12}(1-\theta)\mu_2'(Y_S) = \tfrac{1}{8}(3-2\theta)$$

and

(13.2) $$\left(1-\frac{1}{12}\,\theta\right)\mu_2'(Y_S)-\frac{7}{12}\,(1-\theta)\mu_2'(Y_L)=\frac{1}{8}\,(1+2\theta)$$

whence

(14.1) $$\mu_2'(Y_L)=\frac{3}{2}\,\frac{37-26\theta}{137-82\theta}\,;$$

and

(14.2) $$\mu_2'(Y_S)=\frac{3}{2}\,\frac{33-18\theta}{137-82\theta}\,.$$

Also

(15.1) $$\mathrm{var}(Y_L)=\frac{85-74\theta}{4(137-82\theta)}\,;$$

and

(15.2) $$\mathrm{var}(Y_S)=\frac{61-26\theta}{4(137-82\theta)}\,.$$

Note that

$$5\,\mu_2'(Y_L)+11\,\mu_2'\,(Y_S)=6\quad(\text{for all }\theta)$$

and

$$\mathrm{var}(Y_L)\gtrless\mathrm{var}(Y_S)\text{ according as }\theta\lessgtr\tfrac{1}{2}.$$

Values of $\mu_4'(Y_L)$ and $\mu_4'(Y_S)$ can be obtained by solving the simultaneous equations

(16.1) $$(80-31\theta)\mu_4'(Y_L)-(1-\theta)\mu_4'(Y_S)=\tfrac{1}{2}(41-5\theta)+46\theta\mu_2'(Y_L)+50(1-\theta)\mu_2'(Y_S),$$

(16.2) $$(80-\theta)\mu_4'(Y_S)-31(1-\theta)\mu_4'(Y_L)=\tfrac{1}{2}(50\theta-9)+46(1-\theta)\mu_2'(Y_L)+50\theta\mu_2'(Y_S),$$

It is easy to visualize further alternative selection procedures along these lines. Writing L for 'longer' and S for 'shorter', the special case just considered takes

$$\Pr[\text{choose L}\,|\,\text{L chosen previously}]=\Pr[\text{choose S}\,|\,\text{S chosen previously}]=\theta,$$

It would be possible to have two separate values – θ_1 and θ_2, say – for these probabilities. The necessarily analysis could be based on the same two random variables, Y_L and Y_S, introduced above. On the other hand, increases in the *order* of the Markov–type dependence, making probabilties depend on earlier, as well as immediately preceding choices would call for additional random variables to achieve a solution. However, these extensions are straightforward, and can easily be implemented.

4. ANOTHER TYPE OF PROCEDURE

As a final example we deal with a different type of selection procedure – namely, that discovered by Kennedy (1988) and described in the Introduction. We need a few preliminary results.

If $Z_1' \le Z_2' \le \dots \le Z_k'$ are the order statistics corresponding to k independent standard uniform (0,1) random variables, then

$$\text{(17.1)} \qquad E[Z_1'^{a}(1-Z_1')^{b}] = k(k+b-1)!a!/(a+k+b)!$$

$$\text{(17.2)} \qquad E[Z_1'^{a}(Z_k'-Z_1')^{b}] = k(k-1)(k+b-2)!a!/(a+k+b)!$$

$$\text{(17.3)} \qquad E[Z_k'^{c}] = k/(k+c).$$

Denoting the limiting position of the left–hand endpoints of the chosen intervals by X, the distribution of X is a mixture of distributions of

$Z_1' + (1-Z_1')X$, $Z_k' X$ and $Z_1' + (Z_k'-Z_1')X$ with probabilities

p, q and r respectively,

where X and (Z_1', Z_k') are mutually independent.

Denoting $E[X^s]$ by μ_s', we have

$$\text{(18)} \quad \mu_s' = p\, E[\{Z_1' + (1-Z_1')X\}^s] + q\, E[(Z_k'X)^s] + r\, E[\{Z_1' + (Z_k'-Z_1')X'^{s}]$$

$$= q\, E[Z_k'^{s}]\mu_s' + \sum_{j=0}^{s} \binom{s}{j}\{p\, E[Z_1'^{s-j}(1-Z_1')^{j}] + r\, E[Z_1'^{s-j}(Z_k'-Z_1')^{s}]\}\mu_j'$$

$$= \frac{qk}{k+s}\mu_s' + \sum_{j=0}^{s} \binom{s}{j} \left\{pk\frac{(s-j)!\,(k+j-1)!}{(k+s)!} + rk(k-1)\frac{(s-j)!\,(k+j-2)!}{(k+s)!}\mu_j'\right\}$$

$$= \frac{qk}{k+s}\mu_s' + \frac{s!k}{(k+s)!}\sum_{j=0}^{s}\frac{(k+j-2)!}{j!}\{(k-1)(p+r) + jp\}\mu_j'$$

Collecting together terms in μ'_s, we find

$$\left\{1 - \frac{qk}{k+s} - \frac{pk}{k+s} - \frac{r\ k(k-1)}{(k+s)(k+s-1)}\right\} \mu'_s$$

$$= \frac{s!k}{(k+s)!} \sum_{j=0}^{s-1} \frac{(k+j-2)!}{j!} \{(k-1)(p+r)+jp\}\mu'_j$$

i.e.

(18)′ $$\mu'_s = \frac{(s-1)!\,k}{(k+s-2)!\{k(1+r)+s-1\}} \sum_{j=0}^{s-1} \frac{(k+j-2)!}{j!} \{(k-1)(p+r) + jp\}\mu'_j$$

$$= A_S \sum_{j=0}^{s-1} B_j \mu'_j$$

with $A_S = \frac{(s-1)!\,k}{(k+s-2)!\{k(1+r) + s-1\}}$; $B_j = \frac{(k+j-2)!}{j!} \{(k-1)(p+r) + jp\}$

Since $\mu'_{s-1} = A_{s-1} \sum_{j=0}^{s-2} B_j \mu'_j$ we have, from (18)′

(19) $$\mu'_s = A_s B_{s-1} \mu'_{s-i} + A_s \sum_{j=0}^{s-2} B_j \mu'_j$$

$$= \frac{k\{(k-1)(p+r) + (s-1)p\} + \{k(1+r) + s-2\}(s-1)}{(k+s-2)\{k(1+r) + s-1\}} \mu'_{s-1}$$

$$= \frac{k\{(k+s-2)p + (k-1)r\} + \{k+s-2 + kr\}(s-1)}{(k+s-2)\{k(1+r) + s-1\}} \mu'_{s-1}$$

$$= \frac{k(p+r) + s-1}{k(1+r) + s-1} \mu'_{s-1}$$

In particular

$$\mu'_1 = \frac{k(p+r)}{k(1+r)} \mu'_0 = \frac{k(p+r)}{k(1+r)}$$

From (19)

(20) $$\mu'_s = \frac{k(p+r)\{k(p+r) + 1\} \ldots \{k(p+r) + s-1\}}{k(1+r)\{k(1+r) + 1\} \ldots \{k(1+r) + s-1\}}$$

which is the s–th moment of the standard beta (k(p+r), k(q+r)) distribution. (Note that (p+r) + (q+r) = 1+r.) This is, therefore, the distribution of X, since it has finite range and so is determined by its moments.

REFERENCES

Chen, R., Goodman, R. and Zame, A. (1984) On the limiting distribution of two random sequences, *J. Multiv. Anal.*, 14, 221–230.

Chen, R., Lin, E. and Zame, A. (1981) Another arc sine law, *Sankhya, Series A*, 43, 371–373.

Johnson, N.L. and Kotz, S. (1990) Randomly weighted averages: Some aspects and extensions, *Amer. Statistician*, 44, 243–249.

Kennedy, D.P. (1988) A note on stochastic search methods for global optimization, *Adv. Appl. Prob.*, 20, 476–478.

O'Connor, C.A., Hook, A. and O'Connor, T.A. (1985) Beta distributions as limiting distributions, *Proceedings, ASA Statistical Computing Section*, 292–295.

Van Assche, W. (1986) Products of 2×2 stochastic matrices with random entries, *J. Appl. Probab.*, 23, 1019–1024.

Van Assche, W. (1987) A random variable uniformly distributed between two random variables, *Sankhya, Series A*, 49, 207–211.