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USE OF MOMENTS IN DERIVING DISTRIBUTIONS, AND SOME CHARACTERIZATIONS

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ABSTRACT

The utility of moment calculations as a simple way of establishing distributional forms is illustrated with several examples, based on some recent problems. Characterization theorems for beta distributions are obtained.

KEY WORDS: beta distribution; limit distributions; recurrence relations

1. Introduction

Recently (Johnson and Kotz (1989)) we have shown how moment methods can provide a way of obtaining distributions and characterizations, without recourse to characteristic functions, or solution of integral equations. In all cases, more sophisticated methods had been used to obtain the relevant results. In this note we briefly recapitulate the methods we have used, and present further applications discussed by Assche (1986), Kennedy (1988) and Chen et al. (1984). In the first of these three applications a characterization of the beta distribution is obtained.

2. Methodology

(I) It is required to find the distribution of

$$Y = \sum_{j=0}^{\infty} (-1)^j \prod_{i=0}^{j} X_i \tag{1}$$

where the X's are mutually independent and have a common beta (a,b) distribution (density function

$$f_X(x) = \{B(a,b)\}^{-1} x^{a-1} (1-x)^{b-1} \qquad (0 < x < 1;\ a,\ b > 0),$$

where $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ is a complete beta function).

The solution presented here is alluded to, briefly, in "Solution..." (1988) (page 564), but no details are given there. Since

$$\Pr[1 > X_0 > X_0X_1 > \ldots > 0] = 1$$

the series in (1) converges almost surely, and $0 \leq Y \leq 1$.

Now

$$\begin{aligned} Y &= X_0 - X_0X_1 + X_0X_1X_2 - \dots \\ &= X_0 - X_0(X_1 - X_1X_2 + \dots) \\ &= X_0(1-Y^*) \end{aligned} \qquad (2)$$

where X_0 and Y^* are mutually independent and <u>Y^* has the same distribution as Y</u>. Using $\mu_s'(\cdot)$ to denote s-th moment, it follows from (2) that

$$E[Y^s] = \mu_s'(Y) = \mu_s'(X_0)\ \mu_s'(1-Y). \qquad (3)$$

Since X_0 has a beta (a,b) distribution

$$\mu_s'(X_0) = \frac{\Gamma(a+s)\Gamma(a+b)}{\Gamma(a)\Gamma(a+b+s)} . \qquad (4)$$

It is natural to inquire whether Y has a beta (c,d) distribution, for appropriate values of c and d. If Y were to have a beta (c,d) distribution, (1-Y) would have a beta (d,c) distribution, and (3) would require

$$\frac{\Gamma(a+s)\Gamma(c+d)}{\Gamma(a)\Gamma(a+d+s)} = \frac{\Gamma(a+s)\Gamma(a+b)}{\Gamma(a)\Gamma(a+b+s)} \cdot \frac{\Gamma(d+s)\Gamma(c+d)}{\Gamma(d)\Gamma(c+d+s)} ;$$

that is,

$$\frac{\Gamma(c+s)\Gamma(d)}{\Gamma(c)\Gamma(d+s)} = \frac{\Gamma(a+s)\Gamma(a+b)}{\Gamma(a)\Gamma(a+b+s)} \quad \text{for all } s. \qquad (5)$$

This is satisfied by c=a, d=a+b. Furthermore, since

(i) using (3) with s=1,2,... sequentially, the $\{\mu_s'(Y)\}$ can be determined from the $\{\mu_s'(X_0)\}$s, and

(ii) the moments of Y determine its distribution, because its support is finite [0,1],

it follows that the distribution of Y must be beta (a,a+b).

The moments of the common distribution of the X's can be determined from those of Y, using (3) with s=1,2,..., so the converse result holds, that if Y has a beta (a,a+b) distribution, the common distribution of the X's must be beta (a,b). These results can be summarized in

<u>Characterization Theorem 1</u>

If $X_0, X_1, \ldots$ are i.i.d. variables, then the distribution of

$$Y = \sum_{j=0}^{\infty} (-1)^j \prod_{i=0}^{j} X_i$$

is beta (a,a+b) if and only if the common distribution of the X's is beta (a,b).

(II) Assche (1986) has considered the distribution of a random variable Z^* 'uniformly distributed between two independent variables X_1 and X_2' defined by

$$\Pr[Z^* \leq z | X_1 = x_1, X_2 = x_2] = \begin{cases} 1 & z > \max(x_1, x_2) \\ 0 & \text{if } z < \min(x_1, x_2) \\ \dfrac{z-x_1}{x_2-x_1} & \text{if } x_1 \leq z \leq x_2 \\ \dfrac{z-x_2}{x_1-x_2} & \text{if } x_2 \leq z \leq x_1 \end{cases}$$

For the special case when X_1 and X_2 have identical distributions, Johnson and Kotz (1989) have shown that the distribution of Z^* is the same as that of

$$Z = W X_1 + (1-W)X_2 \tag{7}$$

where W has a uniform [0,1] distribution and is independent of X_1 and X_2. Hence with $\mu'_s(X) = \mu'_s(X_1) = \mu'_s(X_2)$,

$$\mu'_s(Z^*) = \mu'_s(Z) = \sum_{j=0}^{s} \binom{s}{j} \mu'_j(X)\mu'_{s-j}(X)\ E[W^j L(1-W)^{s-j}]$$

$$= \sum_{j=0}^{s} \binom{s}{j} \mu'_j(X)\mu'_{s-j}(X)\ B(j+1,\ s-j+1)$$

$$= (s+1)^{-1} \sum_{j=0}^{s} \mu'_j(X)\mu'_{s-j}(X) \qquad (s=1,2,\dots) \tag{8}$$

Using (8) for $s=1,2,\dots$, the moments $\{\mu'_s(Z^*)\}$ can be determined from the moments $\{\mu'_s(X)\}$ and conversely. If the moments determine the distributions, then the common distribution of the X's is characterized by that of Z^*, and conversely. In particular, noting from (7), that if the common distribution of the X's has finite support so does that of Z^*, we have

Characterization Theorem 2

If X_1 and X_2 are i.i.d. variables with finite support, then their common distribution is characterized by that of Z^*, as defined in (6), and conversely. (See Johnson and Kotz (1989) for some special cases.)

3. New Applications

(I) In Assche (1987), the following problem is posed. Consider the sequence of stochastic matrices

$$\left\{\begin{bmatrix} X_n & 1-X_n \\ Y_n & 1-Y_n \end{bmatrix}\right\} \quad (n = 1,2,\dots)$$

formed by the recurrence formula

$$\begin{bmatrix} X_n & 1-X_n \\ Y_n & 1-Y_n \end{bmatrix} = \begin{bmatrix} \alpha_n & 1-\alpha_n \\ \beta_n & 1-\beta_n \end{bmatrix} \begin{bmatrix} X_{n-1} & 1-X_{n-1} \\ Y_{n-1} & 1-Y_{n-1} \end{bmatrix} \tag{9}$$

with $X_1 = \alpha_1$, $Y_1 = \beta_1$.

The α's and β's are two sets of i.i.d. variables with support [0,1]. Assche shows (see below) that $Y = \lim_{n\to\infty} Y_n$ and $X = \lim_{n\to\infty} X_n$ exist and are equal. What is the distribution of Y (or X)?

From (9)

$$X_n = \alpha_n X_{n-1} + (1-\alpha_n)Y_{n-1} \tag{10.1}$$

and
$$Y_n = \beta_n X_{n-1} + (1-\beta_n)Y_{n-1}, \tag{10.2}$$

so that

$$X_n - Y_n = (\alpha_n - \beta_n)(X_{n-1} - Y_{n-1}) \tag{11.1}$$

and
$$X_n + Y_n = (\alpha_n + \beta_n)(X_{n-1} - Y_{n-1}) + 2Y_{n-1}. \tag{11.2}$$

From (11.1), remembering that $X_1 = \alpha_1$, $Y_1 = \beta_1$,

$$X_n - Y_n = \prod_{j=1}^{n} (\alpha_j - \beta_j) \tag{12}$$

and from (11.1) and (11.2)

$$Y_n = \beta_n(X_{n-1} - Y_{n-1}) + Y_{n-1} = \beta_n \prod_{j=1}^{n-1} \gamma_j + Y_{n-1} \tag{13}$$

with $\gamma_j = \alpha_j - \beta_j$.

Provided

$$\max(\Pr[\alpha_i < 1], \Pr[\beta_i < 1]) > 0 \qquad \text{for all } i,$$

then (using (12))

$$Y = \lim_{n\to\infty} Y_n = \sum_{j=1}^{\infty} \beta_j \prod_{i=1}^{j-1} \gamma_i \qquad \left(\text{with } \prod_{i=1}^{0} \text{ interpreted as } 1\right)$$

almost surely exists and is equal to $X = \lim_{n\to\infty} X_n$ (cf Assche (1987)).

Now

$$\begin{aligned} Y &= \beta_1 + \beta_2 \gamma_1 + \beta_3 \gamma_1\gamma_2 + \cdots \\ &= \beta_1 + \gamma_1(\beta_2 + \beta_3\gamma_2 + \beta_4\gamma_2\gamma_4 + \ldots) \\ &= \beta_1 + \gamma_1 Y^* \\ &= \alpha_1 Y^* + \beta_1(1-Y^*) \end{aligned} \tag{14}$$

where α_1, β_1 and Y^* are mutually independent, and <u>Y^* has the same distribution as Y</u>. Hence

$$\mu'_s(Y) = \sum_{j=0}^{s} \binom{s}{j} \mu'_j(\alpha)\mu'_{s-j}(\beta)\ E[Y^j(1-Y)^{s-j}] \qquad (s=1,2,\ldots), \qquad (15)$$

where $\mu'_s(\alpha)$, $\mu'_s(\beta)$ are the s-th moments about zero of the distributions of the α's and β's, respectively.

Equation (15) expresses $\mu'_s(Y)$ as a linear function of $\mu'_{s-1}(Y)$, $\mu'_{s-2}(Y),\ldots,\mu'_1(Y)$. From this equation with s=1,2,... the values of all moments of Y can be expressed in terms of the moments of the distributions of α and β. For example, with s=1, (15) gives

$$\mu'_1(Y) = \mu'_1(\beta)\ \{1-\mu'_1(Y)\} + \mu'_1(\alpha)\mu'_1(Y)$$

whence

$$\mu'_1(Y) = \frac{\mu'_1(\alpha) + \mu'_1(\beta)}{1-\mu'_1(\alpha) + \mu'_1(\beta)} .$$

This value is then used in (15) with s=2, from which a formula for $\mu'_2(Y)$ is obtained, and so on.

Assche (1987) considers the case when the α's and β's have a common beta (a,a) distribution, so that

$$\mu'_s(\alpha) = \mu'_s(\beta) = \frac{\Gamma(a+s)\ \Gamma(2a)}{\Gamma(a)\ \Gamma(2a+s)} .$$

Then (15) becomes

$$\mu'_s(Y) = \left\{\frac{\Gamma(2a)}{\Gamma(a)}\right\}^2 \sum_{j=0}^{s} \binom{s}{j} \frac{\Gamma(a+j)\ \Gamma(a+s-j)}{\Gamma(2a+j)\ \Gamma(2a+s-j)} E[Y^j(1-Y)^{s-j}]. \qquad (16)$$

We now show that (16) is satisfied if Y has a beta (2a,2a) distribution. The left-hand side of (16) would then be

$$\frac{\Gamma(2a+s)\ \Gamma(4a)}{\Gamma(2a)\ \Gamma(4a+s)} . \qquad (17)$$

Since we would have

$$E[Y^j(1-Y)^{s-j}] = \frac{\Gamma(2a+j)\ \Gamma(2a+s-j)}{\Gamma(4a+s)}\ \frac{\Gamma(4a)}{\{\Gamma(2a)\}^2},$$

the right hand side of (16) would be equal to

$$\frac{\Gamma(4a)}{\{\Gamma(a)\}^2}\ \frac{1}{\Gamma(4a+s)}\ \sum_{j=0}^{s} \binom{s}{j}\ \Gamma(a+j)\ \Gamma(a+s-j). \tag{18}$$

Now,

$$\begin{aligned}\sum_{j=0}^{s} \binom{s}{j}\ \Gamma(a+j)\ \Gamma(a+s-j) &= \sum_{j=0}^{s} \binom{s}{j}\ \Gamma(2a+s)\ B(a+j,\ a+s-j) \\ &= \Gamma(2a+s) \sum_{j=0}^{s} \binom{s}{j} \int_0^1 t^{a+j-1}(1-t)^{a+s-j-1}dt \\ &= \Gamma(2a+s) \int_0^1 t^{a-1}(1-t)^{a-1} \{\sum_{j=0}^{s} \binom{s}{j}\ t^s(1-t)^{s-j}\}dt \\ &= \Gamma(2a+s)\ B(a,a) = \Gamma(2a+s)\ \{\Gamma(a)\}^2/\Gamma(2a).\end{aligned} \tag{19}$$

From (18) and (19) we see that the right-hand side of (16) agrees with the value (17) for the left-hand side. Thus (16) is satisfied if Y has a beta (2a,2a) distribution. The distribution of Y must be beta (2a,2a) because the equations (16) have only one solution.

It may appear that this derivation requires prior "knowledge of the answer." However, one could be naturally led to examine the possibility of a beta (2a,2a) distribution for Y by determining the first three or four moments from (16) with s=1,2,3,(4). (Formula (16) takes quite a simple form when $\mu'_s(\alpha) = \mu'_s(\beta)$ for all α and β.)

We note that if $\mu'_s(\alpha) = \mu'_s(\beta) = \mu'_s$, say, for all s, then the values of $\{\mu'_s\}$ can be determined (using (15)) from the values of $\{\mu'_s(Y)\}$, if these are known.

Our results can be summarized in:

Characterization Theorem 3

If $\{\alpha_n, \beta_n\}$ (n=1,2,...) are i.i.d. random variables

$$\begin{bmatrix} X_n & 1-X_n \\ Y_n & 1-Y_n \end{bmatrix} = \begin{bmatrix} \alpha_n & 1-\alpha_n \\ \beta_n & 1-\beta_n \end{bmatrix} \begin{bmatrix} X_{n-1} & 1-X_{n-1} \\ Y_{n-1} & 1-Y_{n-1} \end{bmatrix} \quad (n=2,3,\dots)$$

with $X_1 = \alpha_1$, $Y_1 = \beta_1$, then the common limiting distribution, as $n \to \infty$, of X_n and Y_n is beta (2a,2a) if and only if the common distribution of the α's and β's is beta (a,a).

(II) Kennedy (1988) has attacked the following problem, in connection with a stochastic search model for global optimization:

"Let $[A_n, B_n]$ be random subintervals of [0,1], defined recursively as follows. Let $A_1 = 0$, $B_1 = 1$ and take C_n, D_n to be the minimum and maximum of k independent random points, each uniformly distributed on $[A_n, B_n]$. Choose $[A_{n+1}, B_{n+1}]$ to be $[C_n, B_n]$, $[A_n, D_n]$ or $[C_n, D_n]$ with probabilities p,q,r respectively. (p+q+r=1)."

Figure 1 may help to clarify the procedure.

Since

$$E[B_{n+1} - A_{n+1} | A_n, B_n] = \{(p+q)\frac{k}{k+1} + r\frac{k-1}{k+1}\}(B_n - A_n) < \frac{k}{k+1}(B_n - A_n)$$

we have

$$E[B_{n+1} - A_{n+1}] < \left(\frac{k}{k+1}\right)^n (B_1 - A_1) = \left(\frac{k}{k+1}\right)^n$$

and $\lim_{n\to\infty} E[B_n - A_n] = 0$.

Thus as $n \to \infty$ the interval $[A_n, B_n]$ converges to a point, Z, say. What is the distribution of Z?

Let $U_1 \leq U_2 \leq \dots \leq U_k$ be order statistics from a random sample of size k from a standard [0,1] uniform distribution. Consider the left-hand end-points of the intervals $[A_n, B_n]$ (which have the same limiting distribution as the right-hand end points). Then Z has the same distribution as a mixture of

$U_1 + Z^*(1-U_1)$	with proportion	p	$((A_2, B_2) \equiv (C_1, B_1))$;
$Z^* U_k$	" "	q	$((A_2, B_2)) \equiv (A_1, D_1))$;
and $U_1 + Z^*(U_k - U_1)$	" "	r	$((A_2, B_2) \equiv (C_2, D_2))$;

where <u>Z^* has the same distribution as Z</u>, and Z^* and $(U_1, \dots, U_k)$ are mutually independent.

Hence

$$\mu_s'(Z) = p\, E[\{U_1 + Z(1-U_1)\}^s] + q\, E[(Z\, U_k)^s] + r\, E[\{U_1 + Z(U_k - U_1)\}^s]$$

$$= \sum_{j=0}^{s} \binom{s}{j} \mu_j'(Z)\{p\, E[U_1^{s-j}(1-U_1)^j] + r\, E[U_1^{s-j}(U_k - U_1)^j]\} + q\, \mu_s'(Z)\, E[U_k^s]. \tag{20}$$

Now, from the distribution of the order statistics $(U_1 \dots, U_k)$ (see, e.g., David (1981), Chapter 3)

$$E[U_k^s] = \frac{k}{k+s} ;$$

$$E[U_1^g(1-U_k)^h] = k\, \frac{g!(k+h-1)!}{(k+g+h)!} ;$$

$$E[U_1^g(U_k-U_1)^h] = k(k-1)\,\frac{g!(k+h-2)!}{(k+g+h)!}\,.$$

Inserting these values in (20), and collecting terms in $\mu_s'(Z)$ on the left-hand side, we obtain

$$[1 - \frac{qk}{k+s} - \frac{pk}{k+s} - \frac{rk(k-1)}{(k+s)(k+s-1)}]\mu_s'(Z) = \sum_{j=0}^{s-1} \binom{s}{j} \frac{(s-j)!(k+j-2)!k}{(k+s)!}$$

$$\times \{(k+j-1)p+(k-1)r\}\mu_j'(Z),$$

i.e.

$$\mu_s'(Z) = \frac{(s-1)!k}{(k+s-2)!\{k(1+r)+s-1\}} \sum_{j=0}^{s-1} \frac{(k+j-2)!}{j!} \{(p+r)(k-1)+pj\}\mu_j'(Z) \quad (21)$$

Equation (21) can be written as

$$\mu_s'(Z) = A_s \sum_{j=0}^{s-1} T_j \quad (21)'$$

with $A_s = \dfrac{(s-1)!k}{(k+s-2)!\{k(1+r)+s-1\}}$; $T_j = \dfrac{(k+j-2)!\{(p+r)(k-1)+pj\}}{j!}\,\mu_j'(Z)$.

Since, also,

$$\mu_{s-1}'(Z) = A_{s-1} \sum_{j=0}^{s-2} T_j$$

we have

$$\mu_s'(Z) = (\frac{A_s}{A_{s-1}})\mu_{s-1}'(Z) + A_s\, T_{s-1}$$

$$= [\frac{A_s}{A_{s-1}} + A_s\, \frac{(k+j-2)!\{(p+r)(k-1)+pj\}}{j!}]\mu_{s-1}'(Z)$$

Inserting the values of A_s and A_{s-1}, the term in square brackets becomes

$$\frac{(s-1)\{k(1+r)+s-2\}}{(k+s-2)\{k(1+r)+s-1\}} + \frac{k\{(p+r)(k-1)+p(s-1)\}}{(k+s-2)\{k(1+r)+s-1\}}$$

$$= \frac{1}{(k+s-2)\{k(1+r)+s-1\}}\,\{(k+s-2)(s-1)+kr(s-1)+kp(s-1)+k(p+r)(k-1)\}$$

$$= \frac{k(p+r)+s-1}{k(1+r)+s-1}\ .$$

So

$$\mu_s'(Z) = \frac{k(p+r)+s-1}{k(1+r)+s-1}\,\mu_{s-1}'(Z). \tag{22}$$

Since $\mu_0'(Z) = 1$,

$$\mu_s'(Z) = \prod_{j=1}^{s} \left\{\frac{k(p+r)+j-1}{k(1+r)+j-1}\right\}\ ,$$

which is the s-th moment about zero of a beta $(k(p+r),\ k(1+r))$ distribution. This is, therefore, the distribution of Z, since the support of Z is finite.

We note that if k=1 and p=q=½ (r=0), then we have a procedure in which a random point W_n is uniformly distributed over (A_n,B_n) and (A_{n+1},B_{n+1}) is chosen by taking (A_n,W_n) and (W_n,B_n) as equally likely. The limiting distribution is then beta (½,½) - the 'arc-sine' distribution (see Chen et al. (1981).

(III) Chen et al. (1984) have considered a more general problem in which the longer of the intervals (A_n,W_n) and (W_n,B_n) is chosen to be (A_{n+1},B_{n+1}) with probability ω and the shorter with probability $(1-\omega)$. It is easy to show that the intervals (A_n,B_n) converge to a point Y_ω. What is the distribution of Y_ω?

Clearly $Y_{½}$ has the same distribution as Z in the last paragraph of (II) of this section. Here we outline a method of determining the moments of Y_ω for general ω. After W_1 has been obtained we have the following situations:

$$\text{If}\quad W_1 \le ½:\quad (A_2,B_2) \equiv \begin{cases} (W_1,1) & \text{with probability } \omega \\ (0,W_1) & \text{with probability } (1-\omega); \end{cases}$$

$$\text{If}\quad W_1 \ge ½:\quad (A_2,B_2) \equiv \begin{cases} (W_1,1) & \text{with probability } (1-\omega) \\ (0,W_1) & \text{with probability } \omega. \end{cases}\ .$$

Noting that the conditional distribution of W_1, given $W_1 \leq \frac{1}{2}$ is the distribution of $\frac{1}{2}U$, where U is a uniform [0,1] random variable; and the conditional distribution of W_1, given $W_1 \geq \frac{1}{2}$, is that of $\frac{1}{2}(1+U)$, we see that the distribution of Y_ω is that of a mixture of

$$\tfrac{1}{2} U + Y_\omega^*(1-\tfrac{1}{2}U) \qquad \text{with proportion } \tfrac{1}{2}\omega;$$

$$\tfrac{1}{2} Y_\omega^* U \qquad \text{with proportion } \tfrac{1}{2}(1-\omega);$$

$$\tfrac{1}{2}(1+U) + \tfrac{1}{2} Y_\omega^*(1-U) \qquad \text{with proportion } \tfrac{1}{2}(1-\omega);$$

and

$$\tfrac{1}{2} Y_\omega^*(1+U) \qquad \text{with proportion } \tfrac{1}{2}\omega;$$

where U and Y_ω^* are mutually independent and <u>Y_ω^* has the same distribution as Y_ω</u>.

Hence
$$\mu_s'(Y_\omega) = \tfrac{1}{2}\,\omega\{E[\{\tfrac{1}{2}U+Y_\omega(1-\tfrac{1}{2}U)\}^s] + E\{\tfrac{1}{2}Y_\omega(1+U)\}^s\}$$

$$+ \tfrac{1}{2}(1-\omega)\{E[\tfrac{1}{2}Y_\omega U)^s] + E[\{\tfrac{1}{2}(1+U) + \tfrac{1}{2}Y_\omega(1-U)\}^s]\} \qquad (23)$$

Calculation of the first and last terms is facilitated by noting that Y_ω has a distribution on [0,1] which is symmetrical about $\frac{1}{2}$, so $(1-Y_\omega)$ and Y_ω have the same distribution. Thus

$$E[\{\tfrac{1}{2}U + Y_\omega(1-\tfrac{1}{2}U)\}^s] = E[\{1-Y_\omega(1-\tfrac{1}{2}U)\}^s]$$

$$= \sum_{j=0}^{s} \binom{s}{j}(-1)^j\, \mu_j'(Y_\omega)E[(1-\tfrac{1}{2}U)_j]$$

$$= \sum_{j=0}^{s} \binom{s}{j}(-1)^j\, \frac{2^{j+1}-1}{2^j(j+1)}\, \mu_j'(Y_\omega).$$

Similarly

$$E[\{½(1+U) + ½\, Y_\omega(1-U)\}^s] = E[\{1-½\, Y_\omega\; 1-U)\}^s]$$

$$= \sum_{j=0}^{s} \binom{s}{j}(-1)^j\, \mu_j'(Y_\omega)E[\frac{1}{2^j}\, j(1-U)^j]$$

$$= \sum_{j=0}^{s} \binom{s}{j}(-1)^j\, \frac{1}{2^j(j+1)}\, \mu_j'(Y_\omega).$$

Inserting in (23) we obtain

$$\mu_s'(Y_\omega) = \sum_{j=0}^{s} \binom{s}{j}(-1)^j\, \frac{1}{2^{j+1}(j+1)}\, \{\omega(2^{j+1}-1) + (1-\omega)\}\mu_j'(Y_\omega)$$

$$+ \frac{1}{2^{s+1}(s+1)}\, \{\omega(2^{s+1}-1) + (1-\omega)\}\mu_s'(Y_\omega). \qquad (24)$$

Note that for s odd, the right hand side of (24) does not contain a term in $\mu_s'(Y_\omega)$. Taking s=1 we have

$$\mu_1'(Y_\omega) = ½(\omega+1-\omega)\mu_0'(Y_\omega) = ½,$$

(as expected, because Y_ω has a distribution symmetrical about ½). Taking s=2,

$$\mu_2'(Y_\omega)\ ½,\ [1-2 \cdot \frac{1}{2^3{\times}3}\,(7\omega+1-\omega)] = ½ - 2 \cdot \frac{1}{2^2{\times}2}\,(3\omega+1-\omega)½$$

whence

$$\mu_2'(Y_\omega) = \frac{3(3-2\omega)}{2(11-6\omega)}\ .$$

The third moment could be derived from s=3, or using symmetry, since the central moment $\mu_3(Y_\omega)$ is zero, wee have

$$\mu_3(Y_\omega) = \mu_3'(Y_\omega) - 3\mu_2'(Y_\omega)\mu_1'(Y_\omega. + 2\{\mu_1'(Y_\omega)\}^3 = 0,$$

whence

$$\mu_3'(Y_\omega) = 3\ \frac{3}{4}\, \frac{3-2\omega}{11-6\omega} - \frac{2}{8} = \frac{4-3\omega}{11-6\omega}\ .$$

4. Concluding Remarks

The foregoing methods depend on the relevant distributions being determined by their moments. In all the cases we considered, this was ensured, because the distributions had finite support. The condition may also be true even when the support is unbounded, so that the method can be applied in a wider field, subject to checking uniqueness.

5. Acknowledgement

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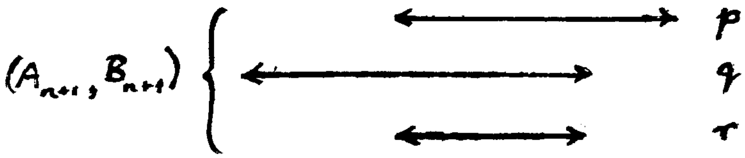
Figure 1



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