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NON-EXISTENCE OF CERTAIN POLYNOMIAL REGRESSIONS IN RANDOM SUM THEORY

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Abstract

It is shown that the regression of a counting variable (X) on the value of a random sum of X independent and identically distributed, non-negative integer-valued random variables cannot, in general, be a polynomial of degree greater than 1.

[Key Words: Bayes' Theorem, Counting Variables, Polynomial Regression,
Random Sums]

Kyriakoussis (1988) considered recently the following model. Let X , Z_1, Z_2, \dots be independent random variables taking nonnegative integer values, and

$$Y = \begin{cases} Z_1 + Z_2 + \dots + Z_X & \text{if } X > 0 \\ 0 & \text{if } X = 0 \end{cases}$$

The Z 's are identically distributed, with $\Pr[Z_j=0] = p_0 \neq 1$, and the regression of X and Y is a polynomial of degree r ,

$$E[X|Y=y] = \sum_{i=0}^r \beta_i y^i, \quad \text{with } \beta_r \neq 0. \quad (1)$$

(Kyriakoussis shows that if $E[Y^{r-1}]$ is finite, then the joint distribution of (X, Y) is uniquely determined by the common distribution of the Z 's, the regression function (1), and the first $(r-1)$ moments of Y .)

The main purpose of this paper is to show that, under the stated conditions, the value of r in (1) cannot exceed 1 - that is, polynomial regression, other than linear, is impossible - unless Y is bounded. In what follows we shall assume that Y is unbounded.

We first note that we must have $\beta_r > 0$ because if β_r is negative, then $E[X|Y=y]$ would be negative for sufficiently large y . Also, the condition that Y is unbounded implies that X is unbounded, too. This is because, if $\Pr[X < \xi] = 1$ for some finite ξ ,

$$E[X|Y=y] < \xi$$

and (1) could not be satisfied for sufficiently large y . We also note that if

$p_0=0$, (so $Z_j \geq 1$, a.s.) then $X \leq Y$ a.s. and so

$$E[X|Y=y] \leq y$$

which again implies that (1) could not be satisfied for sufficiently large y .

So without loss of generality, we take $0 < p_0 < 1$.

Now write

$$X = X_+ + X_0$$

where

X_+ = number of Z 's which are nonzero

and

X_0 = number of Z 's which are zero.

Since the minimum possible value for each of the X_+ Z 's is 1, it follows (as above) that

$$X_+ \leq y.$$

Using Bayes' theorem

$$\begin{aligned} \Pr[X=x|X_+=x_+] &= \frac{P_x \binom{x}{x_+} p_0^{x-x_+} (1-p_0)^{x_+}}{\sum_{w=x_+}^{\infty} P_w \binom{w}{x_+} p_0^{w-x_+} (1-p_0)^{x_+}} \\ &= \frac{P_x \binom{x}{x_+} p_0^{x-x_+}}{\sum_{w=0}^{\infty} P_w \binom{x_+}{w} p_0^w} \end{aligned}$$

where $P_x = \Pr[X=x]$ and $a^{(b)} = a(a-1)\dots(a-b+1)$, as usual. (Note that $w^{(x_+)} = 0$ for $w < x_+$.)

Hence

$$E[X|X_+=x_+] = \frac{\sum_{x=0}^{\infty} x P_x \binom{x}{x_+} p_0^{x-x_+}}{\sum_{x=0}^{\infty} P_x \binom{x}{x_+} p_0^{x-x_+}} = g(x_+), \text{ say.}$$

We now show that (as is to be expected) $g(x_+)$ is a nondecreasing function of x_+ . Using the relation $x^{(x_++1)} = (x-x_+)x^{(x_+)}$ we have

$$g(x_++1) - g(x_+) = \frac{\sum_{x=0}^{\infty} x(x-x_+)\alpha_x}{\sum_{x=0}^{\infty} (x-x_+)\alpha_x} - \frac{\sum_{x=0}^{\infty} x \alpha_x}{\sum_{x=0}^{\infty} \alpha_x}$$

where $\alpha_x = P_x x^{(x_+)} p_0^x$.

Continuing,

$$g(x_++1) - g(x_+) = \frac{\left[\sum_{x=0}^{\infty} \alpha_x \right] \left\{ \sum_{x=0}^{\infty} x^2 \alpha_x - \left[\sum_{x=0}^{\infty} x \alpha_x \right]^2 \right\}}{\left\{ \sum_{x=0}^{\infty} (x-x_+)\alpha_x \right\} \left[\sum_{x=0}^{\infty} \alpha_x \right]}$$

$$= \frac{\sum_{x=0}^{\infty} \alpha_x (x-\bar{x})^2}{\sum_{x=0}^{\infty} (x-x_+)\alpha_x} \geq 0$$

where $\bar{x} = \left[\sum_{x=0}^{\infty} x \alpha_x \right] / \left[\sum_{x=0}^{\infty} \alpha_x \right]$. (Note that $\alpha_x = 0$ for $x < x_+$.)

So $g(x_+)$ is a nondecreasing function of the integer-valued variable x_+ .

Since $E[X|X = x_+] = g(x_+)$ and the maximum possible value of X_+ , given $Y=y$, is y , we have

$$E[X|Y=y] \leq g(y) = \frac{\sum_{x=0}^{\infty} x P_x x^{(y)} p_0^x}{\sum_{x=0}^{\infty} P_x x^{(y)} p_0^x}.$$

Since

$$x x^{(y)} = (y + \overline{x-y})x^{(y)} = y x^{(y)} + x^{(y+1)},$$

$$g(y) = y + \frac{\sum_{x=0}^{\infty} P_x x^{(y+1)} p_0^x}{\sum_{x=0}^{\infty} P_x x^{(y)} p_0^x} .$$

So

$$E[X|Y=y] \leq y + H(y) \quad (2)$$

where

$$\begin{aligned} H(y) &= \frac{\sum_{x=0}^{\infty} P_x x^{(y+1)} p_0^x}{\sum_{x=0}^{\infty} P_x x^{(y)} p_0^x} \\ &= \frac{D_t^{y+1} h(t) |_{t=1}}{D_t^y h(t) |_{t=1}} \end{aligned}$$

with $D_t \equiv \frac{d}{dt}$ and $h(t) = \sum_{x=0}^{\infty} P_x (p_0 t)^x$.

Note that $h(t)$ and $D_t^s h(t)$ converge for $0 \leq p_0 t < 1$ (i.e. $0 \leq t < p_0^{-1}$) because $\sum_{x=0}^{\infty} P_x (=1)$ converges, and $x^{(s)} (p_0 t)^x \rightarrow 0$ as $x \rightarrow \infty$.

Hence the Taylor expansion of $h(t)$ about $t = 1$

$$h(t) = \sum_{y=0}^{\infty} \frac{(t-1)^y}{y!} D_t^y h(t) |_{t=1}$$

converges for $1 \leq t < p_0^{-1}$. The ratio of the $(y+1)$ -term to the y -term in the series is

$$\frac{t-1}{y+1} H(y) .$$

By Cauchy convergence criterion the series would not converge if $H(y)/(y+1)$ tends to a limit exceeding 1 as $y \rightarrow \infty$. This would certainly be the case of $H(y)$ were of order y^2 - in fact the ratio would tend to infinity as $y \rightarrow \infty$. The same argument shows that we cannot have $H(y) = O(y^{1+\epsilon})$ for any fixed $\epsilon > 0$.

while if (1) were valid we would have $E[X|Y=y] = O(y^r)$.

So (1) can only be true for $r=1$, (unless Y is bounded). From (2) we see that $E[X|Y=y] \leq o(y^{1+\epsilon})$.

Remarks: 1) Similar arguments can be applied when the common distribution of the Z 's is not restricted to the nonnegative integers, but only satisfies the conditions

(i) Z is nonnegative.

(ii) Z can take the value zero, but no other values in an interval $[0, \zeta)$ from some $\zeta > 0$.

In this case we have $X_+ \leq [y/\zeta] + 1$, where $[]$ denotes "integer part of"; this value replaces y in the subsequent analysis.

Moreover, the distribution of Z may be continuous for $Z > \zeta$.

Also, the Z_i 's need not even have a common distribution, provided each satisfies (i) and (ii) above, the ζ 's have a lower bound greater than zero, and there is a common value (p_0) for $\Pr[Z_i=0]$. Subject to these limitations, also, the distribution of each Z_i may depend on the value of X (though the Z_i 's must still be independent of one another).

2) It is, of course, possible for the regression function $E[X|Y=y]$ to be curvilinear. (Xekalaki (1980) gives examples.) Indeed, linearity of regression is a basis for a number of characterizations, just because it is not in general a feature of the joint distribution of X and Y . The seminal paper of Cacoullos and Papageorgiu (1983) contains much useful information on these matters. (It is straightforward, though tedious, to calculate the value of $H(y)$ for the examples in that paper, which involve binomial, Pascal and Poisson distributions.)

However, as we have shown, curvilinearity of the regression cannot be expressed as a polynomial function unless, possibly, if Y is bounded.

3) If the Z 's have a common Bernoulli distribution (so that Y , given $X=x$, has a binomial distribution with parameters $(x, 1-p_0)$) then the only nonzero value of Z_i is 1, and $X_+ = y$ if $Y=y$. In this case, inequality (2) becomes an equality, from which the regression function $E[X|Y=y]$ can be found. (This result does not depend on assumption that X is unbounded.)

For example, if X has a binomial (n, p) distribution, straightforward calculation shows that $H(y) = \frac{n-y}{1-p+pp_0}$, whence

$$E[X|y] = \frac{(1-p)y + npp_0}{1-p + pp_0}$$

a result which is implicit in Cacoullos and Papageorgiu (1983).

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