THE ASYMPOTOTIC DISTRIBUTION OF EXTREME SUMS

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Let $X_{1,n} \leq \cdots \leq X_{n,n}$ be the order statistics of $n$ independent random variables with a common distribution function $F$ and let $k_n$ be positive integers such that $k_n \to \infty$ and $k_n/n \to \alpha$ as $n \to \infty$, where $0 \leq \alpha < 1$. We find necessary and sufficient conditions for the existence of normalizing and centering constants $A_n > 0$ and $C_n$ such that the sequence

$$
E_n = \frac{1}{A_n} \left\{ \sum_{i=1}^{k_n} X_{n+1-i,n} - C_n \right\}
$$

converges in distribution along subsequences of the integers $\{n\}$ to non-degenerate limits and completely describe the possible subsequential limiting distributions. We also give a necessary and sufficient condition for the existence of $A_n$ and $C_n$ such that $E_n$ be asymptotically normal along a given subsequence, and with suitable $A_n$ and $C_n$ determine the limiting distributions of $E_n$ along the whole sequence $\{n\}$ when $F$ is in the domain of attraction of an extreme value distribution.

1. Introduction and Statements of Results. Let $X, X_1, X_2 \ldots$ be a sequence of independent non-degenerate random variables with a common distribution function $F(x) = P\{X \leq x\}, x \in R$, and for each integer $n \geq 1$ let $X_{1,n} \leq \cdots \leq X_{n,n}$ denote the order statistics based on the sample $X_1, \ldots, X_n$. Throughout the paper $k_n$ will be a sequence of integers such that

$$(1.1) \quad 1 \leq k_n \leq n, k_n \to \infty, \text{ and } k_n/n \to 0 \text{ as } n \to \infty; \text{ or } k_n = \lfloor n\alpha \rfloor \text{ with } 0 < \alpha < 1,$$

where $\lfloor \cdot \rfloor$ denotes integer part. (We shall refer to the first case as the case $\alpha = 0$.) The study of the asymptotic distribution of the (properly normalized and centered) sums of extreme values

$$(1.2) \quad \sum_{i=1}^{k_n} X_{n+1-i,n} = \sum_{i=n-k_n+1}^{n} X_{i,n}$$

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was initiated in [6] for the case when $\alpha = 0$ in (1.1) and under the restrictive assumption that $F$ belongs to the domain of attraction of a non-normal stable law. Later the problem was solved in [7] assuming that $F$ has a regularly varying upper tail, and by Lo [13] for all $F$ which are in the domain of attraction of a Gumbel distribution in the sense of extreme value theory. When put together, these results say that whenever $F$ is in the domain of attraction of any one of the three possible limiting extreme value distributions for the maximum $X_{n,n}$ and $\alpha = 0$ in (1.1), then the sums in (1.2), with suitable centering and normalization, have a limiting distribution which is either non-normal stable or normal. (See Corollary 2 below.) The first aim of the present paper is to give an exhaustive study of the problem of the asymptotic distribution of the sums in (1.2) for an arbitrary $F$.

All three papers [6], [7], and [13] mentioned above employ a direct probabilistic approach based upon the asymptotic behavior of the uniform empirical distribution function in conjunction with the behavior of the inverse or quantile function of $F$ defined as

$$Q(s) = \inf\{x : F(x) \geq s\}, \quad 0 < s \leq 1, \quad Q(0) = Q(0+) .$$

This quantile-transform method for handling the whole sums $\sum_{j=1}^{n} X_j$ was first used in [2] and [3] to obtain probabilistic proofs of the sufficiency parts of the normal and stable convergence criteria, respectively. In [3] the effect of trimming off a finite number of the smallest and largest summands is also considered. A refined version of this method in [5] produces a complete asymptotic distribution theory of the finitely trimmed sums $T_n(m, k) = \sum_{i=m+1}^{n-k} X_{i,n}$, where $m \geq 0$ and $k \geq 0$ are arbitrarily fixed integers. Included in this paper is a new description in terms of the quantile function of the classical theory concerning domains of attraction, domains of partial attraction and stochastic compactness of the whole sum $T_n(0,0)$.

The paper [6] also initiated the study of trimmed sums of the form $T_n(m_n, k_n)$, where $k_n$ is as in (1.1) with $\alpha = 0$ and $m_n$ satisfies the same condition as $k_n$. This was done in [6] only in the domain of attraction case. A different refinement of the quantile-transform method in [4] has established the complete asymptotic distributional theory for the sums $T_n(m_n, k_n)$.

The second aim of this paper is to completely round off our study of sums of order statistics by means of the quantile-transform method, so that papers [4], [5], and the present one together constitute a complete and unified general theory of the asymptotic distribution of sums of order statistics of independent, identically distributed random variables.

We emphasize very strongly that the quantile-transform method itself is by no means new. It has been in wide use in nonparametric statistics for many decades and scattered applications of it can be found in probability as well. A good source for its earlier use is the book [19]. It is the approximation result for the uniform empirical and quantile processes in weighted supremum metrics in [2] in combination with Poisson approximation techniques for extremes that has made this old method especially feasible for the treatment of problems of the asymptotic distribution of various ordered portions of the sums of independent, identically distributed random variables. The method was augmented in [4] by a general pattern of necessity proofs which has already been applied in [5] and [15].

There are, of course, several other methods for studying sums of order statistics. Besides classical approaches, various new methods have been invented recently to deal with
a number of different types of trimmed sums and the influence of extremes on the whole sum. For descriptions of these methodologies along with discussions of their advantages and disadvantages, including ours, see the monograph [11] edited by Hahn and Weiner, where extensive lists of references can also be found.

Now we introduce some notation. The basic function for the present paper is the left-continuous function

\[ H(s) = -Q((1 - s) -), \quad 0 \leq s < 1. \]

Notice that if \( U_{1,n} \leq \cdots \leq U_{n,n} \) are the order statistics of a sample of size \( n \) from the uniform distribution on \((0,1)\) then for each \( n \geq 1 \) we have the distributional equality

\[ (X_{1,n}, \ldots, X_{n,n}) =_d (-H(U_{n,n}), \ldots, -H(U_{1,n})). \]

For \( 0 \leq s \leq t \leq 1 \), consider the truncated variance function

\[ \sigma^2(s,t) = \int_s^t \int_s^t (u \wedge v - uv)dH(u)dH(v), \]

where \( u \wedge v = \min(u,v) \), and for a given sequence \( k_n \) satisfying (1.1) set \( b_n = \sigma(1/n, k_n/n) \) and

\[ a_n = \begin{cases} b_n, & \text{if } b_n > 0, \\ 1, & \text{otherwise}. \end{cases} \]

Note that \( \sigma^2(s,t) \) is the variance of \( \int_t B(u)dH(u) \), where \( B(\cdot) \) is a Brownian bridge. Such random variables will be seen to enter the picture quite naturally.

Choose and fix any sequence of positive constants \( \delta_n \) such that \( n\delta_n < n \) and \( n\delta_n \to 0 \) as \( n \to \infty \). Then we have \( P\{\delta_n \leq U_{1,n} \leq U_{n,n} \leq 1 - \delta_n\} \to 1 \), as \( n \to \infty \). The following two sequences of functions will govern the asymptotic behavior:

\[ \psi_n(x) = \begin{cases} k_n^{1/2} \left\{ H\left( \frac{k_n}{n} + \frac{1/2}{n^{1/2}a_n} \right) - H\left( \frac{k_n}{n} \right) \right\}, & \text{if } -\frac{k_n^{1/2}}{2} \leq x \leq -\frac{k_n^{1/2}}{2}, \\
\psi_n\left( -\frac{k_n^{1/2}}{2} \right), & \text{if } -\infty < x < -\frac{k_n^{1/2}}{2}, \\
\psi_n\left( \frac{k_n^{1/2}}{2} \right), & \text{if } \frac{k_n^{1/2}}{2} < x < \infty, \end{cases} \]

and

\[ \varphi_n(y) = \begin{cases} \frac{H(y/n) - H(1/n)}{n^{1/2}a_n}, & \text{if } 0 < y \leq n - n\delta_n, \\
\frac{H(1-\delta_n) - H(1/n)}{n^{1/2}a_n}, & \text{if } n - n\delta_n < y < \infty. \end{cases} \]

The crucial (necessary and sufficient) conditions these functions will have to satisfy are the following:

**Conditions.** Assume that there exists a subsequence \( \{n_1\} \) of the positive integers \( \{n\} \) and a sequence of positive constants \( A_{n_1} \) such that, for the sequence \( \{k_{n_1}\} \) given in (1.1),
(I) there exists a non-decreasing, left-continuous function $\psi$ defined on $(-\infty, \infty)$ with $\psi(0) \leq 0$ and $\psi(0+) \geq 0$ necessarily holding such that

$$\psi_{n_1}(x) = \frac{n_1}{A_{n_1}} \psi_{n_1}(x) \to \psi(x), \quad \text{as } n_1 \to \infty,$$

at every continuity point $x$ of $\psi$;

(II) there exists a non-decreasing, left-continuous function $\varphi$ defined on $(0, \infty)$ with $\varphi(1) \leq 0$ and $\varphi(1+) \geq 0$ necessarily holding such that

$$\varphi_{n_1}(y) = \frac{n_1}{A_{n_1}} \varphi_{n_1}(y) \to \varphi(y), \quad \text{as } n_1 \to \infty,$$

at every continuity point $y$ of $\varphi$;

(III) there exists a constant $0 \leq a < \infty$ such that

$$n_1^{1/2} b_{n_1} / A_{n_1} \to a \quad \text{as } n_1 \to \infty.$$

Conditions (I) and (II) are not directly related to each other. Condition (II) controls the largest extremes while condition (I) governs the behavior of the smallest terms in the sum (1.2). Condition (III) will say that there are two qualitatively different ways to normalize this sum according to the two cases when $a > 0$ or $a = 0$. The exact probabilistic meaning of the conditions along with equivalent forms expressed through $F$ are discussed in Section 3 where illustrative examples are also constructed.

It will be shown in Lemma 2.5 in the next section that if conditions (II) and (III) both hold, then $\varphi(y) \leq a$ for all $y \in (0, \infty)$. Therefore the finite limit $\varphi(\infty) := \lim_{y \to \infty} \varphi(y) \leq a$ exists and, as Lemma 2.5 will also show, for the non-decreasing, left-continuous, nonpositive function $\varphi(\cdot) - \varphi(\infty)$ defined on $(0, \infty)$ we have

$$\int_\varepsilon^\infty (\varphi(y) - \varphi(\infty))^2 dy < \infty \quad \text{for all } \varepsilon > 0.$$  

(1.4)

Consider now a standard (intensity one) right-continuous Poisson process $N(t), 0 < t < \infty$, and two independent standard normal random variables $Z_1$ and $Z_2$ such that $(Z_1, Z_2)$ is also independent of $N(\cdot)$. Given a function $\psi$ as in condition (I), a function $\varphi$ as in condition (II) satisfying (1.4), and constants $0 \leq b < \infty$ and $0 \leq r \leq (1 - \alpha)^{1/2}$, where $\alpha$ is the limit in (1.1), consider the random variable

$$V(\varphi, \psi, b, r, \alpha) := \int_1^\infty (N(t) - t) d\varphi(t) + \int_0^1 N(t) d\varphi(t) - \varphi(1) + b Z_1 + \int_{-Z(r, \alpha)}^{0} \psi(x) dx,$$

with $Z(r, \alpha) = -r Z_1 + (1 - \alpha - r^2)^{1/2} Z_2$, where the first integral exists almost surely as an improper Riemann integral by (1.4). Finally, a natural centering sequence will be seen to be

$$\mu_n := -n \int_{1/n}^{k_n/n} H(u) du - H(\frac{1}{n}).$$
Our principle results are contained in the following two theorems, where $\rightarrow_{D}$ denotes convergence in distribution.

**THEOREM 1.** If the conditions (I), (II) and (III) are satisfied, then there exist a subsequence $\{n_2\} \subset \{n_1\}$ and a sequence of positive numbers $l_{n_2}$ satisfying $l_{n_2} \rightarrow \infty$ and $l_{n_2}/k_{n_2} \rightarrow 0$, as $n_2 \rightarrow \infty$, such that either $\sigma(l_{n_2}/n_2, k_{n_2}/n_2) > 0$ for all $n_2$, in which case for some $0 \leq b \leq a$ and $0 \leq r \leq (1 - \alpha)^{1/2}$,

$$
\frac{1}{A_{n_2}} \left\{ \sum_{i=1}^{k_{n_2}} X_{n_2+1-i, n_2} - \mu_{n_2} \right\} \rightarrow_{D} V(\varphi, \psi, b, r, \alpha)
$$

as $n_2 \rightarrow \infty$, where $\alpha$ is zero or positive according to the two cases in (1.1), $\varphi$ necessarily satisfies (1.4) and $\psi$ necessarily satisfies

$$
\psi(x) \geq -ar/(1 - \alpha), \quad -\infty < x < \infty.
$$

Moreover, if $a > 0$ and $\varphi \equiv 0$ then $b = a$ in (1.5), while if $a = 0$ then $\varphi(y) = 0$ for all $y > 1$.

**THEOREM 2.** If there exist a subsequence $\{n_1\} \subset \{n\}$ and two sequences $A_{n_1} > 0$ and $C_{n_1}$ along it such that

$$
\frac{1}{A_{n_1}} \left\{ \sum_{i=1}^{k_{n_1}} X_{n_1+1-i, n_1} - C_{n_1} \right\} \rightarrow_{D} V,
$$

where $V$ is a non-degenerate random variable, then there exists a subsequence $\{n_2\} \subset \{n_1\}$ such that conditions (I), (II) and (III) hold along the subsequence $\{n_2\}$ for $A_{n_2}$ in (1.8) and for appropriate functions $\psi$ and $\varphi$ and some constant $0 \leq a < \infty$, with $\varphi$ satisfying (1.4) and $\psi$ satisfying (1.7). The random variable $V$ in (1.8) is of the form $V(\varphi, \psi, b, r, \alpha) + c$ for appropriate constants $0 \leq b \leq a, 0 \leq r \leq (1 - \alpha)^{1/2}$, and $-\infty < c < \infty$. Moreover, either $\varphi \neq 0$ or $\psi \neq 0$ or $b > 0$.

Just as in the case of full sums in [5], the method of proof makes it possible to see the effect on the limiting distribution of deleting a finite number of the largest summands from the extreme sums $T_n(n - k_n, 0) = \sum_{i=1}^{k_n} X_{n+1-i, n}$ at each stage $n$, both in the
sufficiency and the necessity directions. Let \( k \geq 0 \) be any fixed integer. Then, replacing 
\[
\sum_{i=1}^{k_n} X_{n+1-i,n} \text{ by}
\]
\[
T_n(n-k_n,k) = \sum_{i=k+1}^{k_n} X_{n+1-i,n} = \sum_{i=n-k_n+1}^{n-k} X_{i,n},
\]
\(
\mu_n \text{ by}
\)
\[
\mu_n(k) = -n \int_{(k+1)/n}^{k+n/n} H(u)du - H \left( \frac{k+1}{n} \right),
\]
and \( V(\varphi, \psi, b, r, \alpha) \) by
\[
V_k(\varphi, \psi, b, r, \alpha) := \int_{S_{k+1}}^{\infty} \left( N(t) - t \right) d\varphi(t) - \int_{1}^{S_{k+1}} t d\varphi(t) + k \varphi(S_{k+1})
\]
\[ - \int_{1}^{k+1} \varphi(t) dt - \varphi(1) + bZ_1 + \int_{-Z(r,\alpha)}^{0} \psi(x) dx,
\]
where \( S_{k+1} \) is the \((k+1)\)st jump-point of the Poisson process \( N(\cdot) \), both Theorem 1 and
Theorem 2 remain true word for word. This can be seen by simple adjustments of the proofs presented in Section 2 for the case \( k = 0 \). (Cf. [5] for details.) That the case \( k = 0 \)
formally agrees with the above theorems, i.e. that \( V_0(\varphi, \psi, b, r, \alpha) = V(\varphi, \psi, b, r, \alpha) \), is
easily seen by noting that
\[
\int_{1}^{\infty} (N(t) - t) d\varphi(t) - \int_{1}^{S_1} t d\varphi(t) = \int_{1}^{\infty} (N(t) - t) d\varphi(t) + \int_{0}^{1} N(t) d\varphi(t).
\]

It is also straightforward to formulate Theorems 1 and 2 (or their generalized versions
just described) for the sum of the lower extremes \( \sum_{i=1}^{m_n} X_{i,n} \), where \( m_n \to \infty \) and \( m_n/n \to 0 \) as \( n \to \infty \), or \( m_n = \lfloor \beta n \rfloor \) with \( 0 < \beta < 1 \). The limiting random variable is of the form
\( -V(\varphi, \psi, b, r, \beta) \) with appropriate ingredients. In fact, if at least one of \( \alpha \) and \( \beta \) is zero
then the two convergence statements hold jointly with the limiting random variables being
independent.

Now we come back to the principal results and formulate some consequences of them.

**COROLLARY 1.** Let \( \{n_1\} \) be any subsequence of the positive integers. There exist
sequences of constants \( A_{n_1}^* > 0 \) and \( C_{n_1} \) such that
\[
(1.10) \quad \frac{1}{A_{n_1}^*} \left\{ \sum_{i=1}^{k_n} X_{n_1+1-i,n_1} - C_{n_1} \right\} \to d Z
\]
holds as \( n_1 \to \infty \) for a non-degenerate normal random variable \( Z \) if and only if (I) and
(II) are satisfied with \( A_{n_1} \equiv n_1^{1/2} a_{n_1}, \psi \equiv 0 \) and \( \varphi \equiv 0 \), in which case (1.10) is true with
the choice \( A_{n_1}^* \equiv n_1^{1/2} a_{n_1} \) and \( C_{n_1} \equiv \mu_{n_1} \) with \( Z \) being standard normal.
On heuristic grounds one expects that the existence of normalizing and centering constants \( d_n > 0 \) and \( c_n \) such that

\[
(1.11) \quad d_n^{-1}(X_{n,n} - c_n) \rightarrow_D Y \quad \text{with } Y \text{ non-degenerate}
\]

implies that the suitably centered and normalized extreme sums \( T_n(n - k_n, 0) \) or \( T_n(n - k_n, k) \) also converge in distribution to a non-degenerate variable along the whole sequence \( \{n\} \). This is the content of our second corollary below.

As was pointed out in [8], results from de Haan [10] imply that (1.11) holds if and only if

\[
(1.12) \quad \lim_{s \downarrow 0} \frac{H(xs) - H(ys)}{H(vs) - H(ws)} = \frac{x^{-c} - y^{-c}}{v^{-c} - w^{-c}} \quad \text{for some } -\infty < c < \infty
\]

for all distinct \( 0 < x, y, v, w < \infty \), where for \( c = 0 \) the limit is understood as \((\log x - \log y)/(\log v - \log w)\). (The case \( c = 0 \), going back to Meijller [16], is explicitly stated as Theorem 2.4.1 in [10].) If condition (1.12) holds, the constants \( d_n \) and \( c_n \) can be chosen so that when \( c > 0 \), then \( P\{Y \leq y\} = \exp(-y^{1/c}), y > 0 \); when \( c = 0 \), then \( P\{Y \leq y\} = \exp(-\exp(-y)), -\infty < y < \infty \); and when \( c < 0 \), then \( P\{Y \leq y\} = \exp(\ -y^{1/c}), y < 0 \), and by Gnedenko's classic theorem these are the only possible limiting types. Whenever (1.12) holds, we write \( F \in \Delta(c) \).

For \( c > 1/2 \) set

\[
D(c) := c \left( \frac{2c - 1}{2c} \right)^{1/2} \left\{ \int_1^\infty (N(t) - t)t^{-c-1}dt + \int_0^1 N(t)t^{-c-1}dt \right\}.
\]

According to Corollary 3 in [5], the random variable \( D(c) \) is stable with index \( 1/c \). This fact can also be derived after some calculations from the representations given by Ferguson and Klass [9] and LePage, Woodroofe, and Zinn [12]. For the case when \( \alpha > 0 \) in (1.1), consider

\[
(1.13) \quad \psi_\alpha(x) := \begin{cases} 0, & \text{if } x \leq 0, \\ \alpha^{1/2}(H(\alpha x) - H(\alpha))/\sigma(0, \alpha), & \text{if } x > 0, \end{cases}
\]

and note that when (1.12) holds then

\[
(1.14) \quad 0 < \sigma(0, \alpha) \begin{cases} < \infty, & \text{if } c < 1/2, \\ = \infty, & \text{if } c > 1/2. \end{cases}
\]

If \( c = 1/2 \), then \( \sigma(0, \alpha) \) can be finite or infinite. Define \( \sigma(0, 0) = \sigma(0, 0+) \) and, finally, put

\[
M(\alpha) := \begin{cases} 0, & \text{if } \alpha = 0, \\ \int_{-Z(\alpha)}^0 \psi_\alpha(x)dx, & \text{if } \alpha > 0, \end{cases}
\]

where \( Z(\alpha) = -r_\alpha Z_1 + (1 - \alpha - r_\alpha^2)^{1/2} Z_2 \) with

\[
r_\alpha := \frac{1 - \alpha}{\alpha^{1/2}\sigma(0, \alpha)} \int_0^\alpha sdH(s) \leq (1 - \alpha)^{1/2}.
\]
Note that for $\alpha > 0$,

$$M(\alpha) = \alpha^{1/2}(H(\alpha) - H(\alpha)^0) \min(0, Z(\alpha))/\sigma(0, \alpha).$$

Versions of the three subcases $c > 1/2; -\infty < c \leq 1/2, c \neq 0$; and $c = 0$ of the case $\alpha = 0$ in the following corollary were proven in [6], [7] and [13], respectively. The full form of the corollary follows presently from Theorem 1.

**COROLLARY 2.** Let $\{k_n\}$ satisfy (1.1). Whenever $F \in \Delta(c)$ for some $-\infty < c < \infty$,

$$\left\{ \sum_{i=1}^{k_n} X_{n+1-i, n} - \mu_n \right\}/(n^{1/2}a_n) \to p \left\{ \begin{array}{l}
D(c) \quad \text{if} \quad c > 1/2, \\
Z_1 + M(\alpha) \quad \text{if} \quad c = 1/2 \text{ and } \sigma(0, \alpha) = \infty,
\end{array} \right. \right.$$

or $c < 1/2$.

If we replace the sum $T_n(n - k_n, 0)$ by $T_n(n - k_n, k)$ and $\mu_n$ by $\mu_n(k)$ in Corollary 2, where $k \geq 0$ is a fixed integer, then the limits remain exactly the same in the cases of $c \leq 1/2$, while if $c > 1/2$, then $D(c)$ should be replaced by

$$D_k(c) = \left( \frac{2c - 1}{2c} \right)^{1/2} \left\{ c \int_{s_{k+1}}^{\infty} (N(t) - t) t^{-c-1} dt - c \int_{1}^{s_{k+1}} t^{-c} dt + k(1 - s_{k+1}^{-c}) 
- \int_{1}^{k+1} (1 - t^{-c}) dt \right\}.$$ 

Again, $D_0(c) = D(c)$.

Of course, if the quantile function $Q$ is continuous at $1 - \alpha$, then $M(\alpha) = 0$ also in the case when $\alpha > 0$. It was Stigler [20] who discovered that such terms generally enter in the limiting distribution of the classical trimmed mean. So the third subcase of the case $\alpha > 0$ in Corollary 2 may be looked upon as a Stigler phenomenon for sums of extreme values. In this regard we also note that in the case when $\alpha > 0$ in condition (1.1) we could have worked with a sequence $k_n$ more general than $k_n = [an]$. Namely, if $k_n$ is a sequence of integers such that $1 \leq k_n \leq n$ and $n^{1/2}(k_n/n - \alpha) \to 0$ as $n \to \infty$ with some $0 < \alpha < 1$, then all the results so far stated remain valid without change provided that we modify the definition of $\psi_n$ by replacing $k_n/n$ by $\alpha$ in the arguments of $H$ in its numerator. However, then the proofs would have to be separated for the cases $\alpha = 0$ and $\alpha > 0$. (For the slight modifications needed for the case $\alpha > 0$ we refer to [15].) In order to keep a unified proof for the two cases we have chosen to work with the special sequence $k_n = [an]$ here. A similar remark applies to our version of Stigler's theorem, formulated as Theorem 5 in [4].

Consider now linear combinations of $k_n$ extreme values of the form

$$L_n = \sum_{i=1}^{k_n} c_{i,n} X_{n+1-i, n},$$

8
where \( c_{i,n}, i = 1, \ldots, n, n \geq 1 \), is a triangular array of weights. Under the same regularity conditions as in [15] one can easily formulate and prove the analogues for \( L_n \) of the results in [15] on the asymptotic distribution of linear combinations of the middle order statistics. The proofs consist of a straightforward technical extension of those in the present paper combined with details from [15]. To keep, however, the main ideas easily accessible to a wider audience we decided to only consider sums of extremes in the present paper.

We take this opportunity to correct some minor oversights and misprints in paper [4]. First, Lemma 2.6 is not correct as stated. The specification that \( f(0) = 0 \) must be changed to \( f(0) \leq 0 \) and \( f(0+) \geq 0 \). For this reason, whenever in the statements of results or in the proofs a function is specified to be zero at zero, this must be changed to the requirement that it be non-positive on \((-\infty, 0]\) and non-negative on \((0, \infty)\). In particular, in Theorem 1 the requirement that \( \Psi_1(0) = \Psi_2(0) = 0 \) should be weakened to read \( \Psi_i(0) \leq 0, \Psi_i(0+) \geq 0, i = 1, 2 \), with analogous changes needed for Theorem 2. Also, all integrals of the form \( \int_{-z}^{-z} (z + x) dg(x) \) should be read as

\[
\int_{-z}^{0} x dg(x) + zg(-z) = \int_{-z}^{0} g(x) dx,
\]

whenever \((z, g) = (Z_i, \Psi_i), (Z_i, \Psi_i^2), (Z_i, \varphi_i), i = 1, 2\). With these corrections the proofs proceed as before. Finally, the \( X_{1,n} \) appearing in Theorems 2, 3, and 5 should be \( X_{i,n} \), the subscript 1 in formulae (1.12), (1.13), and (1.15) should be \( i \), and the expressions for \( r_1 \) and \( r_2 \) on page 678 should be divided by \( o(\alpha, \beta) \).

2. Proofs. Introducing the empirical distribution function

\[
G_n(u) = \frac{1}{n} \sum_{i=1}^{n} I(U_{i,n} \leq u), \quad 0 \leq u \leq 1,
\]

where \( I \) is the indicator function, using (1.3), and integrating by parts

\[
\sum_{i=1}^{k_n} X_{n+1-i,n} - \mu_n = \sum_{i=1}^{k_n} H(U_{i,n}) - \mu_n
\]

\[
= - \int_{0}^{1/n} nH(u) dG_n(u) + \int_{1/n}^{1} nH(u) du + H(1/n)
\]

\[
= -nG_n(U_{k_n,n}) H(U_{k_n,n}) + \int_{0}^{1/n} nG_n(u) dH(u)
\]

\[
+ k_n H(k_n/n) - \int_{1/n}^{1} nuduH(u)
\]

\[
= \int_{0}^{1/n} nG_n(u) dH(u) + \int_{1/n}^{k_n/n} n(G_n(u) - u) dH(u)
\]

\[
+ \int_{k_n/n}^{1} n\left( G_n(u) - \frac{k_n}{n} \right) dH(u).
\]
Here the sum of the first two terms can be written as

\[
\int_{l_{1,n}}^{1/n} (nG_n(u) - 1)dH(u) + H(1/n) - H(U_{1,n})
\]
\[
+ \int_{l_{1,n}}^{t_{1,n}} n(G_n(u) - u)dH(u) + \int_{l_{1,n}}^{k_{n}/n} n(G_n(u) - u)dH(u)
\]
\[
= \int_{l_{1,n}}^{1/n} (nu - 1)dH(u) + H(1/n) - H(U_{1,n}) + \int_{l_{1,n}}^{k_{n}/n} n(G_n(u) - u)dH(u)
\]
\[
= \int_{l_{1,n}}^{1/n} (nu - 1)dH(u) + H(1/n) - H(U_{1,n}) + \int_{l_{1,n}}^{m_{n}/n} n(G_n(u) - u)dH(u)
\]
\[
+ \int_{m_{n}/n}^{l_{n}/n} n(G_n(u) - u)dH(u) + \int_{l_{n}/n}^{k_{n}/n} n(G_n(u) - u)dH(u),
\]

where, for the time being, \( m_n \) and \( l_n \) are any real numbers such that \( 1 \leq m_n \leq l_n \leq k_n \). Hence

\[
\frac{1}{A_n} \left\{ \sum_{i=1}^{k_n} X_{n+1-i,n} - \mu_n \right\} = \Delta_n^{(1)}(m_n) + \Delta_n^{(2)}(m_n, l_n) + \Delta_n^{(3)}(l_n, k_n),
\]

where

\[
\Delta_n^{(1)}(m_n) = \int_{nU_{1,n}}^{m_{n}/n} (nG_n(u/n) - u) d\frac{H(u/n) - H(1/n)}{A_n}
\]
\[
+ \int_{nU_{1,n}}^{1/n} (u - 1) d\frac{H(u/n) - H(1/n) - H(nU_{1,n}/n) - H(1/n)}{A_n},
\]

\[
\Delta_n^{(2)}(m_n, l_n) = \int_{m_{n}/n}^{l_{n}/n} (nG_n(u/n) - u) d\frac{H(u/n) - H(1/n)}{A_n},
\]

and

\[
\Delta_n^{(3)}(l_n, k_n) = \int_{l_{n}/n}^{k_{n}/n} n(G_n(u) - u) d\frac{H(u)}{A_n} + \int_{k_{n}/n}^{t_{n}/n} n(G_n(u) - \frac{k_n}{n}) d\frac{H(u)}{A_n}.
\]

This distributional equality is of course true without regard to the underlying probability space where the order statistics \( U_{1,n}, \ldots, U_{n,n} \) are defined. In the proof of Theorem 1 we shall be working on a specially constructed space \((\Omega, \mathcal{A}, P)\) described in [2,3,5]. It carries two independent sequences \( \{Y_n^{(j)}, n \geq 1\}, j = 1, 2 \), of independent, exponentially distributed random variables with mean one and a sequence \( \{B_n(t), 0 \leq t \leq 1; n \geq 1\} \) of Brownian bridges with the following property: for the \( G_n \) in (2.1) and the uniform quantile function \( U_n(s) = U_{i,n} \) for \((i-1)/n < s \leq i/n, i = 1, \ldots, n, U_n(0) = U_{1,n} \), determined by the order statistics \( U_{k,n} = \tilde{S}_k(n)/\tilde{S}_{n+1}(n), k = 1, \ldots, n \), given by \( \tilde{S}_k(n) = \tilde{Y}_1(n) + \cdots + \tilde{Y}_n(n) \).
where \( \tilde{Y}_j(n) = Y_j^{(1)} \) for \( j = 1, \ldots, \lfloor n/2 \rfloor \) and \( \tilde{Y}_j(n) = Y_{n+2-j}^{(2)} \) for \( j = \lceil n/2 \rceil - 1, \ldots, n+1 \), we have

\[
(2.3) \quad \Delta_n(\nu) = \sup_{n^{-1} \leq s \leq 1-n^{-1}} |n^{1/2} \{ G_n(s) - s \} - B_n(s) | (s(1-s))^{1/2-\nu} = O_P(n^{-\nu})
\]

and

\[
(2.4) \quad \sup_{n^{-1} \leq s \leq 1-n^{-1}} |n^{1/2} (s - U_n(s)) - B_n(s) | (s(1-s))^{1/2} \nu = O_P(n^{-\nu})
\]

for any fixed \( 0 \leq \nu < 1/4 \). Note that it is justified to call the above \( U_{k,n} \) order statistics since it is well known that \( (\tilde{S}_1(n)/\tilde{S}_{n+1}(n)), \ldots, \tilde{S}_n(n)/\tilde{S}_{n+1}(n) \) equals in distribution to the vector of order statistics of \( n \) independent random variables uniformly distributed on \( (0,1) \). We will also need the Poisson process \( N(\cdot) \) with jump-points \( S_n = S_n^{(1)} = Y_1^{(1)} + \cdots + Y_n^{(1)} \), \( n \geq 1 \), defined to be right-continuous by setting

\[
(2.5) \quad N(t) = \sum_{k=1}^{\infty} I(S_k \leq t), \quad 0 \leq t < \infty.
\]

The behavior of the term \( \Delta_{n}^{(3)}(l_n, k_n) \) is described in Lemma 2.4 below which requires three preparatory lemmas, the first of which is crucial at many other places too.

**Lemma 2.1.** If \( 0 < s < t \leq 1 - \varepsilon \), where \( 0 < \varepsilon < 1 \), then

\[
s(H(t) - H(s))^2 / \sigma^2(s,t) \leq 1/\varepsilon,
\]

where \( 0/0 := 1 \).

**Proof.** If \( s \leq u \leq v < t \) or \( s \leq v \leq u < t \), then \( u \wedge v - uv \geq \varepsilon s \). Hence

\[
\sigma^2(s,t) = \int_s^t \int_s^t (u \wedge v - uv)dH(u)dH(v) \geq \varepsilon s (H(t) - H(s))^2,
\]

from which the result follows upon noting also that \( \sigma(s,t) = 0 \) if and only if \( H(s) = H(t) \), which justifies the definition \( 0/0 := 1 \).

**Lemma 2.2.** For all \( \beta \neq 1/2 \) and \( 0 < s < t \leq 1 - \varepsilon \), where \( 0 < \varepsilon < 1 \),

\[
\int_s^t u^\beta dH(u) / \sigma(s,t) \leq \frac{1}{\varepsilon^{1/2}} \left\{ \frac{\beta^{\beta - 1/2}}{\beta - 1/2} + \frac{s^{\beta - 1/2}}{1 - 2\beta} \right\}.
\]

**Proof.** Integrating by parts we see that

\[
\int_s^t u^\beta dH(u) = s^\beta \{ H(t) - H(s) \} + \beta \int_s^t \{ H(t) - H(u) \} u^{\beta - 1} du.
\]
Thus the lemma follows from Lemma 2.1.

In the following two lemmas $k_n$ is as in (1.1) and $l_n$ is any sequence of positive numbers such that

\begin{equation}
0 \leq l_n \leq n, \ l_n \to \infty \text{ and } k_n/l_n \to \infty \text{ as } n \to \infty.
\end{equation}

**Lemma 2.3.** Suppose there exists a subsequence $\{n_2\} \subset \{n\}$ such that $\sigma^2(l_{n_2}/n_2, k_{n_2}/n_2) > 0$ for all $n_2 \in \{n_2\}$ and consider the two-dimensional random vectors

\[
\left( Z_{n_2}^{(1)}, Z_{n_2}^{(2)} \right) := \left( \int_{l_{n_2}/n_2}^{k_{n_2}/n_2} B_{n_2}(s) dH(s), -\frac{n_2}{k_{n_2}} B_{n_2}(k_{n_2}/n_2) \right).
\]

Then there exist a subsequence $\{n_3\} \subset \{n_2\}$ and a number $0 \leq r \leq (1 - \alpha)^{1/2}$ such that, as $n_3 \to \infty$,

\[
\left( Z_{n_3}^{(1)}, Z_{n_3}^{(2)} \right) \to_{\mathbb{D}} (Z_1, -rZ_1 + (1 - \alpha - r^2)^{1/2} Z_2) =_{\mathbb{D}} N \left( (0, 0), \begin{pmatrix} 1 & r \\ r & 1 - \alpha \end{pmatrix} \right).
\]

**Proof.** First notice that $(Z_{n_2}^{(1)}, Z_{n_2}^{(2)})$ is, for each $n_2$, a bivariate normal random variable with mean vector zero and covariance matrix

\[
\begin{pmatrix}
1 & -r_{n_2} \\
-r_{n_2} & 1 - k_{n_2}/n_2
\end{pmatrix},
\]

where

\[
0 \leq r_{n_2} = \left( \frac{n_2}{k_{n_2}} \right)^{1/2} \frac{(1 - k_{n_2}/n_2)}{\sigma(l_{n_2}/n_2, k_{n_2}/n_2)} \int_{l_{n_2}/n_2}^{k_{n_2}/n_2} sdH(s) \leq \left( 1 - \frac{k_{n_2}}{n_2} \right)^{1/2}.
\]

Thus by (1.1) there exist a subsequence $\{n_3\} \subset \{n_2\}$ and a number $0 \leq r \leq (1 - \alpha)^{1/2}$ such that $r_{n_3} \to r$ as $n_3 \to \infty$, which implies the lemma.

**Lemma 2.4.** Suppose that conditions (I) and (III) hold along some $\{n_1\}$, and let $k_n$ and $l_n$ be as in (1.1) and (2.6). Then there exist a subsequence $\{n_3\} \subset \{n_1\}$ and numbers $0 \leq b \leq a$ and $0 \leq r \leq (1 - \alpha)^{1/2}$ such that

\[
\Delta_{n_3}^{(3)}(l_{n_3}, k_{n_3}) \to_{\mathbb{D}} V_3(\psi, b, r, \alpha) := bZ_1 + \int_{-Z(r, \alpha)}^{Z(r, \alpha)} \psi(x) dx
\]

where $Z(r, \alpha) = -rZ_1 + (1 - \alpha - r^2)^{1/2} Z_2$.

**Proof.** There are two cases. First we deal with
Case 1: There exists a subsequence \( \{n_2\} \subset \{n_1\} \) such that \( \sigma(l_{n_2}/n_2, k_{n_2}/n_2) > 0 \) for all \( n_2 \in \{n_2\} \). Now let \( \{n_3\} \subset \{n_2\} \) be the subsequence given by the proof of Lemma 2.3.

To see the behavior of the first term in \( \Delta_{n_3}^{(3)}(l_{n_3}, k_{n_3}) \) in (2.2), fix any \( 0 < \nu < 1/4 \). Then for the random variable \( \tilde{Z}_{n_3} \) defined by the equation

\[
\int_{l_{n_3}/n_3}^{k_{n_3}/n_3} n_3(G_{n_3}(u) - u) d\frac{H(u)}{A_{n_3}} = n_3^{1/2} \sigma(l_{n_3}/n_3, k_{n_3}/n_3) \tilde{Z}_{n_3}
\]

by Lemma 2.2 and (2.3) we have

\[
|\tilde{Z}_{n_3} - Z_{n_3}^{(1)}| \leq \Delta_{n_3}(\nu) \int_{l_{n_3}/n_3}^{k_{n_3}/n_3} s^{1/2-\nu} dH(s)/\sigma(l_{n_3}/n_3, k_{n_3}/n_3) = O_P(n_3^{-\nu})O((l_{n_3}/n_3)^{-\nu}) = O_P(l_{n_3}^{-\nu}) = o_P(1)
\]

by (2.6) as \( n_3 \to \infty \). By condition (III) we can assume without loss of generality that \( \{n_3\} \) is chosen such that for some \( 0 \leq b \leq a \) we also have

\[
n_3^{1/2} \sigma(l_{n_3}/n_3, k_{n_3}/n_3)/A_{n_3} \to b \quad \text{as} \quad n_3 \to \infty
\]

and hence we have

\[
\Delta_{n_3}^{(3)}(l_{n_3}, k_{n_3}) = b Z_{n_2}^{(1)} + \int_{l_{n_3}/n_3}^{k_{n_3}/n_3} n_3(G_{n_3}(u) - \frac{k_{n_3}}{n_3}) d\frac{H(u)}{A_{n_3}} + o_P(1).
\]

Note that this step corresponds to Lemma 2.2 in [4].

Let \( 0 < M < \infty \) be fixed. Adapting the proof of (2.6) in Lemma 2.3 in [4] to the present situation (i.e. replacing the function \( \psi_{1,n} \) there by the function \( \psi_{n}^* \) of assumption (I) but otherwise proceeding line by line in exactly the same way), by (2.3) and (2.4) we obtain

\[
\int_{l_{n_3}/n_3}^{k_{n_3}/n_3} n_3(G_{n_3}(u) - \frac{k_{n_3}}{n_3}) d\frac{H(u)}{A_{n_3}} I(|Z_{n_3}^{(2)}| < M)
\]

\[
= \int_0^{-Z_{n_3}^{(2)}} (Z_{n_3}^{(2)} + z) d\psi_{n_3}^*(z) I(|Z_{n_3}^{(2)}| < M) + o_P(1)
\]

as \( n_3 \to \infty \).

Using now Lemma 2.3, we obtain exactly as in the proof of Lemma 2.4 in [4] that

\[
b Z_{n_2}^{(1)} + \int_0^{-Z_{n_3}^{(2)}} (Z_{n_3}^{(2)} + z) d\psi_{n_3}^*(z) I(|Z_{n_3}^{(2)}| < M)
\]

\[
\to_P b Z_1 + \int_{Z_{(r,a)}}^0 \psi(x) dx I(|Z(r,a)| < M).
\]
Now (2.7), (2.8), (2.9), the simple little argument finishing the proof of the first part of Theorem 1 in [4], and Theorem 4.2 in Billingsley [1] give the lemma in the first case.

Case 2. For all \( n_1 \in \{ n_1 \} \) sufficiently large, \( \sigma(l_n, n_1, k_n, n_1) = 0 \). Then the first term of \( \Delta^{(3)}_{n_1}(l_n, k_n) \) is almost surely zero for all these \( n_1 \), and \( a = 0 \) in condition (III) in this case. Thus, with \( b = r = 0 \), a simplified version of the argument in Case 1 gives

\[
\Delta^{(3)}_{n_1}(l_n, k_n) \to_{\mathbb{P}} \int_{Z(0, \alpha)} \psi(x)dx.
\]

Now we turn to the behavior of the terms \( \Delta^{(1)}_n \) and \( \Delta^{(2)}_n \) in (2.2). First we need the following.

**Lemma 2.5.** Suppose that conditions (II) and (III) hold along some subsequence \( \{ n_1 \} \). Then \( \varphi(y) \leq a \) for all \( 0 < y < \infty \) and (1.4) holds true.

**Proof.** Since \( a \geq 0 \) and \( \varphi(y) \leq 0 \) for \( 0 < y \leq 1 \), it is sufficient to consider \( y > 1 \). Then for all sufficiently large \( n_1 \), whenever \( \sigma(1/n_1, y/n_1) > 0 \),

\[
\varphi^*_n(y) = \frac{n_1^{1/2}b_{n_1}}{A_{n_1}} \frac{\sigma(1/n_1, y/n_1)}{\sigma(1/n_1, k_n/n_1)} \frac{H(y/n_1) - H(1/n_1)}{n_1^{1/2}a(1/n_1, y/n_1)} \leq (1 + \varepsilon)n_1^{1/2}b_{n_1}/A_{n_1}
\]

by an application of Lemma 2.1, where \( \varepsilon > 0 \) is any preassigned number. (Of course, when \( \sigma(1/n_1, y/n_1) = 0 \) then \( \varphi^*_n(y) = 0 \)). The first statement is therefore clear.

Since \( \varphi \) is non-decreasing, the limit \( \varphi(\infty) := \lim_{y \to \infty} \varphi(y) \leq a \) exists. Thus \( \bar{\varphi}(y) := \varphi(y) - \varphi(\infty) \) is a non-positive, left-continuous, non-decreasing function on \( (0, \infty) \), and the fact that conditions (II) and (III) imply

\[
\int_{s}^{\infty} \int_{s}^{\infty} (u \land v)d\bar{\varphi}(u)d\bar{\varphi}(v) < \infty \quad \text{for all} \quad 1 < s < \infty
\]

can be shown exactly as in the proof of Lemma 2.5 in [5]. Since \( \bar{\varphi}(y) \to 0 \) as \( y \to \infty \), this is sufficient to conclude that (1.4) indeed follows. \( \square \)

**Lemma 2.6.** Suppose that conditions (II) and (III) hold along some subsequence \( \{ n_1 \} \) and \( k_n \) is as in (1.1). Then there exist sequences of positive constants \( m_{n_1} \) and \( l_n \) such that \( 1 \leq m_{n_1} \leq l_n \leq k_n \) and \( m_{n_1} \to \infty, l_n/m_{n_1} \to \infty, k_n/l_n \to \infty \),

\[
\Delta^{(1)}_{n_1}(m_{n_1}) \to_{\mathbb{P}} V_1(\varphi) := \int_{S_1}^{\infty} (N(t) - t)d\varphi(t) - \int_{1}^{S_1} td\varphi(t) - \varphi(1)
\]

with \( \varphi \) satisfying (1.4), and \( \Delta^{(2)}_{n_1}(m_{n_1}, l_n) \to_{\mathbb{P}} 0 \) as \( n_1 \to \infty \). Moreover, if \( \varphi \equiv 0 \) then \( \{ l_n \} \) can be chosen so that

\[
n_1^{1/2} \sigma(1/n_1, l_n/n_1)/A_{n_1} \to 0.
\]
Proof. Let $1 < m < l$ be arbitrarily fixed continuity points of $\varphi$. Then, noting that in [5] the proofs of Lemmas 2.1, 2.2 and 2.3 require only the assumption of the weak convergence of functions corresponding to the present sequence $\{\varphi_{n_1}\}$, condition (II) and these lemmas together with (2.10) in the proof of Lemma 2.2, all in [5], directly imply that

$$
\Delta_{n_1}^{(1)}(m) \to_p \int_{S_1}^{m} (N(t) - t) d\varphi(t) + \int_{S_1}^{1} (t - 1) d\varphi(t) - \varphi(S_1)
= \int_{S_1}^{m} (N(t) - t) d\varphi(t) - \int_{1}^{S_1} t d\varphi(t) : \varphi(1)
$$

and

$$
\Delta_{n_1}^{(2)}(m, l) \to_p V_{m, l} := \int_{m}^{l} (N(t) - t) d\varphi(t).
$$

Since we have (1.4) by Lemma 2.5,

$$
\limsup_{l, m \to \infty, l > m} EV_{m, l}^2 \leq \lim_{m \to \infty} \int_{m}^{\infty} \int_{m}^{\infty} (u \wedge v - uv) d\varphi(u) d\varphi(v) = 0.
$$

Thus $V_{m, l} \to_p 0$ as $m, l \to \infty, l \geq m$.

Using these findings, the construction of the sequences $\{m_{n_1}\}$ and $\{l_{n_1}\}$ is accomplished by a routine diagonal selection procedure.

Finally, choose any number $d \geq 1$ and let $c_1$ and $c_2$ be continuity points of $\varphi$ such that $c_1 \leq d, 1 \leq c_2$. Then, using (II) and a weak convergence argument, we obtain

$$
\limsup_{n_1 \to \infty} n_1 \sigma^2(1/n_1, d/n_1) / A_{n_1}^2 \leq \lim_{n_1 \to \infty} n_1 \int_{c_1/n_1}^{c_2/n_1} \int_{c_1/n_1}^{c_2/n_1} (u \wedge v - uv) dH(u) dH(v) / A_{n_1}^2
= \int_{c_1}^{c_2} \int_{c_1}^{c_2} (s \wedge t) d\varphi(s) d\varphi(t),
$$

so that if $\varphi \equiv 0$, then for each number $d \geq 1$,

$$
n_1^{1/2} \sigma(1/n_1, d/n_1) / A_{n_1} \to 0 \text{ as } n_1 \to \infty.
$$

From this the last statement of the lemma also follows. \qed

Proof of Theorem 1. Choose $\{m_{n_1}\}$ and $\{l_{n_1}\}$ according to Lemma 2.6. Since $\sigma(l_{n_1}/n_1, k_{n_1}/n_1) \leq b_{n_1}$, by condition (III) and Lemmas 2.4 and 2.6 there exists a subsequence $\{n_2\} \subset \{n_1\}$ such that we have (1.5) and

$$
\Delta_{n_2}^{(1)}(m_{n_2}) \to_p V_1(\varphi), \Delta_{n_2}^{(2)}(m_{n_2}, l_{n_2}) \to_p 0, \text{ and } \Delta_{n_2}^{(3)}(l_{n_2}, k_{n_2}) \to_p V_3(\psi, b_1, r, \alpha)
$$
as $n_2 \to \infty$, where $\varphi$ satisfies (1.4). Moreover, $\{n_2\}$ can clearly be chosen so that in addition either $\sigma(l_{n_2}/n_2, k_{n_2}/n_2) > 0$ for all $n_2$, or $\sigma(l_{n_2}/n_2, k_{n_2}/n_2) = 0$ for all $n_2$. 15
Since $l_{n_1}/m_{n_1} \to \infty$, an elementary argument similar to the one used in the proof of Theorem 2 of Mason [14] based on Satz 4 of Rossberg [18] shows that $\Delta^{(1)}_{n_2}(m_{n_2})$ and $\Delta^{(3)}_{n_2}(l_{n_2}, k_{n_2})$ are asymptotically independent. Therefore, on account of (2.2) and the above convergence relations, (1.6) follows since by (1.9),

$$V(\varphi, \psi, b, r, \alpha) = V_1(\varphi) + V_3(\psi, b, r, \alpha).$$

If $\varphi \equiv 0$ and $0 < a < \infty$ in (III), then by Lemma 2.6 the sequence $\{l_{n_1}\}$ can be chosen so that (2.10) holds. It is easy to see that this implies (1.5) with $b = a$. Also, since $\varphi(1^+) \geq 0$, if $a = 0$ then $\varphi(y) = 0$ for all $y > 1$ by Lemma 2.5.

It remains to establish the lower bound in (1.7). Since $\psi(0^+) \geq 0$, it is enough to deal with $x < 0$. If $n_2 = n$ is large enough,

$$\psi_n^*(x) = -\frac{n^{1/2}a_n}{A_n} \left( \frac{k_n}{n} \right)^{1/2} \int_{k_n/n + z(k^{1/2}/n)}^{k_n/n} dH(u)/a_n$$

$$\geq -\frac{n^{1/2}a_n}{A_n} \left( \frac{k_n}{n} \right)^{1/2} \left( \frac{n}{k_n^{1/2}} \left( 1 + \frac{x}{k_n^{1/2}} \right) \right)^{-1} \int_{k_n/n + z(k^{1/2}/n)}^{k_n/n} udH(u)/a_n$$

$$\geq -\frac{n^{1/2}a_n}{A_n} \left( \frac{n}{k_n} \right)^{1/2} \int_{l_n/n}^{k_n/n} udH(u)/a(l_n/n, k_n/n)$$

$$= -\frac{n^{1/2}a_n}{A_n} \left( \frac{n}{k_n} \right)^{1/2} \frac{1}{1 + x/k_n^{1/2}} \frac{1}{1 - k_n/n} \rho_n,$$

where $\rho_n$ is as in the proof of Lemma 2.3 and the formulation of the theorem. Hence (1.7) follows by (1.1), condition (III) and the proof of Lemma 2.3.

Proof of Theorem 2. Before starting the proof we note that by Theorem 3(i) in [5] the random variable $V_1(\varphi)$ is degenerate if and only if $\varphi \equiv 0$. Also, taking into account the bound (1.7), an application of the second part of Proposition 2 in [4] shows that $V_3(\psi, b, r, \alpha)$ is degenerate if and only if $b = 0$ and $\psi \equiv 0$. Since $V_1$ and $V_3$ are independent, by (2.11) we see that the limiting random variable $V(\varphi, \psi, b, r, \alpha)$ is degenerate if and only if $\varphi \equiv 0, \psi \equiv 0$, and $b = 0$. Hence if we prove the first two statements of the theorem, the third will be automatically true.

We distinguish three cases.

Case 1: For all $n_1$ large enough $b_{n_1} > 0$,

$$\lim_{n_1 \to \infty} \sup_{x < \infty} |\psi_n(x)| < \infty \quad \text{and} \quad \lim_{n_1 \to \infty} \sup_{y > 0} |\varphi_n(y)| < \infty,$$

Then by the Helly-Bray theorem we can select a subsequence $\{n_2\} \subset \{n_1\}$ such that $\psi_{n_2} \to \psi$ weakly and $\varphi_{n_2} \to \varphi$ weakly as $n_2 \to \infty$, where $\psi$ and $\varphi$ have all the usual inherent properties. By Theorem 1, along a further subsequence $\{n_3\} \subset \{n_2\}$,

$$T_{n_3} := \frac{1}{n_3^{1/2}a_{n_3}} \left\{ \sum_{i=1}^{k_{n_3}} X_{n_3+1-i, n_3} - \mu_{n_3} \right\}$$

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converges in distribution to \( \bar{V} := V(\varphi, \check{\psi}, b, r, \alpha) \), \( 0 \leq b \leq 1 \), \( 0 \leq r \leq (1 - \alpha)^{1/2} \). If \( \varphi \equiv 0 \) then \( b = 1 \), so that \( \bar{V} \) is non-degenerate. Hence by the convergence of types theorem we have (III) with some \( \alpha > 0 \), and thus (I) and (II) with \( \psi = a\check{\psi} \) and \( \varphi = a\varphi \), and \( V \) is of the form stated with \( c \) being the limit of \( (\mu_{n_n} - C_{n_n})/A_{n_n} \) as \( n \to \infty \).

**Case 2:** There exists a subsequence \( \{n_2\} \subset \{n_1\} \) such that \( b_{n_2} > 0 \) for all \( n_2 \) and

\[
\lim_{n_2 \to \infty} |\psi_{n_2}(x)| = \infty \quad \text{for some} \quad -\infty < x < \infty
\]

or

\[
\lim_{n_2 \to \infty} |\varphi_{n_2}(y)| = \infty \quad \text{for some} \quad 0 < y < \infty.
\]

First we note that by the argument at the end of the proof of Theorem 1,

\[
\limsup_{n \to \infty} |\psi_n(x)| \leq (1 - \alpha)^{-1/2} \quad \text{for all} \quad x \leq 0,
\]

and by the first part of the proof of Lemma 2.5,

\[
\limsup_{n \to \infty} |\varphi_n(y)| \leq 1 \quad \text{for all} \quad 1 \leq y < \infty.
\]

At the beginning of the present section we saw that for \( T_n \) in (2.12),

\[
T_n = R_n^{(1)} + W_n + R_n^{(2)}
= \frac{n^{1/2}}{a_n} \int_0^{1/n} G_n(u) dH(u) + \frac{n^{1/2}}{a_n} \int_{1/n}^{k_n/n} (G_n(u) - u) dH(u)
+ \frac{n^{1/2}}{a_n} \int_{k_n/n}^{U_{k_n,n}} (G_n(u) - \frac{k_n}{n}) dH(u).
\]

We have

\[
\lim_{M \to \infty} \liminf_{n \to \infty} P\{|R_n^{(i)}| < M\} > 0 \quad \text{for} \quad i = 1, 2.
\]

The case \( i = 1 \) is trivial because \( R_n^{(1)} = 0 \) if \( U_{1,n} > n^{-1} \), and hence

\[
P\{|R_n^{(1)}| < M\} \geq P\{nU_{1,n} > 1\} \to e^{-1} \quad \text{as} \quad n \to \infty,
\]

for all \( M > 0 \). The case \( i = 2 \) follows exactly as in the proof of Lemma 2.8 in [4], using (2.15) and noting only that in the present generality of having \( \alpha \geq 0 \) in (1.1) one has to replace \( c \) in the limit in relation (2.24) of [4] by \( (1 - \alpha)^{1/2} c \).

The next step is to derive from Satz 4 of Rossberg [18] again that

\[
|R_n^{(1)}| \quad \text{and} \quad |R_n^{(2)}| \quad \text{are asymptotically independent.}
\]
Since \( b_{n_2} > 0 \), we have \( \text{Var}(W_{n_2}) = 1 \) for each \( n_2 \), so that \( W_{n_2} \) is obviously stochastically bounded. Combining this fact, (2.17), (2.18) and the general Lemma 2.10 in [4], and proceeding exactly as in the proof of Lemma 2.11 in [4], we see that if the left-hand side of (1.8) is stochastically bounded then

\[
(2.19) \quad n_2^{1/2} a_{n_2} R_n^{(1)}/(n_2^{1/2} a_{n_2} \lor A_{n_2}) = O_P(1), \quad i = 1, 2,
\]

where \( x \lor y = \max(x, y) \). Thus assumption (1.8) implies (2.19).

Now we claim the following three properties:

\[
(2.20) \quad \lim_{n_2 \to \infty} n_2^{1/2} a_{n_2}/A_{n_2} = 0,
\]

\[
(2.21) \quad \limsup_{n_2 \to \infty} |\psi_{n_2}^*(x)| < \infty \quad \text{for all} \quad -\infty < x < \infty,
\]

\[
(2.22) \quad \limsup_{n_2 \to \infty} |\varphi_{n_2}^*(y)| < \infty \quad \text{for all} \quad 0 < y < \infty.
\]

Note first that by (2.15) and (2.16) the relations (2.13) and (2.14) can only occur for some \( x > 0 \) and \( 0 < y < 1 \), respectively. If (2.13) holds, we first work on the set

\[
\Omega_{n_2}(x, t) := \left\{ \frac{k_{n_2}}{n_2} + t \frac{k_{n_2}^{1/2}}{n_2} \leq U_{k_{n_2}} - \frac{|xk_{n_2}^{1/2}|}{k_{n_2}^{1/2}} \psi_{n_2}(t) \right\},
\]

where \( t \geq x \). On this, similarly as in the proof of Lemma 2.12 in [4], one has

\[
(2.23) \quad \frac{n_2^{1/2} a_{n_2} R_n^{(2)}}{n_2^{1/2} a_{n_2} \lor A_{n_2}} \leq \frac{n_2^{1/2} a_{n_2}}{n_2^{1/2} a_{n_2} \lor A_{n_2}} - \frac{|xk_{n_2}^{1/2}|}{k_{n_2}^{1/2}} \psi_{n_2}(t).
\]

Since \( \psi_{n_2}(t) \to \infty \), as \( n_2 \to \infty \), by (2.13) and since, with \( N(0, 1) \) standing for a standard normal variable,

\[
\lim_{n_2 \to \infty} P\{\Omega_{n_2}(x, t)\} = P\{N(0, 1) \geq (1 - \alpha)^{1/2}(t + x)\} > 0,
\]

we see that (2.19) and (2.23) imply (2.20) and (2.21). To get (2.22), note that on the set \( \Omega_{n_2}(y) = \{n_2 U_{1, n_2} < y\} \) we have

\[
(2.24) \quad \frac{n_2^{1/2} a_{n_2} R_n^{(1)}}{n_2^{1/2} a_{n_2} \lor A_{n_2}} \geq \frac{n_2^{1/2} a_{n_2}}{n_2^{1/2} a_{n_2} \lor A_{n_2}} \varphi_{n_2}(y).
\]

Since the right-hand side here is \( -\varphi_{n_2}^*(y) \) for all \( n_2 \) large enough and since \( P\{\Omega_{n_2}(y)\} \to 1 - e^{-y} > 0 \), as \( n_2 \to \infty \), we see that (2.19) and (2.24) imply (2.22).
If (2.14) holds then we work first on $\Omega_{n_2}(y)$ to get (2.20) and (2.22), and afterwards on $\Omega_{n_2}(x,t)$ to get (2.21).

Clearly, (2.20), (2.21) and (2.22) imply conditions (I), (II) and (III) with $\alpha = 0$ along a further subsequence of $\{n_2\}$.

Case 3: There exists a subsequence $\{n_2\} \subset \{n_1\}$ such that $b_{n_2} = 0$ for all $n_2$. In this case the left side of (1.8) equals in distribution to

$$\frac{n_2}{A_{n_2}} \int_0^{1/n_2} G_{n_2}(u) dH(u) + \frac{n_2}{A_{n_2}} \int_{k_{n_2}/n_2}^{U_{k_{n_2}/n_2}} \left( G_{n_2}(u) - \frac{k_{n_2}}{n_2} \right) dH(u)$$

$$+ \frac{\mu_{n_2} - C_{n_2}}{A_{n_2}}$$

for all such $n_2$. If we now denote the first two terms here by $R_{n_2}^{(1)}$ and $R_{n_2}^{(2)}$, respectively, then of course we still have (2.17) for $i = 1$. Since $H$ has no mass on $(1/n_2, k_{n_2}/n_2]$, we have $R_{n_2}^{(2)} = 0$ if $1/n_2 < U_{k_{n_2}/n_2} < k_{n_2}/n_2$, and we see that

$$P\{|R_{n_2}^{(2)}| < M\} \geq P\{1/n_2 < U_{k_{n_2}/n_2} < k_{n_2}/n_2\} \rightarrow \frac{1}{2}$$

as $n_2 \to \infty$, so that (2.17) is also true for $j = 2$ along $\{n_2\}$. Since (2.18) is still obviously true along $\{n_2\}$ in the present case, we obtain as in Case 2 that $R_{n_2}^{(2)} - O_p(1)$ as $n_2 \to \infty$, $j = 1, 2$. A simplified form of the corresponding argument above now yields (2.21) and (2.22). Thus, again, we obtain conditions (I), (II) and (III) with $\alpha = 0$ along a subsequence of $\{n_2\}$. The theorem is completely proved. □

In the proof of Corollary 1 we shall require the following.

**Lemma 2.7.** Let the functions $\varphi$ and $\psi$ be as in conditions (I) and (II) and consider the constants $0 \leq b < \infty$, $0 \leq \alpha < 1$ and $0 \leq r \leq (1 - \alpha)^{1/2}$. Assume that $\varphi$ satisfies (1.4) and $\psi$ satisfies (1.7). Then the random variable $V(\varphi, \psi, b, r, \alpha)$ is non-degenerate normal if and only if $\varphi \equiv 0$, $\psi \equiv 0$ and $b > 0$, in which case $V(\varphi, \psi, b, r, \alpha)$ is $N(0, b^2)$.

**Proof.** The sufficiency part is trivial. Suppose that $V(\varphi, \psi, b, r, \alpha) = V_1(\varphi) + V_3(\psi, b, r, \alpha)$, where we use (2.11), is non-degenerate normal. Since $V_1(\varphi)$ and $V_3(\psi, b, r, \alpha)$ are independent, the Cramér characterization forces both to be normal. Theorem 3 (i) in [5] says that $V_1(\varphi)$ has an infinitely divisible distribution without a normal component. Hence the only way it can be normal is when it is degenerate which again by Theorem 3 in [5] implies $\varphi \equiv 0$. Since then in fact $V_1(\varphi) = 0$, $V_3(\psi, b, r, \alpha)$ must be non-degenerate normal. Using now condition (1.7), Proposition 1 from [4] implies that this can happen only if $\psi \equiv 0$ and $b > 0$. □

**Proof of Corollary 1.** First we prove the 'if' part. Let $\{n_2\}$ be an arbitrary subsequence of $\{n_1\}$. Since conditions (I), (II) and (III) hold along $\{n_2\}$ with $A_{n_2} \equiv n_2^{1/2} a_{n_2}$, $\varphi \equiv
0, \psi \equiv 0 \text{ and } a = 1, \text{ Theorem 1 and Lemma 2.7 imply the existence of a subsequence } \{n_3\} \text{ such that}
\sum_{i=1}^{k_{n_3}} X_{n_3 + 1 - i, n_2} - \mu_{n_3} / n_3^{1/2} a_{n_2} \rightarrow_d N(0, 1).

This of course implies that the same is true along the original \{n_1\}.

Now suppose that (1.10) holds. Let again \{n_2\} \subset \{n_1\} be arbitrary. By Theorem 2 there exists a further subsequence \{n_3\} \subset \{n_2\} such that (I), (II) and (III) hold along \{n_3\} with \(A_{n_3} = A_{n_3}^*\) and appropriate functions \(\varphi\) and \(\psi\) satisfying conditions (1.4) and (1.7), respectively, and a constant \(0 \leq a < \infty\), and the distribution of \(Z\) is necessarily that of \(V(\varphi, \psi, b, r, a) + c\) with some constant \(0 \leq b \leq a\), \(0 \leq r \leq (1 - \alpha)^{1/2}\) and \(-\infty < c < \infty\). Thus by Lemma 2.7, \(\varphi \equiv 0\), \(\psi \equiv 0\) and \(b > 0\). Hence \(a > 0\), yielding that (I) and (II) hold along \{n_3\} with \(A_{n_3} = A_{n_3}^{1/2} a_{n_3}\), \(\varphi \equiv 0\) and \(\psi \equiv 0\). Since \{n_2\} was arbitrary, the same must be true along the original sequence \{n_1\}.

The proof of Corollary 2 requires some preparations concerning the asymptotic behavior of the functions \(\psi_n\) and \(\varphi_n\). First of all we note that, inverting the results of Section 2.3 in de Haan [10],

\(F \in \Delta(c)\) with \(0 < c < \infty\) if and only if \(-H(s) = s^{-c} L(s), 0 < s < 1\),
for some function \(L\) slowly varying at zero; and

\(F \in \Delta(c)\) with \(-\infty < c < 0\) if and only if \(-H(s) = A - s^{-c} L(s), 0 < s < 1\),
for some function \(L\) slowly varying at zero and some finite constant \(A\).

The following lemma is a slight extension of Lemma 2 in [6].

**Lemma 2.8.** Let \(L\) be any function defined on \((0, 1)\), bounded on compact subintervals and slowly varying at zero. Let \(\{k_n\}\) be a not-necessarily integer-valued sequence satisfying (1.1) and \(\{l_n\}\) be any sequence of positive numbers such that \(k_n/l_n \to \infty\) as \(n \to \infty\). Then for any \(0 < \nu < \infty\) we have

\[
\lim_{n \to \infty} \left\{ \left( \frac{k_n}{n} \right)^{-\nu} L \left( \frac{k_n}{n} \right) \right\} / \left\{ \left( \frac{l_n}{n} \right)^{-\nu} L \left( \frac{l_n}{n} \right) \right\} = 0.
\]

**Proof.** The two cases \(\alpha > 0\) and \(\alpha = 0\) follow from properties 1 and 2 of Corollary 1.2.1 in [10], respectively.

**Lemma 2.9.** If \(F \in \Delta(c)\) for some \(c \neq 0\) and (1.1) holds then with the respective \(L\) functions from (2.25) and (2.26) we have the asymptotic equalities

\[
\sigma(1/n, k_n/n) \sim K_c(1/n)^{1/2} L(1/n), \text{ if } c > 1/2 \text{ and } \alpha \geq 0;
\]

\[
\sigma(1/n, k_n/n) \sim \left( \int_{1/n}^{k_n/n} u^{-1} L^2(u) du \right)^{1/2}, \text{ if } c = 1/2 \text{ and } \alpha = 0; \text{ and}
\]

\[
\sigma(1/n, k_n/n) \sim K_c(k_n/n)^{1/2} L(k_n/n), \text{ if } c < 1/2, c \neq 0, \text{ and } \alpha = 0,
\]

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as \( n \to \infty \), where

\[
K_c = \begin{cases} 
(2c^2/((1-c)(1-2c)))^{1/2} & , \text{if } c < 1/2, \ c \neq 0, \\
(2c/(2c - 1))^{1/2} & , \text{if } c > 1/2.
\end{cases}
\]

Proof. The case \( c > 1/2 \) can be inferred from Lemma 1 in [6] and Lemma 2.8. The cases \( c = 1/2 \) and \( 0 < c < 1/2 \) are proven in Lemma 6 in [7], and the same proof given there for \( 0 < c < 1/2 \) also works for \( c < 0 \). \( \square \)

**Lemma 2.10.** Whenever \( F \in \Delta(0) \) and (1.1) holds with \( \alpha = 0 \),

\[
\lim_{s \to 0} \frac{s^{1/2} \{H(\lambda s) - H(s)\}}{\sigma(0,s)} = 2^{-1/2} \log \lambda \quad \text{for all } \quad 0 < \lambda < \infty
\]

and

\[
\sigma(0,s) = s^{1/2}l(s) \quad \text{for some function } l \text{ slowly varying at zero.}
\]

Proof. Assertion (2.27) follows directly from Lemmas 4 and 6, while (2.28) from Lemmas 2 and 6 of Lo [13]. \( \square \)

**Lemma 2.11.** Assume \( F \in \Delta(c) \) for some \( c \) and that (1.1) holds. If \( \alpha > 0 \), then with the function \( \psi_\alpha \) defined in (1.13), for all \( x \in R \),

\[
\lim_{n \to \infty} \psi_n(x) = \begin{cases} 
0 & , \text{if } c > 1/2, \ \text{or } c = 1/2 \text{ and } \sigma(0,\alpha) = \infty, \\
\psi_\alpha(x) & , \text{if } c < 1/2, \ \text{or } c = 1/2 \text{ and } \sigma(0,\alpha) < \infty.
\end{cases}
\]

while if \( \alpha = 0 \) then \( \psi_n(x) \to 0 \) for all \( x \in R \) as \( n \to \infty \), for any \( c \).

Proof. The case \( \alpha > 0 \) follows directly from (1.14) and the definition of \( \psi_n \).

Consider now \( \alpha = 0 \). In this case we claim that for all \( 0 < \lambda < \infty \) the following limiting relations hold which clearly imply the second statement:

\[
\lim_{n \to \infty} \frac{k_n^{1/2}\{H(\lambda \frac{k_n}{n}) - H(\frac{k_n}{n})\}}{n^{1/2}a_n} = \begin{cases} 
K_c^{-1}(1-\lambda^{-c})\text{sign}(c) & , \text{if } c < 1/2, \ c \neq 0, \\
2^{-1/2} \log \lambda & , \text{if } c = 0, \\
0 & , \text{if } c \geq 1/2.
\end{cases}
\]

Here the cases \( c < 0 \) and \( 0 < c < 1/2 \) follow from (2.26) and (2.25) and the third statement of Lemma 2.9, the case \( c = 1/2 \) follows by (2.25), the second statement of Lemma 2.9 and by an application of Lemma 4 in [7], and the case \( c > 1/2 \) follows from (2.25) and the first statement of Lemma 2.9 via an application of Lemma 2.8. Finally, the case \( c = 0 \) will follow directly from (2.27) of Lemma 2.10 once we can show that

\[
(2.29) \quad \text{ when } c = 0, \quad \sigma(1/n,k_n/n) \sim \sigma(0,k_n/n) \quad \text{as } \ n \to \infty.
\]
But this is immediate from the inequalities $\sigma^2(0, k_n/n) \geq \sigma^2(1/n, k_n/n) \geq \sigma^2(0, k_n/n) - \sigma^2(0, 1/n)$ and (2.28) of Lemma 2.10 via another application of Lemma 2.8.

**LEMMA 2.12.** If $F \in \Delta(c)$ for some $c$ and (1.1) holds then

\[ \lim_{n \to \infty} \varphi_n(y) = \begin{cases} K_c^{-1}(1 - y^{-c}) & \text{for all } y > 0, \text{ if } c > 1/2, \\ 0 & \text{for all } y > 0, \text{ if } c \leq 1/2. \end{cases} \]

**Proof.** The case $c > 1/2$ follows directly from (2.25) and the first statement of Lemma 2.9 for any $0 \leq \alpha < 1$.

In the case of $c \leq 1/2$, we first consider the subcase $\alpha = 0$. Then for $c = 1/2$ the assertion of the lemma follows from (2.25), the second statement of Lemma 2.9 and Lemma 4 in [7]. For $c < 1/2$, $c \neq 0$, the assertion follows by (2.25), (2.26), the third statement of Lemma 2.9 and an application of Lemma 2.8. When $c = 0$, the assertion follows from (2.29) and the two statements of Lemma 2.10, via one more application of Lemma 2.8.

The subcase $\alpha > 0$ of the case $c \leq 1/2$ follows from the former subcase when $\alpha = 0$ if we only notice that for $a_n = \sigma(1/n, k_n/n)$ in the denominator of $\varphi_n$ we have $a_n \geq \sigma(1/n, m_n/n)$ for all large enough $n$ for any sequence $\{m_n\}$ such that $m_n \to \infty$ and $m_n/n \to 0$ as $n \to \infty$. \qed

**Proof of Corollary 2.** The boundary case $c = 1/2$ and $\sigma(0, \alpha) = \infty$ follows directly from Lemmas 2.11 and 2.12 combined with Corollary 1.

Next, when $c < 1/2$, or $c = 1/2$ and $\sigma(0, \alpha) < \infty$, one notes that for $r_n$ in the proof of Lemma 2.3 or in the formulation of Theorem 1 and for $r_\alpha$ in the definition of $M(\alpha)$ we always have $r_n \to r_\alpha$ as $n \to \infty$ along the whole sequence $\{n\}$ of the positive integers. Thus by Lemmas 2.11 and 2.12 and Theorem 1, applied with $A_n \equiv n^{1/2}a_n$, $\psi \equiv \psi_0$, where $\psi_0 \equiv 0$, $\varphi \equiv 0$, and $\alpha = 1$, for any subsequence $\{n_1\} \subset \{n\}$ there exists a further subsequence $\{n_2\} \subset \{n_1\}$ such that the statement of Corollary 2 holds along $\{n_2\}$. Hence it holds along the whole sequence $\{n\}$.

Finally, when $c > 1/2$, it can be shown using Lemma 1 in [6] and Lemma 2.8 that for any sequence $\{l_n\}$ of positive numbers such that $l_n \to \infty$ and $k_n/l_n \to \infty$ as $n \to \infty$ we have $\sigma(l_n/k_n/n) \sim K_c(l_n/n)^{1/2 - c}L(l_n/n)$ as an extension of the first statement of Theorem 2.9, which by (2.25) and a final application of Lemma 2.8 yields

\[ \sigma(l_n/k_n/n)/\sigma(1/n, k_n/n) \to 0 \quad \text{as } n \to \infty. \]

Thus by Lemmas 2.11 and 2.12 and Theorem 1, applied with $A_n \equiv n^{1/2}a_n$, $\psi \equiv 0$, $\varphi(y) = K_c^{-1}(1 - y^{-c})$ and $b = 0$, we obtain that for every subsequence $\{n_1\} \subset \{n\}$ there exists a further subsequence $\{n_2\} \subset \{n_1\}$ such that the statement of Corollary 2 holds along $\{n_2\}$. This of course gives the same statement along the whole sequence $\{n\}$. \qed

3. Discussion of the Conditions and Examples. The two propositions below provide equivalent forms for conditions (I) and (II), respectively. The first of these forms is
a probabilistic interpretation of the respective condition. The second one, following from the first, is a reformulation of the condition in terms of the underlying distribution function \( F \). That the latter reformulations might be possible was suggested to us by a referee.

For ease of notation we set
\[
c_n = -H \left( \frac{1}{n} \right) = Q \left( \left( 1 - \frac{1}{n} \right) - \right) \quad \text{and} \quad c_n(k_n) = -H \left( \frac{k_n}{n} \right) = Q \left( \left( 1 - \frac{k_n}{n} \right) - \right),
\]
and for a non-decreasing function \( h \) defined on a subset \( S \) of the real line \( \mathbb{R} \) we define its left-continuous inverse by
\[
h^{-1}(x) = \inf\{ s \in S : h(s) \geq x \}, \quad x \in \mathbb{R},
\]
where we agree that the infimum of the empty set is \(+\infty\). Let \( Z \) denote a standard normal random variable and \( Y \) denote an exponential random variable with mean 1, and let \( \{k_n\} \) be a sequence of integers satisfying (1.1) with \( \alpha = 0 \).

**Proposition 1.** Let \( \psi \) be a non-decreasing, left-continuous function on \( (-\infty, \infty) \) such that \( \psi(0) \leq 0 \) and \( \psi(0+) \geq 0 \). Condition (I) holds along \( \{n_1\} \) with this \( \psi \) if and only if
\[
\frac{k_n^{1/2}}{A_{n_1}} \left\{ X_{n_1+1-k_n,n_1} - c_{n_1}(k_n) \right\} \rightarrow_D -\psi(Z) \quad \text{as} \quad n_1 \rightarrow \infty,
\]
which happens if and only if
\[
\frac{1}{k_n^{1/2}} \left( n_1 \{1 - F(c_{n_1}(k_n)) - x k_n^{-1/2} A_{n_1} \} - k_{n_1} \right) \rightarrow \psi^{-1}(x) \quad \text{as} \quad n_1 \rightarrow \infty
\]
for every continuity point \( x \) of the distribution function of \(-\psi(Z)\).

**Proposition 2.** Let \( \varphi \) be a non-decreasing left-continuous function on \((0, \infty)\) such that \( \varphi(1) \leq 0 \) and \( \varphi(1+) \geq 0 \). Condition (II) holds along \( \{n_1\} \) with this \( \varphi \) if and only if
\[
\frac{1}{A_{n_1}} \left\{ X_{n_1,n_1} - c_{n_1} \right\} \rightarrow_D -\varphi(Y) \quad \text{as} \quad n_1 \rightarrow \infty,
\]
which happens if and only if
\[
n_1 \{1 - F(c_{n_1} - x A_{n_1})\} \rightarrow \varphi^{-1}(x) \quad \text{as} \quad n_1 \rightarrow \infty
\]
for every continuity point \( x \) of the distribution function of \(-\varphi(Y)\).

**Proof.** First we consider Proposition 1. For the uniform \((0, 1)\) order statistics \( U_{1,n} \leq \ldots \leq U_{n,n} \) as in (1.3) we introduce
\[
W_{k_n,n} = \frac{n}{k_n^{1/2}} \left\{ U_{k_n,n} - \frac{k_n}{n} \right\}, \quad n \geq 1.
\]
Note that by (1.3)

\[ X_{n+1-k_n},n = P(1 - U_{k_n},n) = - II(U_{k_n},n), \]

and hence, as \( n \to \infty \),

\[ \frac{k_n^{1/2}}{A_n} \{ X_{n+1-k_n},n - c_n(k_n) \} = P \left\{ \frac{n^{1/2}a_n}{A_n} \psi_n(W_{k_n},n) + o_P(1) \right\}. \]

On the other hand,

\[ \sup_B \left| P \left\{ - \frac{n^{1/2}a_n}{A_n} \psi_n(W_{k_n},n) \in B \right\} - P \left\{ - \frac{n^{1/2}a_n}{A_n} \psi_n(Z) \in B \right\} \right| \]

\[ \leq \sup_B \left| P\{W_{k_n},n \in B\} - P\{Z \in B\} \right|, \]

where the supremum is taken over all Borel sets \( B \) on the real line, and this upper bound goes to zero as \( n \to \infty \) by Proposition 2.10 of Reiss [17], where earlier references concerning this result can also be found. Hence (3.1) holds if and only if

\[ \psi_{n_1}^*(Z) = - \frac{n_1^{1/2}a_{n_1}}{A_{n_1}} \psi_{n_1}(Z) \to \psi(Z) \]

almost surely as \( n_1 \to \infty \), and it is easily checked that this happens if and only if condition (1) holds along \{n_1\}.

The equivalence of (3.1) and (3.2) follows by a standard argument as given on p. 654 of Watts, Rootzén, and Leadbetter [21].

The proof of the first statement of Proposition 2 is the same as above upon noting that

\[ \frac{1}{A_n} \{ X_{n,n} - c_n \} = P \left\{ - \frac{n^{1/2}a_n}{A_n} \varphi_n(nU_{1,n}) + o_P(1) \right\} \]

and, with the supremum taken again over all Borel sets \( B \) on the line,

\[ \sup_B |P\{nU_{1,n} \in B\} - P\{Y \in B\}| \to 0 \quad \text{as} \quad n \to \infty. \]

The latter follows from Theorem 2.6 of Reiss [17].

The equivalence of (3.3) and (3.4) is an easy exercise well known in extreme value theory.

We note that since \( \psi \) and \( \varphi \) can only be constants if they are zero, the limits in (3.1) and (3.3) are non-degenerate if and only if \( \psi \equiv 0 \) and \( \varphi \equiv 0 \), respectively. If \( \psi \equiv 0 \), then we have (3.2) with

\[ \psi^{-1}(x) = \begin{cases} -\infty, & \text{if } x < 0, \\ \infty, & \text{if } x > 0, \end{cases} \]

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and if $\varphi \equiv 0$, then we have (3.4) with

$$
\varphi^{-1}(x) = \begin{cases} 
0, & \text{if } x < 0, \\
\infty, & \text{if } x > 0.
\end{cases}
$$

Our first example shows that limiting distributions in Theorem 1 can arise along subsequences of $\{n\}$ with $\varphi \not\equiv 0$, $\psi \not\equiv 0$ and $b > 0$.

**EXAMPLE 1.** Let $1 \leq k_n \leq n$ be integers such that $k_n \to \infty$ and $k_n/n \to 0$ as $n \to \infty$, let $\{n_1, n_2, \ldots\}$ be a subsequence of $\{n\}$ and $\{d_{n_j}, j \geq 1\}$ be a sequence of arbitrary positive numbers. For $j$ large enough to make $k_{n_j}/n_j < 1/2$, consider

$$
H(s) := \begin{cases} 
H \left( \frac{k_{n_j}}{n_j} \right), & \text{if } \frac{k_{n_j}}{n_j} - \frac{k_{n_j}^{1/2}}{4n_j} < s \leq \frac{k_{n_j}}{n_j} - \frac{k_{n_j}^{1/2}}{4n_j}, \\
H \left( \frac{k_{n_j}}{n_j} \right) - d_{n_j}, & \text{if } \frac{1}{2n_j} < s \leq \frac{k_{n_j}}{n_j} - \frac{k_{n_j}^{1/2}}{4n_j}, \\
H \left( \frac{1}{k_{n_j}} \right) - k_{n_j}^{1/2} d_{n_j}, & \text{if } \frac{1}{n_j k_{n_j}^{1/2}} < s \leq \frac{1}{2n_j}.
\end{cases}
$$

Clearly, a quantile function $Q$ with this corresponding $H$ function exists as long as

$$
\frac{2k_{n_{j+1}}}{n_{j+1}} \leq \frac{1}{n_j k_{n_{j+1}}},
$$

that is

$$
k_{n_{j+1}}^2 \leq \frac{1}{2n_j}
$$

for all large enough $j$. (This is the case, for example, if $n_j = [2^j \log j]$ and $k_{n_j} = [2^{3^{-1} \log j}]$.)

Noting the asymptotic equality

$$
\sigma^2 \left( \frac{1}{n_j}, \frac{k_{n_j}}{n_j} \right) = \left\{ \left( \frac{k_{n_j}}{n_j} - \frac{k_{n_j}^{1/2}}{4n_j} \right)^2 - \left( \frac{k_{n_j}}{n_j} - \frac{k_{n_j}^{1/2}}{4n_j} \right)^2 \right\} \sim \frac{k_{n_j}^2 d_{n_j}^2}{n_j} \quad \text{as } j \to \infty,
$$

it is easy to show that

$$
\lim_{j \to \infty} \psi_{n_j}(x) = \psi^*(x) := \begin{cases} 
-1, & \text{if } -\infty < x \leq -1/4, \\
0, & \text{if } -1/4 < x < \infty,
\end{cases}
$$

and that

$$
\lim_{j \to \infty} \sigma(l_{n_j}/k_{n_j}, n_j)/\sigma(1/n_j, k_{n_j}/n_j) = 1
$$

for any sequence $\{t_{n_j}\}$ such that $t_{n_j} \to \infty$ and $l_{n_j}/k_{n_j} \to 0$ as $j \to \infty$. Moreover, $r_{n_j} \to 1$ as $j \to \infty$. Thus we have

$$
k_{n_j}^{-1/2} d_{n_j}^{-1} \left\{ \sum_{i=1}^{k_{n_j}} X_{n_j+1-i, n_j} - \mu_{n_j} \right\} \to \mathcal{D} \left( \varphi^*, \psi^*, 1, 1, 0 \right), \quad \text{as } j \to \infty,
$$

as

$$
\lim_{j \to \infty} \sigma(l_{n_j}/k_{n_j}, n_j)/\sigma(1/n_j, k_{n_j}/n_j) = 1
$$

for any sequence $\{t_{n_j}\}$ such that $t_{n_j} \to \infty$ and $l_{n_j}/k_{n_j} \to 0$ as $j \to \infty$. Moreover, $r_{n_j} \to 1$ as $j \to \infty$. Thus we have

$$
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$$

as

$$
\lim_{j \to \infty} \sigma(l_{n_j}/k_{n_j}, n_j)/\sigma(1/n_j, k_{n_j}/n_j) = 1
$$

for any sequence $\{t_{n_j}\}$ such that $t_{n_j} \to \infty$ and $l_{n_j}/k_{n_j} \to 0$ as $j \to \infty$. Moreover, $r_{n_j} \to 1$ as $j \to \infty$. Thus we have

$$
k_{n_j}^{-1/2} d_{n_j}^{-1} \left\{ \sum_{i=1}^{k_{n_j}} X_{n_j+1-i, n_j} - \mu_{n_j} \right\} \to \mathcal{D} \left( \varphi^*, \psi^*, 1, 1, 0 \right), \quad \text{as } j \to \infty,
$$

as

$$
\lim_{j \to \infty} \sigma(l_{n_j}/k_{n_j}, n_j)/\sigma(1/n_j, k_{n_j}/n_j) = 1
$$

for any sequence $\{t_{n_j}\}$ such that $t_{n_j} \to \infty$ and $l_{n_j}/k_{n_j} \to 0$ as $j \to \infty$. Moreover, $r_{n_j} \to 1$ as $j \to \infty$. Thus we have

$$
k_{n_j}^{-1/2} d_{n_j}^{-1} \left\{ \sum_{i=1}^{k_{n_j}} X_{n_j+1-i, n_j} - \mu_{n_j} \right\} \to \mathcal{D} \left( \varphi^*, \psi^*, 1, 1, 0 \right), \quad \text{as } j \to \infty,
where $V(\varphi^*, \psi^*, 1, 1, 0) = N(1/2) + \max(Z_1, 1/4)$ and, by simple computation, $\mu_{n_j} = -k_{n_j}H(k_{n_j}/n_j) + d_{n_j}(k_{n_j} - k_{n_j}^{1/2}/4)$.

We emphasize that the numbers $d_{n_j} > 0$ determining the jump sizes of the quantile function $Q$ are arbitrary in the above example. Therefore, they can be chosen so that the underlying distribution has moments of arbitrarily high order.

The second example relates the convergence in distribution of extreme sums along subsequences to that of the whole sum.

**EXAMPLE 2.** Suppose that $F(0) = 0$ and there exist a subsequence $\{n_1, n_2, \ldots\} \subset \{n\}$ and constants $A_{n_j} > 0$ and $B_{n_j}$ such that

(3.5) \[ A_{n_j}^{-1} \left\{ \sum_{i=1}^{n_j} X_i - B_{n_j} \right\} \rightarrow_d W \text{ as } j \to \infty, \]

where $W$ is an infinitely divisible random variable with a non-degenerate non-normal component. It is shown in [5] that we can assume that

\[ B_{n_j} := n_j \int_0^{1-1/n_j} Q(s)ds. \]

Note that necessarily $\text{Var}(X) = \infty$, so that we must have

(3.6) \[ \lim_{j \to \infty} A_{n_j}/n_j^{1/2} = \infty. \]

By Theorem 5 in [4], for each $0 < \beta < 1$,

\[ \sum_{i=1}^{[\beta n]} X_{i,n} - n \int_0^{[\beta n]/n} Q(s)ds = O_P(n^{1/2}) \text{ as } n \to \infty. \]

Thus by (3.6), for each $0 < \beta < 1$,

\[ A_{n_j}^{-1} \left\{ \sum_{i=1}^{[\beta n_j]} X_{i,n_j} - n_j \int_{[\beta n_j]/n_j}^{1-1/n_j} Q(s)ds \right\} \rightarrow_d W \text{ as } j \to \infty. \]

Hence by a simple diagonal selection procedure we can find a sequence $\{k_{n_j}\}$ such that $k_{n_j} \to \infty$ and $k_{n_j}/n_j \to 0$ and

\[ A_{n_j}^{-1} \left\{ \sum_{i=1}^{k_{n_j}} X_{n_j+1-i,n_j} - n_j \int_{1-k_{n_j}/n_j}^{1-1/n_j} Q(s)ds \right\} \rightarrow_d W \text{ as } j \to \infty. \]

Our last example connects conditions (I), (II), and (III) with the notion of stochastic compactness for sums and maxima and provides a relatively general situation when the
scaling factor $A_n$ is the same for extreme sums, whole sums, and maxima. We say that $F$ is stochastically compact, written $F \in SC$, if there are sequences $A_n > 0$ and $B_n$ such that for every subsequence $\{m_j\} \subset \{n\}$ there exists a further subsequence $\{n_j\} \subset \{m_j\}$ such that (3.5) holds with a non-degenerate $W$. We call $\{X_{n,n}\}$ stochastically compact if there exists a sequence $C_n > 0$ such that for every subsequence $\{n'\} \subset \{n\}$ there exists a further subsequence $\{n''\} \subset \{n'\}$ such that $X_{n'',n''}/C_n''$ converges in distribution to a non-degenerate random variable as $n'' \to \infty$.

**Example 3.** Assume that $F(0) = 0$ and $F$ is not in the domain of partial attraction of a normal law. Corollary 12 in [5] says that $F \in SC$ if and only if $X_{n,n}$ is stochastically compact. In this case one can choose, according to the same corollary, $C_n = A_n = a(n) = n^{1/2}\sigma\left(\frac{1}{n},1 - \frac{1}{n}\right)$, where $\sigma(\cdot,\cdot)$ is defined below (1.3). Also, for such an $F$ it is readily inferred from Corollary 10 in [5] that $F \in SC$ if and only if

$$\limsup_{s \to 0} s^{1/2} Q(1 - \lambda s)/\sigma(s,1-s) < \infty \quad \text{for all} \quad 0 < \lambda < \infty.$$ 

Combining these two facts with the simple observation that $a(n) \geq n^{1/2}a_n$, $n \geq 1$, for any sequence $\{k_n\}$ satisfying (1.1) with $\alpha = 0$ or $\alpha > 0$, we easily see that whenever such an $F$ is in $SC$, each subsequence of $\{n\}$ contains a further subsequence along which conditions (I), (II), and (III) are simultaneously satisfied with $A_n = a(n)$ for some $\psi$, $\varphi$, and $0 \leq \alpha < \infty$. In fact the stochastic compactness of the maxima forces $\varphi \neq 0$.

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