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# THE DOMAIN OF PARTIAL ATTRACTION OF A POISSON LAW

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Groshev gave a characterization of the union of domains of partial attraction of all Poisson laws in 1941. His classical condition is expressed by the underlying distribution function and disguises the role of the mean $\lambda$ of the attracting distribution. In the present paper we start out from results of the recent 'probabilistic approach' and derive characterizations for any fixed $\lambda > 0$ in terms of the underlying quantile function. The approach identifies the portion of the sample that contributes the limiting Poisson behavior of the sum, delineates the effect of extreme values, and leads to necessary and sufficient conditions all involving $\lambda$. It turns out that the limiting Poisson distributions arise in two qualitatively different ways depending upon whether $\lambda > 1$ or $\lambda < 1$. A concrete construction, illustrating all the results, also shows that in the boundary case when $\lambda = 1$ both possibilities may occur.

KEY WORDS: Poisson law; domain of partial attraction; quantiles.

## 1. INTRODUCTION, THE RESULTS, AND DISCUSSION

Let $X_1, X_2, \ldots$ be independent random variables with the common distribution function $F(x) = P\{X \leq x\}$, $x \in \mathbb{R}$, and corresponding left-continuous inverse distribution or quantile function

$$Q(s) = \inf\{x : F(x) \geq s\},\ 0 < s \leq 1,\ Q(0) = Q(0+).$$

We say that $F$ is in the domain of partial attraction of the Poisson distribution with mean $\lambda > 0$ if there exist a subsequence $\{n'\}$ of the sequence $\{n\}$ of the positive integers and

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normalizing and centering constants $A_{n'} > 0$ and $C_{n'} \in \mathbb{R}$ such that

$$\frac{1}{A_{n'}}\left\{\sum_{j=1}^{n'} X_j - C_{n'}\right\} \xrightarrow{\mathcal{D}} Y_\lambda \quad \text{as} \quad n' \to \infty, \tag{1.1}$$

where $\xrightarrow{\mathcal{D}}$ denotes convergence in distribution and $Y_\lambda$ is a Poisson random variable with mean $\lambda$, that is,

$$P\{Y_\lambda = k\} = \frac{\lambda^k}{k!}e^{-\lambda},\ k = 0, 1, 2, \ldots .$$

In this case we shall write $F \in D_p$ (Poisson ($\lambda$)). It is well known that $\{n'\}$ cannot be the whole sequence $\{n\}$ in (1.1) because only stable distributions have non-empty domains of attraction (cf. Gnedenko and Kolmogorov[4] or Csörgő, Haeusler, and Mason[2], abbreviated as CsHM from here on). On the other hand, every infinitely divisible distribution has a non-empty domain of partial attraction by a basic theorem of Khinchin (Gnedenko and Kolmogorov[4], p. 184) and hence $D_p$ (Poisson($\lambda$)) is not void. Furthermore, the Poisson distribution is the most important infinitely divisible distribution in the sense that the class of all infinitely divisible distributions is the closure of the distributions of random variables of the form $c_1 Y_{\lambda_1} + \ldots + c_m Y_{\lambda_m}$, where $c_1, \ldots, c_m$ are constants and $Y_{\lambda_1}, \ldots, Y_{\lambda_m}$ are independent, and closure is meant with respect to weak convergence (Gnedenko and Kolmogorov[4], pp. 74-75). Therefore, the problem of the characterization of $D_p$ (Poisson ($\lambda$)) has a distinctive theoretical appeal.

Starting out from the classical general criterion of convergence to an infinitely divisible distribution that involves a condition to set the variance of the normal component and two conditions to set the Lévy measures of the canonical Lévy form of the characteristic function of the limiting distribution (Gnedenko and Kolmogorov[4], p. 124), Groshev[5] has proved that

$$F \in \bigcup_{\lambda > 0} D_p(\text{Poisson}(\lambda))$$

if and only if

$$\liminf_{h\to\infty} \int_{|x-1|>\varepsilon} \frac{x^2}{1+x^2}dF(hx) \bigg/ \int_{|x-1|<\varepsilon} dF(hx) = 0 \quad \text{for any} \quad \varepsilon > 0.$$

While nice lookong, it is not immediate to see in what ways this condition (cited also in Gnedenko and Kolmogorov[4], p. 190) restricts F.

The recent study CsHM[2] of a 'probababilistic approach' to the problem of convergence of centered and normalized sums of the form in (1.1) revealed that whatever infinitely divisible random variable we have as a limit in (1.1), there are two basic possibilities concerning the size of the normalizing factor $A_{n'}$. One is when, informally speaking, $A_{n'}$ is comparable to

$$a(n') = \sqrt{n'}\sigma(1/n'), \tag{1.2}$$

where the 'truncated variance' function $\sigma^2(\cdot)$ is defined as

$$\sigma^2(s) = \int_s^{1-s}\int_s^{1-s}(u \wedge v - uv)dQ(u)dQ(v), \quad 0 < s < \frac{1}{2}, \tag{1.3}$$

with $u \wedge v = \min(u, v)$, while the other is when a sequence $A_{n'}$ diverging to infinity faster than $a(n')$ is needed, that is when $a(n')/A_{n'} \to 0$ as $n' \to \infty$. For example, it is shown in CsHM[2] that for stochastically compact $F's$ the correct normalizing sequence is always $\{a(n')\}$, but a rather complicated construction also shows that the second possibility also occurs. While the program of characterizing $D_p$ (Poisson $(\lambda)$) within the probabilistic approach of CsHM[2] has some interest in itself, and in fact requires some augmentations of the original theory given in Cs[1], one of our primary motivations for the research reported here was to see whether $A_{n'} = a(n')$ is always sufficient in (1.1) or not.

Let $X_{1,n} \leq \ldots \leq X_{n,n}$ denote the order statistics based on the sample $X_1, \ldots, X_n$. The probabilistic approach in CsHM[2] generally allows to see which portions of the sum $\sum_{j=1}^{n'} X_{j,n'}$ contribute the ingredients of the limiting infinitely divisible law or do not contribute anything at all. At the same time, it also delineates the effect of extreme values. In order to cover these in the present situation, we need some more notation.

Let $E_1, E_2, \ldots$ be independent random variables having the exponential distribution with mean 1 and consider the standard left-continuous Possion process

(1.4) $$N(u) = \sum_{j=1}^{\infty} I(S_j < u), \quad u \geq 0,$$

with jump-points $S_j = E_1 + \ldots + E_j$, $j = 1, 2, \ldots$, where $I(\cdot)$ is the indicator function. Also, set

(1.5) $$V_k(\lambda) = \sum_{j=k+1}^{\infty} I(S_j < \lambda) + \min(k+1, \lambda) - \lambda, \ \lambda > 0; k = 0, 1, 2, \ldots,$$

so that we have the distributional equality

(1.6) $$V(\lambda) := V_0(\lambda) \stackrel{\mathcal{D}}{=} Y_\lambda + \min(1, \lambda) - \lambda.$$

For a given $\lambda > 0$, we finally introduce

(1.7) $$r_1(\lambda) = \min\{r : r \text{ integer and } r\lambda > 1\},$$

so that $r_1(\lambda) = 1$ whenever $\lambda > 1$ and $r_1(1) = 2$.

Our first result, connecting the special Poisson situation with the general theory in CsHM[2] and Cs[1] , is the following.

**THEOREM 1.** *$F \in D_p$ (Poisson ($\lambda$)) if and only if for each $r \geq r_1(\lambda)$ there exists a genuine subsequence $\{n'\}$ of the positive integers such that*

(1.8) $$\frac{1}{a(rn')} Q(\frac{s}{n'}) \to 0 \quad \textit{for each} \quad s > 0,$$

(1.9) $$\frac{1}{a(rn')} Q(1 - \frac{s}{n'}) \to \begin{cases} \alpha_r, & \textit{if } 0 < s < \lambda, \\ 0, & \textit{if } s > \lambda, \end{cases}$$

*as $n' \to \infty$, where $\alpha_r > 0$ is some constant, and*

(1.10a) $$\lim_{h\to\infty} \limsup_{n'\to\infty} \frac{\sigma(h/n')}{\sigma(1/rn')} = \lim_{h\to\infty} \limsup_{n'\to\infty} \frac{\sqrt{n'}\sigma(h/n')}{a(rn')} = 0.$$

*If conditions (1.8), (1.9) and (1.10a) are satisfied, then*

$$(1.11)\qquad \frac{1}{a(rn')}\left\{\sum_{j=1}^{n'} X_j - n'\int_{1/rn'}^{1-1/rn'} Q(u)du\right\} \xrightarrow{\mathcal{D}} \alpha_r\left(Y_\lambda + \frac{1-r\lambda}{r}\right)$$

*as $n' \to \infty$, and there exists a subsequence $\{n''\} \subset \{n'\}$ such that*

$$(1.12)\qquad \frac{1}{a(rn'')}\left\{\sum_{j=1}^{n''} X_j - n''\int_{1/n''}^{1-1/n''} Q(u)du\right\} \xrightarrow{\mathcal{D}} \alpha_r V(\lambda)$$

*and, furthermore, there exists a sequence $\{l_{n''}\}$ of positive integers such that $l_{n''} \to \infty$, $l_{n''}/n'' \to 0$ and for each fixed integer $k \geq 0$,*

$$(1.13)\qquad \frac{1}{a(rn'')}\left\{\sum_{j=1}^{n''-l_{n''}} X_{j,n''} - n''\int_{1/n''}^{1-(l_{n''}+1)/n''} Q(u)du\right\} \xrightarrow{\mathrm{P}} 0$$

*and*

$$(1.14)\qquad \frac{1}{a(rn'')}\left\{\sum_{j=n''-l_{n''}+1}^{n''-k} X_{j,n''} - n''\int_{1-(l_{n''}+1)/n''}^{1-(k+1)/n''} Q(u)du\right\} \xrightarrow{\mathcal{D}} \alpha_r V_k(\lambda)$$

*as $n'' \to \infty$.*

We note that condition (1.10a) may be replaced by

$$(1.10\mathrm{b})\qquad \lim_{h\to\infty}\liminf_{n'\to\infty}\frac{\sigma(h/n')}{\sigma(1/rn')} = \lim_{h\to\infty}\liminf_{n'\to\infty}\frac{\sqrt{n'}\sigma(h/n')}{a(rn')} = 0$$

Under (1.8), (1.9) and (1.10b) we generally have (1.11) only along a subsequence $\{n''\} \subset \{n'\}$ already, and (1.12), (1.13) and (1.14) hold along a further subsequence $\{n'''\} \subset \{n''\}$. This will be straightforward to see from the proof given in Section 2, where an analysis will lead us to the following main result of the paper.

**THEOREM 2.** *There exists a subsequence $\{n'\}$ of the positive integers such that conditions (1.8), (1.9) and (1.10a) hold, and hence $F \in D_p$ (Poisson ($\lambda$)), if and only if*

$$(1.15)\qquad \liminf_{s\downarrow 0}\frac{\sigma^2(\varepsilon s) + \sigma^2((\lambda+\varepsilon)s) + sQ^2(\varepsilon s)}{\sigma^2((\lambda-\varepsilon)s)} = 1 \quad \text{for every } 0 < \varepsilon \leq \frac{\lambda}{2}.$$

Using the fact that $\sigma^2(\varepsilon s)/\sigma^2((\lambda-\varepsilon)s) \geq 1$ for any $0 < \varepsilon \leq \lambda/2$ and $s > 0$ for which $(\lambda-\varepsilon)s < 1/2$, and hence that this liminf condition is equivalent to three liminf conditions as in (1.18)-(1.20) below, the monotonicity of the functions $\sigma(\cdot)$ and $Q(\cdot)$ easily imply that (1.15) is equivalent to

$$(1.16) \qquad \liminf_{n\to\infty} \frac{\sigma^2(\varepsilon/n) + \sigma^2((\lambda+\varepsilon)/n) + Q^2(\varepsilon/n)/n}{\sigma^2((\lambda-\varepsilon)/n)} = 1 \quad \text{for every } 0 < \varepsilon \leq \frac{\lambda}{2}.$$

Using again the fact that for any $n > 3\lambda$ and $0 < \varepsilon \leq \lambda/2$ this whole ratio is never smaller than 1, and for each $n$ large enough it is non-increasing function of $\varepsilon$ on $(0, \lambda/2]$ whenever $Q(\cdot)$ is negative near enough to 0 (otherwise $Q^2(\varepsilon/n)/n \to 0$ as $n \to \infty$ and hence the third term in the numerator can completely be neglected) a trivial modification of the argument in the proof of Corollary 2 in CsHM[(2)] shows that condition (1.16) in turn holds if and only if there exists a subsequence $\{n'\}$ of the positive integers such that

$$(1.17) \qquad \lim_{n'\to\infty} \frac{\sigma^2(\varepsilon/n') + \sigma^2((\lambda+\varepsilon)/n') + Q^2(\varepsilon/n')/n'}{\sigma^2((\lambda-\varepsilon)/n')} = 1 \quad \text{for every } 0 < \varepsilon \leq \frac{\lambda}{2},$$

which holds if and only if the three conditions

$$(1.18) \qquad \lim_{n'\to\infty} \sigma^2(\varepsilon/n')/\sigma^2((\lambda-\varepsilon)/n') = 1, \quad 0 < \varepsilon \leq \lambda/2,$$

$$(1.19) \qquad \lim_{n'\to\infty} \sigma^2((\lambda+\varepsilon)/n')/\sigma^2((\lambda-\varepsilon)/n') = 0, \quad 0 < \varepsilon \leq \lambda/2,$$

$$(1.20) \qquad \lim_{n'\to\infty} Q^2(\varepsilon/n')/(n'\sigma^2((\lambda-\varepsilon)/n')) = 0, \quad 0 < \varepsilon \leq \lambda/2$$

hold simultaneously along the same $\{n'\}$.

The proofs will show that if we have (1.18)-(1.20) or, equivalently, (1.17) along some $\{n'\}$ then (1.8)-(1.10a) and hence (1.11) hold along some subsequence $\{n''\}$ of the given $\{n'\}$. Conversely, if (1.1) holds along some $\{n'\}$, then (1.8)-(1.10a) hold along a subsequence $\{n''\}$ of the given $\{n'\}$ and (1.17) or, what is the same, (1.18)-(1.20) are satisfied along the same subsequence $\{n''\} \subset \{n'\}$. Then (1.18) and (1.19), holding along subsequences,will easily imply the second statement of the following.

**COROLLARY.** *Suppose* (1.1). *If* $\lambda > 1$, *then for each subsequence* $\{n''\} \subset \{n'\}$ *there exist a further subsequence* $\{n'''\} \subset \{n''\}$ *and a finite constant* $\delta = \delta_{\{n'''\}} > 0$ *such that* $a(n''')/A_{n'''} \to \delta$ *as* $n''' \to \infty$. *On the other hand, if* $\lambda < 1$, *then* $a(n')/A_{n'} \to 0$ *as* $n' \to \infty$.

The corollary demonstrates that limiting Poisson distributions in (1.1) arise in two qualitatively different ways depending on whether $\lambda > 1$ or $\lambda < 1$. The latter case provides what is probably the simplest possible example for showing that Theorem 2 in CsHM[2] is not empty.

In Section 3, we illustrate the above results by a concrete construction of a quantile function in the domain of partial attraction of a Poisson ($\lambda$) with particular reference to the Corollary. This construction works for all $\lambda > 0$ and it turns out that the case when $\lambda = 1$ is a boundary case in the sense that both possibilities in the behavior of the normalizing sequence may in fact show up.

It follows from the transitivity theorem of Gnedenko[3] (cited also in Gnedenko and Kolmogorov[4], p. 189; for a new proof see Cs[1]) that if $F \in D_p$ (Poisson ($\lambda$)) for some $\lambda > 0$, then $F$ is also in the domain of partial attraction $D_p(2)$ of the normal law. Thus our necessary and sufficient conditions should somehow imply that

$$(1.21) \qquad \liminf_{n\to\infty} \sigma^2(s/n)/\sigma^2(1/n) = 1 \quad \text{for all } 0 < s < 1,$$

which, according to Corollary 2 in CsHM[2] is necessary and sufficient for $F \in D_p$ (2). At this point we meet again the principal difference between the two cases $\lambda > 1$ and $\lambda \le 1$. Namely, if $\lambda > 1$, then (1.21) follows directly from (2.10) below, that is from conditions (1.8), (1.9) and (1.10a) with $r = 1$. The implication is not so direct if $\lambda \le 1$ and can be seen only by extra work.

While conditions expressed in terms of the underlying quantile function are completely natural in our approach, it would perhaps be interesting to obtain equivalent conditions expressed through $F$. These are probably uninformative just as Groshev's condition above and to produce them appears to be a non-trivial analytic problem.

As a final remark we note that, starting out from Theorem 12 in Cs[1], it is possible to characterize in the manner of the present paper the set of infinitely divisible laws partially attracted by a given Poisson law. We will consider this elsewhere.

## 2. PROOFS

Let $\psi$ be a non-positive, non-decreasing, right-continuous function on $(0,\infty)$ such that

$$\int_{\varepsilon}^{\infty} \psi^2(s)ds < \infty \quad \text{for any } \varepsilon > 0.$$

In the notation of CsHM[2] and Cs[1], for each integer $k \geq 0$ consider the random variable

$$\begin{aligned} V_{0,k}(0,\psi,0) &= \int_{S_{k+1}}^{\infty} (N(u)-u)d\psi(u) - \int_{1}^{S_{k+1}} ud\psi(u) \\ &\quad + k\psi(S_{k+1}) - \int_{1}^{k+1} \psi(u)du - \psi(1) \\ &= \int_{1}^{\infty} (N(u)-u)d\psi(u) + \int_{S_{k+1}}^{1} N(u)d\psi(u) \\ &\quad + k\psi(S_{k+1}) - \int_{1}^{k+1} \psi(u)du - \psi(1), \end{aligned}$$

where $N(\cdot)$ is the Poisson process defined in (1.4). For $k=0$, this is a spectrally one-sided infinitely divisible random variable without a normal component. For $\lambda > 0$, set

$$\psi_\lambda(u) = \begin{cases} -1, & \text{if } 0 < u < \lambda, \\ 0, & \text{if } u \geq \lambda. \end{cases}$$

If $\lambda \le 1$, then we have

$$\begin{aligned}
V_{0,k}(0,\psi_\lambda,0) &= \int_{S_{k+1}}^{1} N(u)d\psi_\lambda(u) + k\psi_\lambda(S_{k+1}) \\
&= I(S_{k+1} < \lambda)N(\lambda) - I(S_{k+1} < \lambda)k \\
&= I(S_{k+1} < \lambda)\left\{\sum_{j=1}^{k} I(S_j < \lambda) - k + \sum_{j=k+1}^{\infty} I(S_j < \lambda)\right\} \\
&= \sum_{j=1}^{k}\{I(S_j < \lambda)I(S_{k+1} < \lambda) - I(S_{k+1} < \lambda)\} \\
&\quad + \sum_{j=k+1}^{\infty} I(S_j < \lambda)I(S_{k+1} < \lambda) \\
&= \sum_{j=k+1}^{\infty} I(S_j < \lambda),
\end{aligned}$$

and if $\lambda > 1$, then, using the above lines in the last step, we have

$$\begin{aligned}
V_{0,k}(0,\psi_\lambda,0) &= (N(\lambda) - \lambda) - \int_{1}^{S_{k+1}} N(u)d\psi_\lambda(u) - kI(S_{k+1} < \lambda) \\
&\quad - \int_{1}^{\min(k+1,\lambda)} \psi_\lambda(u)du + 1 \\
&= N(\lambda) - \lambda - N(\lambda)I(S_{k+1} \ge \lambda) - kI(S_{k+1} < \lambda) + \min(k+1,\lambda) \\
&= I(S_{k+1} < \lambda)N(\lambda) - I(S_{k+1} < \lambda)k + \min(k+1,\lambda) - \lambda \\
&= \sum_{j=k+1}^{\infty} I(S_j < \lambda) + \min(k+1,\lambda) - \lambda.
\end{aligned}$$

Putting the two cases together and recalling (1.5), we see that

$$V_{0,k}(0,\psi_\lambda,0) = V_k(\lambda), \quad \lambda > 0;\ k = 0,1,2,\ \ldots\ . \tag{2.1}$$

In particular, by (1.6),

$$V_{0,0}(0,\psi_\lambda,0) = V(\lambda) \stackrel{\mathcal{D}}{=} Y_\lambda + \min(1,\lambda) - \lambda, \quad \lambda > 0. \tag{2.2}$$

Now we are ready to use Theorem 6 of Cs[(1)] to prove Theorem 1 here. If $\lambda > 1$, we are in case (ii) of that theorem, while if $\lambda \le 1$, we are in case (iii). The ambiguity of the latter case will presently be resolved by the well-known fact that if $Y_\lambda^{(1)}, \ldots, Y_\lambda^{(r)}$ are independent copies of $Y_\lambda$, then

$$Y_\lambda^{(1)} + \ldots + Y_\lambda^{(r)} \overset{\mathcal{D}}{=} Y_{r\lambda}. \tag{2.3}$$

**Proof of Theorem 1.** To prove necessity, suppose (1.1). Take any $r \geq r_1(\lambda)$, and let $\{X_j^{(m)}\}_{j=1}^\infty$ be independent copies of the sequence $\{X_j\}_{j=1}^\infty, m = 1, \ldots, r$. Then by (1.1) and (2.3),

$$\frac{1}{A_{n'}}\left\{\sum_{j=1}^{n'}\left(\sum_{m=1}^{r} X_j^{(m)}\right) - rC_{n'}\right\} \overset{\mathcal{D}}{\to} Y_{r\lambda} \quad \text{as } n' \to \infty.$$

Writing $Z_1, Z_2, \ldots$ for the sequence $X_1^{(1)}, \ldots, X_1^{(r)}, X_2^{(1)}, \ldots, X_2^{(r)}, \ldots$, that is, writing

$$Z_{(j-1)r+m} = X_j^{(m)},\ 1 \leq m \leq r,\ j = 1, 2, \ldots, \tag{2.4}$$

and introducing $C_{n'}^* = rC_{n'} - (1 - r\lambda)A_{n'}$, the convergence relation can be rewritten as

$$\frac{1}{A_{n'}}\left\{\sum_{j=1}^{rn'} Z_j - C_{n'}^*\right\} \overset{\mathcal{D}}{\to} V(r\lambda).$$

Therefore, by (2.2), the fact that $r\lambda > 1$, and by case (ii) of Theorem 6 in Cs[1] there exist an $\{n''\} \subset \{n'\}$ and a constant $\alpha_r > 0$ such that

$$\lim_{n'' \to \infty} Q\left(\frac{t}{rn''}\right)/a(rn'') = 0, \quad t > 0,$$

$$\lim_{n'' \to \infty} Q\left(1 - \frac{t}{rn''}\right)/a(rn'') = \begin{cases} \alpha_r, & 0 < t < r\lambda, \\ 0, & t > r\lambda, \end{cases}$$

and

$$\lim_{h \to \infty} \limsup_{n'' \to \infty} \sigma\left(\frac{h}{rn''}\right)/\sigma\left(\frac{1}{rn''}\right) = 0.$$

However, these three conditions are clearly equivalent to (1.8), (1.9) and (1.10a) with $n'$ replaced by $n''$.

Now we turn to the sufficiency statements. Assume (1.8), (1.9) and (1.10a). Then by (2.2) and the sufficiency part of case (ii) of Theorem 6 in Cs[1] we have

$$\frac{1}{a(rn')}\left\{\sum_{j=1}^{rn'} Z_j - rn' \int_{1/rn'}^{1-1/rn'} Q(u)du\right\} \xrightarrow{\mathcal{D}} \alpha_r V(r\lambda)$$

as $n' \to \infty$, where $Z_1, Z_2, \ldots$ are independent with the common distribution fuction $F$. Breaking up the sequence $\{Z_j\}_{j=1}^{\infty}$ into the union of $r$ independent sequences $\{X_j^{(m)}\}_{j=1}^{\infty}$ of independent variables, $m = 1, \ldots, r$, according to the rule in (2.4), this can be rewritten as

$$\sum_{m=1}^{r}\left[\frac{1}{\alpha_r a(rn')}\left\{\sum_{j=1}^{n'} X_j^{(m)} - n' \int_{1/rn'}^{1-1/rn'} Q(u)du\right\}\right] \xrightarrow{\mathcal{D}} \sum_{m=1}^{r}\left[Y_\lambda^{(m)} + \frac{1-r\lambda}{r}\right]$$

as $n' \to \infty$. This clearly implies (1.11) and hence also that $F \in D_p$ (Poisson ($\lambda$)).

To prove the further statements, we rewrite (1.11) as

$$\frac{1}{\alpha_r a(rn')}\left\{\sum_{j=1}^{n'} X_j - \tilde{C}_{n'}\right\} \xrightarrow{\mathcal{D}} V(\lambda), \quad n' \to \infty,$$

where

$$\tilde{C}_{n'} = n' \int_{1/rn'}^{1-1/rn'} Q(u)du + \left(\frac{1-r\lambda}{r} - \min(1,\lambda) + \lambda\right)\alpha_r a(rn'),$$

and recall (2.2) again. If $\lambda > 1$, and hence we are in case (ii) of Theorem 6 in Cs[1] or if $\lambda \le 1$ but we are in the first subcase of case (iii) of the same theorem, then by necessity there exist a subsequence $\{n''\} \subset \{n'\}$ and a constant $\delta_r > 0$ such that

$$\frac{1}{a(n'')} Q\left(\frac{s}{n''}\right) \to 0, \quad s > 0,$$

$$\frac{1}{a(n'')} Q\left(1 - \frac{s}{n''}\right) \to \frac{1}{\delta_r}\psi_\lambda(s), \quad s > 0,\ s \neq \lambda,$$

$$\frac{a(n'')}{\alpha_r a(rn'')} \to \delta_r,$$

and

$$\frac{1}{a(n'')}\left\{n'' \int_{1/n''}^{1-1/n''} Q(u)du - \tilde{C}_{n''}\right\} \to 0$$

as $n'' \to \infty$, and

$$\lim_{h\to\infty} \limsup_{n''\to\infty} \frac{\sigma(h/n'')}{\sigma(1/n'')} = 0.$$

If, on the other hand, $\lambda \leq 1$ and we are in the second subcase of case (iii) of Theorem 6 in Cs[1] then, along some subsequence $\{n''\} \subset \{n'\}$ we have

$$\frac{1}{a(rn'')} Q(\frac{s}{n''}) \to 0, \quad s > 0,$$

$$\frac{1}{a(rn'')} Q(1 - \frac{s}{n''}) \to \alpha_r \psi_\lambda(s), \quad s > 0,\ s \neq \lambda,$$

$$a(n'')/a(rn'') \to 0,$$

and

$$\frac{1}{a(rn'')} \left\{ n'' \int_{1/n''}^{1-1/n''} Q(u)du - \tilde{C}_{n''} \right\} \to 0.$$

Using now the sufficiency part of Theorem 6 in Cs[1] once more, we obtain all the theree statements (1.12) -(1.13) along the chosen $\{n''\}$ and hence the whole theorem. ■

The proof of Theorem 2 requires some preliminary lemmas, the first of which is of some independent interest.

**LEMMA 1.** *If $0 < s < t < 1/2$, then for an arbitrary quantile function*

$$\begin{aligned}\sigma^2(s) - \sigma^2(t) \leq & t(Q(t) - Q(s))^2 + t(Q(1-s) - Q(1-t))^2 \\ & + 2t(Q(t) - Q(s))(Q(1-s) - Q(t)) \\ & + 2t(Q(1-s) - Q(1-t))(Q(1-t) - Q(t))\end{aligned}$$

*and*

$$\begin{aligned}\sigma^2(s) - \sigma^2(t) \geq & \frac{s}{2}(Q(t) - Q(s))^2 + \frac{s}{2}(Q(1-s) - Q(1-t))^2 \\ & + 2s^2(Q(t) - Q(s))(Q(1-s) - Q(t)) \\ & + 2st(Q(1-s) - Q(1-t))(Q(1-t) - Q(t)).\end{aligned}$$

**Proof.** Using the definition in (1.3), as in identity (2.29) in CsHM[2], we have

$$
\begin{aligned}
\sigma^2(s) - \sigma^2(t) = &\int_s^t \int_s^t (u \wedge v - uv) dQ(u) dQ(v) \\
&+ \int_{1-t}^{1-s} \int_{1-t}^{1-s} (u \wedge v - uv) dQ(u) dQ(v) \\
&+ 2\left\{ \int_t^{1-s} (1-u) \left( \int_s^t v dQ(v) \right) dQ(u) \right. \\
&+ \left. \int_t^{1-t} u \left( \int_{1-t}^{1-s} (1-v) dQ(v) \right) dQ(u) \right\} \\
=: &\, I_1(s,t) + I_2(s,t) + 2\{I_3(s,t) + I_4(s,t)\}.
\end{aligned}
$$

Crearly, $I_1(s,t) \leq t(Q(t) - Q(s))^2$ and

$$
\begin{aligned}
I_1(s,t) &= \int_s^t (1-u) \left( \int_s^u v dQ(v) \right) dQ(u) + \int_s^t u \left( \int_u^t (1-v) dQ(v) \right) dQ(u) \\
&\geq s(1-t) \left\{ \int_s^t (Q(u) - Q(s)) dQ(u) + \int_s^t (Q(t) - Q(u)) dQ(u) \right\} \\
&= s(1-t) \{ Q(t)(Q(t) - Q(s)) - Q(s)(Q(t) - Q(s)) \} \\
&\geq \frac{s}{2} (Q(t) - Q(s))^2.
\end{aligned}
$$

Exactly the same way, using $u \wedge v - uv \leq 1 - u \wedge v$ for the upper bound, we obtain

$$
\frac{s}{2} (Q(1-s) - Q(1-t))^2 \leq I_2(s,t) \leq t(Q(1-s) - Q(1-t))^2.
$$

Similarly,

$$
s^2 (Q(t) - Q(s))(Q(1-s) - Q(t)) \leq I_3(s,t) \leq t(Q(t) - Q(s))(Q(1-s) - Q(t))
$$

and

$$
\begin{aligned}
st(Q(1-s) - Q(1-t))(Q(1-t) - Q(t)) &\leq I_4(s,t) \\
&\leq t(Q(1-s) - Q(1-t))(Q(1-t) - Q(t)).
\end{aligned}
$$

Collecting the upper and lower bounds, the lemma follows. ■

Let $\{n'\}$ be a subsequence of the positive integers tending to infinity, let $x \geq 1$ be any fixed number and $s > 0$ be an arbitrary number. Replacing $s$ by $s/xn'$ and $t$ by $s/n'$ and using (1.2), it will be advantageous to rewrite the inequalities of Lemma 1 as

$$
\begin{aligned}
\frac{\sigma^2(\frac{s}{xn'}) - \sigma^2(\frac{s}{n'})}{\sigma^2(1/n')} \leq & s\left(\frac{Q(\frac{s}{n'}) - Q(\frac{s}{xn'})}{a(n')}\right)^2 + s\left(\frac{Q(1-\frac{s}{xn'}) - Q(1-\frac{s}{n'})}{a(n')}\right)^2 \\
& + 2s\frac{Q(\frac{s}{n'}) - Q(\frac{s}{xn'})}{a(n')}\frac{Q(1-\frac{s}{xn'}) - Q(\frac{s}{n'})}{a(n')} \\
& + 2s\frac{Q(1-\frac{s}{xn'}) - Q(1-\frac{s}{n'})}{a(n')}\frac{Q(1-\frac{s}{n'}) - Q(\frac{s}{n'})}{a(n')}
\end{aligned} \tag{2.5}
$$

and

$$
\begin{aligned}
\frac{\sigma^2(\frac{s}{xn'}) - \sigma^2(\frac{s}{n'})}{\sigma^2(1/n')} \geq & \frac{s}{2x}\left(\frac{Q(\frac{s}{n'}) - Q(\frac{s}{xn'})}{a(n')}\right)^2 \\
& + \frac{s}{2x}\left(\frac{Q(1-\frac{s}{xn'}) - Q(1-\frac{s}{n'})}{a(n')}\right)^2 \\
& + 2\frac{s^2}{x^2n'}\frac{Q(\frac{s}{n'}) - Q(\frac{s}{xn'})}{a(n')}\frac{Q(1-\frac{s}{xn'}) - Q(\frac{s}{n'})}{a(n')} \\
& + 2\frac{s^2}{xn'}\frac{Q(1-\frac{s}{xn'}) - Q(1-\frac{s}{n'})}{a(n')}\frac{Q(1-\frac{s}{n'}) - Q(\frac{s}{n'})}{a(n')},
\end{aligned} \tag{2.6}
$$

which hold true for all $n'$ large enough.

**LEMMA 2.** *Let $\Lambda > 1$ be a fixed number and suppose that*

$$Q(s/n')/a(n') \to 0, \quad s > 0, \tag{2.7}$$

$$Q(1-\frac{s}{n'})/a(n') \to \begin{cases} \alpha, & \text{if } 0 < s < \Lambda, \\ 0, & \text{if } s > \Lambda, \end{cases} \tag{2.8}$$

*as $n' \to \infty$, where $\alpha > 0$ is some constant, and*

$$\lim_{h\to\infty} \limsup_{n'\to\infty} \sigma^2(h/n')/\sigma^2(1/n') = 0. \tag{2.9}$$

*Then, as $n' \to \infty$,*

$$\sigma^2(h/n')/\sigma^2(1/n') \to 1, \quad 0 < h < \Lambda, \tag{2.10}$$

and

$$\sigma^2(h/n')/\sigma^2(1/n') \to 0, \quad h > \Lambda. \tag{2.11}$$

**Proof.** Notice first that if both $s$ and $s/x$ are on one and the same side of $\Lambda$, that is either $s < \Lambda$ or $s/x > \Lambda$, then by (2.7) and (2.8) the upper bound in (2.5) goes to zero as $n' \to \infty$.

Put $s = 1$ in (2.5). Then we get

$$\sigma^2(\frac{1}{xn'})/\sigma^2(\frac{1}{n'}) \to 1 \quad \text{as} \quad n' \to \infty$$

for all $x \geq 1$, which implies (2.10) for all $0 < h \leq 1$. If $1 < h < \Lambda$, then putting $s = h$ and $x = h$ in (2.5) gives (2.10) for $1 < h < \Lambda$.

To prove (2.11), let $h > \Lambda$ and put $s = xh$ in (2.5), where $x \geq 1$ as always. Then, since $s/x > \Lambda$, we get

$$0 \leq \frac{\sigma^2(h/n')}{\sigma^2(1/n')} - \frac{\sigma^2(xh/n')}{\sigma^2(1/n')} \to 0 \quad \text{as} \quad n' \to \infty$$

which implies that

$$\bar{v}(h) := \limsup_{n' \to \infty} \frac{\sigma^2(h/n')}{\sigma^2(1/n')} = \limsup_{n' \to \infty} \frac{\sigma^2(xh/n')}{\sigma^2(1/n')} =: \bar{v}(xh)$$

for each $h > \Lambda$ and $x \geq 1$. Now for each fixed $h > 0$, $\bar{v}(xh) \to 0$ as $x \to \infty$ by (2.9), and hence $\bar{v}(h) = 0$ for each $h > \Lambda$ and this is nothing but (2.11). ∎

**LEMMA 3.** *Let $\Lambda > 1$ be a fixed number and suppose (2.10), (2.11) and that*

$$Q(1/n')/a(n') \to 0 \quad \text{as} \quad n' \to \infty. \tag{2.12}$$

*Then there exist a subsequence $\{n''\} \subset \{n'\}$ and a constant $\alpha > 0$ such that (2.7), (2.8) and (2.9) are satisfied with $n'$ replaced by $n''$.*

**Proof.** Take any $s < \Lambda$. Since also $s/x < \Lambda$, (2.10) implies that the left-side of (2.6) goes to zero as $n' \to \infty$. Because all the terms in the lower bound there are non-negative, they all go to zero separately. In particular,

$$(2.13) \qquad (Q(\frac{s}{n'}) - Q(\frac{s}{xn'}))/a(n') \to 0, \quad 0 < s < \Lambda, \ x \geq 1,$$

$$(2.14) \qquad (Q(1 - \frac{s}{xn'}) - Q(1 - \frac{s}{n'}))/a(n') \to 0, \quad 0 < s < \Lambda, \ x \geq 1,$$

as $n' \to \infty$. Putting $s = 1$ in (2.13), by (2.12) we obtain

$$Q(\frac{1}{xn'})/a(n') \to 0, \quad \text{as } n' \to \infty, \text{ for all } x \geq 1.$$

Now if $s > 0$ is arbitrary then we can choose $x \geq 1$ so that $s > 1/x$ and, provided that $Q(\cdot)$ is negative near enouh to zero, we get

$$|Q(\frac{s}{n'})|/a(n') \leq |Q(\frac{1}{xn'})|/a(n') \to 0.$$

If $Q(\cdot)$ is never negative, then of course (2.7) is trivial, and hence we have (2.7) along the original $\{n'\}$.

Consider now an $s > \Lambda$. Then, using (2.11),

$$\frac{Q(1 - \frac{s}{n'})}{a(n')} = \frac{Q(1 - \frac{s}{n'})}{\sqrt{n'})\sigma(s/n')} \frac{\sigma(s/n')}{\sigma(1/n')} \to 0 \quad \text{as } n' \to \infty,$$

on account of the fact that the first ratio on the right side is bounded by Lemma 2.4 in CsHM[(2)]. Thus we have (2.8) for $s > \Lambda$, still along the original $\{n'\}$.

Now we come to (2.8) for the case $s < \Lambda$. Again by Lemma 2.4 from CsHM[(2)] we can choose a subsequence $\{n''\} \subset \{n'\}$ such that

$$Q(1 - \frac{1}{n''})/a(n'') \to \alpha \quad \text{as } n'' \to \infty,$$

where $\alpha \geq 0$. Setting $s = 1$ in (2.14), we then have

$$Q(1 - \frac{1}{xn''})/a(n'') \to \alpha \quad \text{as } n'' \to \infty, \ x \geq 1,$$

which implies

$$(2.15) \qquad Q(1-\frac{s}{n''})/a(n'') \to \alpha, \quad 0 < s \le 1,$$

as $n'' \to \infty$. If $1 < s < \Lambda$, then we find an $x > 1$ so that $s/x \le 1$. Using (2.15) in conjunction with (2.14), we see that (2.15) also holds for $1 < s < \Lambda$. Thus we have (2.8) also for all $0 < s < \Lambda$, along $\{n''\}$, with a possibly zero limit $\alpha$.

We claim that $\alpha > 0$. Suppose, on the contrary, that $\alpha = 0$. Then the already proved (2.7) and (2.8), holding along $\{n''\}$, the latter with $\alpha = 0$, imply in accordance with case (i) of Theorem 1 in CsHM[(2)] or Theorem 1* in Cs[(1)] that

$$\frac{1}{a(n'')}\left\{\sum_{j=1}^{n''} X_j - n'' \int_{1/n''}^{1-1/n''} Q(u)du\right\} \xrightarrow{\mathcal{D}} N(0,1) \quad \text{as } n'' \to \infty.$$

Thus, by an application of the augmented case (ii) of the same theorem (Theorem 1* in Cs[(1)]), we also have

$$\lim_{h\to\infty} \liminf_{n''\to\infty} \frac{\sigma^2(h/n'')}{\sigma^2(1/n'')} = 1 = \lim_{h\to\infty} \limsup_{n''\to\infty} \frac{\sigma^2(h/n'')}{\sigma^2(1/n'')}.$$

Since $\{n''\} \subset \{n'\}$, this obviously contradicts (2.11). Therefore $\alpha > 0$, and we have (2.7) and (2.8) along $\{n''\}$.

Finally, it is trivial that (2.11) implies (2.9) along the original $\{n'\}$, which then holds along $\{n''\}$ *a fortiori*. ■

**LEMMA 4.** *Suppose that (2.10), (2.11) and (2.12) hold true for some* $\Lambda > 1$ *and* $\{n'\}$. *Then we have (1.18), (1.19) and (1.20), and hence (1.17), for the same* $\{n'\}$ *and with* $\lambda$ *replaced by* $\Lambda$.

**Proof.** Consider any $0 < \varepsilon \le \Lambda/2$. Then by (2.10),

$$\frac{\sigma^2(\varepsilon/n')}{\sigma^2((\Lambda-\varepsilon)/n')} = \frac{\sigma^2(\varepsilon/n')}{\sigma^2(1/n')}\left(\frac{\sigma^2((\Lambda-\varepsilon)/n')}{\sigma^2(1/n')}\right)^{-1} \to 1 \quad \text{as} \quad n' \to \infty,$$

that is, we have (1.18) with $\lambda = \Lambda$. Further, by (2.10) and (2.11),

$$\frac{\sigma^2((\Lambda+\varepsilon)/n')}{\sigma^2((\Lambda-\varepsilon)/n')} = \frac{\sigma^2((\Lambda+\varepsilon)/n')}{\sigma^2(1/n')}\left(\frac{\sigma^2((\Lambda-\varepsilon)/n')}{\sigma^2(1/n')}\right)^{-1} \to 0$$

as $n' \to \infty$, that is, we have (1.19) with $\lambda = \Lambda$. Finally, we know from the proof of Lemma 3 that (2.10) and (2.12) imply that (2.7) holds along $\{n'\}$. Therefore, by (2.7) and (2.10) again,

$$\frac{Q^2(\varepsilon/n')}{n'\sigma^2((\Lambda-\varepsilon)/n')} = \frac{Q^2(\varepsilon/n')}{a^2(n')}\left(\frac{\sigma^2((\Lambda-\varepsilon)/n')}{\sigma^2(1/n')}\right)^{-1} \to 0 \quad \text{as } n' \to \infty,$$

which is (1.20) with $\lambda = \Lambda$. ■

**LEMMA 5.** *Suppose that (1.17), and hence (1.18), (1.19) and (1.20) hold true for some $\lambda = \Lambda > 2$ and $\{n'\}$. Then we have (2.10), (2.11) and (2.12) for the same $\{n'\}$ and $\Lambda$.*

**Proof.** Since $\Lambda - 1 > 1$ or $\Lambda - 1 > \Lambda/2$, by (1.18) with $\lambda = \Lambda$ we have

$$1 \le \frac{\sigma^2(\varepsilon/n')}{\sigma^2(1/n')} \le \frac{\sigma^2(\varepsilon/n')}{\sigma^2((\Lambda-\varepsilon)/n')} \to 1, \quad 0 < \varepsilon \le 1,$$

and

$$1 \le \frac{\sigma^2(1/n')}{\sigma^2(\varepsilon/n')} \le \frac{\sigma^2(1/n')}{\sigma^2((\Lambda-1)/n')} \to 1, \quad 1 < \varepsilon \le \Lambda/2,$$

which together give

$$\frac{\sigma^2(\varepsilon/n')}{\sigma^2(1/n')} \to 1 \quad \text{as } n' \to \infty,\ 0 < \varepsilon \le \Lambda/2. \tag{2.16}$$

Also, by (1.18) again and (2.16),

$$\frac{\sigma^2((\Lambda-\varepsilon)/n')}{\sigma^2(1/n')} = \left(\frac{\sigma^2(\varepsilon/n')}{\sigma^2((\Lambda-\varepsilon)/n')}\right)^{-1}\frac{\sigma^2(\varepsilon/n')}{\sigma^2(1/n')} \to 1, \quad 0 < \varepsilon \le \Lambda/2, \tag{2.17}$$

as $n' \to \infty$, and now (2.16) and (2.17) together imply (2.10).

Furthermore, using (1.19) with $\lambda = \Lambda$ and (2.17),

$$\frac{\sigma^2((\Lambda+\varepsilon)/n')}{\sigma^2(1/n')} = \frac{\sigma^2((\Lambda+\varepsilon)/n')}{\sigma^2((\Lambda-\varepsilon)/n')}\frac{\sigma^2((\Lambda-\varepsilon)/n')}{\sigma^2(1/n')} \to 0, \quad 0 < \varepsilon \le \Lambda/2,$$

as $n' \to \infty$. Since for any $h > \Lambda$ one can find a small $\varepsilon > 0$ such that $\Lambda + \varepsilon < h$, this clearly implies (2.11).

Finally, putting $\varepsilon = 1 < \Lambda/2$ in (1.20), assumed with $\lambda = \Lambda$, we have

$$\frac{Q^2(1/n')}{a^2(n')} = \frac{Q^2(1/n')}{n'\sigma^2(1/n')} \le \frac{Q^2(1/n')}{n'\sigma^2((\Lambda-1)/n')} \to 0 \quad \text{as } n' \to \infty,$$

which is (2.12). ∎

**Proof of Theorem 2.** First assume (1.8), (1.9) and (1.10a) for some $\lambda > 0$, a fixed $r \ge r_1(\lambda)$ of (1.7) and some $\{n'\}$. Then we have

(2.18) $$Q(\frac{t}{rn'})/a(rn') \to 0, \quad t > 0,$$

(2.19) $$Q(1 - \frac{t}{rn'})/a(rn') \to \begin{cases} \alpha_r, & 0 < t < r\lambda, \\ 0, & t > r\lambda, \end{cases}$$

as $n' \to \infty$, and

(2.20) $$\lim_{h\to\infty} \limsup_{n'\to\infty} \sigma^2(\frac{h}{rn'})/\sigma^2(\frac{1}{rn'}) = 0.$$

Using Lemma 2 for the subsequence $\{rn'\}$ and $\Lambda = r\lambda > 1$, we obtain

(2.21) $$\sigma^2(\frac{h}{rn'})/\sigma^2(\frac{1}{rn'}) \to \begin{cases} 1, & 0 < h < r\lambda, \\ 0, & h > r\lambda, \end{cases}$$

and, as a special case of (2.18),

(2.22) $$Q(\frac{1}{rn'})/a(rn') \to 0$$

as $n' \to \infty$. Lemma 4, also used with $\{rn'\}$ and $\Lambda = r\lambda$, now gives

(2.23) $$\lim_{n'\to\infty} \frac{\sigma^2(\frac{\eta}{rn'}) + \sigma^2(\frac{r\lambda+\eta}{rn'}) + \frac{1}{n'}Q^2(\frac{\eta}{rn'})}{\sigma^2(\frac{r\lambda-\eta}{rn'})} = 1, \quad 0 < \eta \le \frac{r\lambda}{2}.$$

Setting $\varepsilon = \eta/r$, this is nothing but (1.17) which implies (1.15).

To prove the converse, assume (1.15). Then we have (1.17) for some $\{n'\}$. Let

$$r = r_2(\lambda) = \min\{m : m \text{ integer and } m\lambda > 2\} > r_1(\lambda)$$

and with this $r$ set $\eta = r\varepsilon$ in (1.17) to obtain (2.23). Applying Lemma 5 with $\Lambda = r\lambda > 2$ and $\{rn'\}$, we obtain (2.21) and (2.22). Now Lemma 3, used for $\{rn'\}$ and $\Lambda = r\lambda > 2$, yields (2.18), (2.19) and (2.20) with $n'$ replaced by $n''$, where $\{n''\} \subset \{n'\}$. Since (1.10a) is equivalent to (2.20), the substitution $s = t/r$ finally gives (1.8), (1.9) and (1.10a) with $n'$ replaced by $n''$, and hence the theorem. ■

**Proof of the Corollary.** The first statement follows from Theorem 5 in CsHM[2] because if $\lambda > 1$, the function $\psi_\lambda$ in (2.2) is not identically zero on the half-line $[1, \infty)$.

As to the second statement, assume (1.1) for some $\{n'\}$ and $\lambda < 1$. Then it holds along an arbitrary subsequence $\{n''\} \subset \{n'\}$. As described before the Corollary, we then have (1.8), (1.9) and (1.10a) along some $\{n'''\} \subset \{n''\}$, and hence also (1.18) and (1.19) along the same $\{n'''\}$. Thus, for any $\varepsilon > 0$ for which $\lambda + \varepsilon < 1$,

$$\frac{\sigma^2(1/n''')}{\sigma^2(\varepsilon/n''')} = \frac{\sigma^2(1/n''')}{\sigma^2((\lambda-\varepsilon)/n''')}\left(\frac{\sigma^2(\varepsilon/n''')}{\sigma^2((\lambda-\varepsilon)/n''')}\right)^{-1}$$

$$\leq \frac{\sigma^2((\lambda+\varepsilon)/n''')}{\sigma^2((\lambda-\varepsilon)/n''')}\left(\frac{\sigma^2(\varepsilon/n''')}{\sigma^2((\lambda-\varepsilon)/n'''}\right)^{-1} \to 0$$

as $n''' \to \infty$. Let

$$r > \max(r_1(\lambda), \frac{1}{1-\lambda}),$$

so that, in particular, $\lambda + (1/r) < 1$. Since (1.1) and (1.11) now hold jointly along $\{n'''\}$, the convergence of types theorem (Gnedenko and Kolomogorov[4], pp. 40-42) implies that $a(rn''')/A_{n'''} \to \delta$ as $n''' \to \infty$, where $0 < \delta < \infty$. Therefore,

$$\frac{a(n''')}{A_{n'''}} = \frac{a(rn''')}{A_{n'''}} \frac{\sigma(1/n''')}{\sqrt{r}\sigma(r^{-1}/n''')} \to 0 \quad \text{as } n''' \to 0.$$

Since $\{n''\} \subset \{n'\}$ was arbitrary, it follows that $a(n')/A_{n'} \to 0$ as $n' \to \infty$. ■

## 3. A CONSTRUCTION

Consider a number $0 < c \leq 1$ and set

$$t_j = t_j(c) = c2^{-2^j}, \quad j = 0, 1, 2, \ldots .$$

Then we have $t_0 = c/2$ and $t_j/t_{j-1} \to 0$ as $j \to \infty$. Also, set

$$b_j = b_j(c) = (t_j - t_{j+1})^{-1} = \frac{2^{2^j}}{c(1 - 2^{-2^j})}$$

and note that $b_j/b_{j-1} \to \infty$ as $j \to \infty$. Now we introduce the quantile function

$$Q(u) = \begin{cases} 0, & \text{if } \; 0 \leq u \leq 1 - t_0, \\ b_k, & \text{if } \; 1 - t_k < u \leq 1 - t_{k+1}, \; k = 0, 1, 2, \ldots, \end{cases}$$

or, what is the same,

$$Q(1-t) = \begin{cases} 0, & \text{if } t_0 \leq t \leq 1, \\ b_k, & \text{if } t_{k+1} \leq t < t_k, k = 0, 1, 2, \ldots \end{cases}$$

Fix $\lambda > 0$. The crucial element of the construction is the choice of the subsequence

$$n_k = n_k(\lambda, c) = \lceil \frac{\lambda}{t_k} \rceil = \min\{l : l \text{ integer}, \; l \geq \frac{\lambda}{t_k}\}, k = 0, 1, 2, \ldots .$$

By elementary considerations we obtain that for all $k$ large enough,

$$\begin{aligned} & t_{k+1} < s/n_k < t_k, \;\; \text{if} s < \lambda, \\ & t_{k+1} < \lambda/n_k < t_k, \;\; \text{if} \lambda/t_k < \lceil \lambda/t_k \rceil, \\ (3.1) \qquad & t_{k+1} < \lambda/n_k = t_k, \;\; \text{if} \lambda/t_k = \lceil \lambda/t_k \rceil \\ & t_k < s/n_k < t_{k-1}, \;\; \text{if} s > \lambda. \end{aligned}$$

Setting now $A_{n_k} = b_k$, we obviously have

$$Q(s/n_k)/A_{n_k} \to 0, \quad s > 0, \tag{3.2}$$

and from (3.1),

$$Q(1 - \frac{s}{n_k})/A_{n_k} \to \begin{cases} 1, & s < \lambda, \\ 0, & s > \lambda, \end{cases} \tag{3.3}$$

as $k \to \infty$.

The next step is to analyse the function $\sigma^2(h/n_k)$, $h > 0$, and in doing so, we have to separate three cases. We always use first formula (2.58) from CsHM[(2)], then the corresponding case of (3.1), and the final asymptotic equalities are obtained by simple computation.

If $h < \lambda$, or $h = \lambda$ and $\lambda/t_k < \lceil \lambda/t_k \rceil$, then for all $k$ large enough,

$$
\begin{aligned}
\sigma^2(\frac{h}{n_k}) &= \frac{h}{n_k} Q^2(1 - \frac{h}{n_k}) + \int_{1-t_0}^{1-h/n_k} Q^2(u)du \\
&\quad - \left( \frac{h}{n_k} Q(1 - \frac{h}{n_k}) + \int_{1-t_0}^{1-h/n_k} Q(u)du \right)^2 \\
&= \frac{h}{n_k} b_k^2 + \sum_{j=1}^{k} (t_{j-1} - t_j) b_{j-1}^2 + (t_k - \frac{h}{n_k}) b_k^2 \\
&\quad - \left( \frac{h}{n_k} b_k + \sum_{j=1}^{k} (t_{j-1} - t_j) b_{j-1} + (t_k - \frac{h}{n_k}) b_k \right)^2 \\
&\sim \frac{1}{c} 2^{2^k}.
\end{aligned}
\tag{3.4}
$$

If $h = \lambda$ and $\lambda/t_k = \lceil \lambda/t_k \rceil$, then for all $k$ large enough,

$$
\begin{aligned}
\sigma^2(\frac{\lambda}{n_k}) &= \frac{\lambda}{n_k} b_{k-1}^2 + \sum_{j=1}^{k} (t_{j-1} - t_j) b_{j-1}^2 \\
&\quad - \left( \frac{\lambda}{n_k} b_{k-1} + \sum_{j=1}^{k} (t_{j-1} - t_j) b_{j-1} \right)^2 \\
&\sim \frac{1}{c} \sum_{j=1}^{k} 2^{2^{j-1}} / (1 - 2^{-2^{j-1}})
\end{aligned}
\tag{3.5}
$$

Finally, if $h > \lambda$, then for all $k$ large enough,

$$\begin{aligned}
\sigma^2\left(\frac{h}{n_k}\right) &= \frac{h}{n_k} b_{k-1}^2 + \sum_{j=1}^{k-1}(t_{j-1} - t_j) b_{j-1}^2 + \left(t_{k-1} - \frac{h}{n_k}\right) b_{k-1}^2 \\
&\quad - \left( \frac{h}{n_k} b_{k-1} + \sum_{j=1}^{k-1}(t_{j-1} - t_j) b_{j-1} + \left(t_{k-1} - \frac{h}{n_k}\right) b_{k-1} \right)^2 \\
&\sim \frac{1}{c} 2^{2^{k-1}}
\end{aligned} \tag{3.6}$$

Now we are in the position to draw the conclusions, distinguishing four cases.

**Case 1** : $\lambda > 1$. Combining (3.6) and (3.4), for all $h > \lambda$ we obtain that

$$\frac{\sigma^2(h/n_k)}{\sigma^2(1/n_k)} \sim 2^{2^{k-1}}/2^{2^k} \to 0$$

and by (3.4),

$$\frac{a^2(n_k)}{A_{n_k}^2} = \frac{n_k \sigma^2(1/n_k)}{b_k^2} \to \lambda, \text{ as } k \to \infty,$$

for all $0 < c \leq 1$. These relations together with (3.2) and (3.3) provide an illustration of Theorem 1 and the first case of the corollary.

**Case 2** : $\lambda < 1$. Again by (3.4), with $h = 1/r < \lambda$, where $r \geq r_1(\lambda)$, we have

$$a(rn_k) \sim \frac{\sqrt{r\lambda}}{c} 2^{2^k} \sim \sqrt{r\lambda}\, b_k = \sqrt{r\lambda}\, A_{n_k},$$

$\sigma(h/n_k)/\sigma(1/rn_k) \to 0$ for all $h > 1$ by (3.4) and (3.6), and, finally,

$$\frac{a^2(n_k)}{A_{n_k}^2} \sim \lambda 2^{2^k} 2^{2^{k-1}}/2^{2^{k+1}} \to 0$$

for all $0 < c \leq 1$, as $k \to \infty$. Together with (3.2) and (3.3) again, these illustrate Theorem 1 and the second case of the corollary.

**Case 3** : $\lambda = 1$ *and we choose* $0 < c < 1$ *so that* $1/c$ *is not an integer.* By (3.4),

$$a^2(n_k) \sim \frac{1}{c^2} 2^{2^{k+1}} \sim b_k^2 = A_{n_k}^2$$

and by (3.6) and (3.4), for all $h > 1$,

$$\sigma^2(h/n_k)/\sigma^2(1/n_k) \sim 2^{2^{k-1}}/2^{2^k} \to 0,$$

as $k \to \infty$. Together with (3.2) and (3.3), these show that we are in the situation of Case 1.

Case 4: $\lambda = 1$ *and we choose* $c = 1$. We have by (3.5) that

$$\frac{a^2(n_k)}{A^2_{n_k}} = \frac{n_k \sigma^2(1/n_k)}{b_k^2} \sim \frac{2^{2^k} \sum_{j=1}^k 2^{2^{j-1}}/(1-2^{-2^{j-1}})}{2^{2^{k+1}}}$$
$$\leq 2k 2^{2^k} 2^{2^{k-1}}/2^{2^{k+1}} \to 0, \text{ as } k \to \infty.$$

Furthermore, for each $r \geq 2$, by (3.4)

$$a^2(rn_k) \sim 2^{2^{k+1}} \sim b_k^2 = A^2_{n_k}$$

and for all $h > 1$, by (3.6) and (3.4),

$$\sigma^2(h/n_k)/\sigma^2(1/rn_k) \sim 2^{2^{k-1}}/2^{2^k} \to 0, \text{ as } k \to \infty.$$

These, taken together with (3.2) and (3.3), show that we are in the situation of Case 2.

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