(#2016)

INTERMEDIATE- AND EXTREME-SUM PROCESSES

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Let $X_{1,n} \leq \dots \leq X_{n,n}$ be the order statistics of n independent random variables with a common distribution function F and let k_n be positive numbers such that $k_n \to \infty$ and $k_n/n \to 0$ as $n \to \infty$. With suitable centering and norming, we investigate the weak convergence of the intermediate-sum process $\sum_{i=\lceil ak_n \rceil+1}^{\lceil tk_n \rceil} X_{n+1-i,n}$, $a \leq t \leq b$, where $0 < a < b < \infty$, and the weak convergence of the extreme-sum process $\sum_{i=1}^{\lceil tk_n \rceil} X_{n+1-i,n}$, $0 \leq t \leq b$. Convergence is with respect to the supremum norm and can take place along a subsequence of the positive intergers $\{n\}$.

order statistics * intermediate-sum processes * extreme-sum processes * weak convergence * extreme-value domain of attraction

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[1]On leave from the University of Szeged, Hungary. Partially supported by the Hungarian National Foundation for Scientific Research, Grants 1808/86 and 457/88.

[2]Partially supported by NSF Grant DMS-8803209 and a Fulbright Grant.

1. Introduction and Statement of Results

Let $X, X_1, X_2, \ldots$ be a sequence of independent non-degenerate random variables with a common distribution function $F(x) = P\{X \le x\}$, $x \in \mathbb{R}$, and for each integer $n \ge 1$ let $X_{1,n} \le \cdots \le X_{n,n}$ denote the order statistics based on the sample $X_1, \ldots, X_n$. Let $\{k_n\}$ be a sequence of positive numbers such that

$$k_n \to \infty, \quad \text{and} \quad k_n/n \to 0 \qquad \text{as } n \to \infty, \tag{1.1}$$

and consider the sum process

$$I_n(a,t) = I_n(a,t;k_n) = \sum_{i=\lceil ak_n\rceil+1}^{\lceil tk_n\rceil} X_{n+1-i,n}, \quad a \le t \le b, \tag{1.2}$$

of intermediate order staistics, where $0 < a < b$, and the sum process

$$E_n(t) = E_n(t;k_n) = \begin{cases} \sum_{i=1}^{\lceil tk_n\rceil} X_{n+1-i,n}, & \frac{1}{k_n} \le t \le b, \\ 0 \quad , & 0 \le t < \frac{1}{k_n}, \end{cases} \tag{1.3}$$

of extreme order statistics, where $\lceil x\rceil$ is the smallest interger not smaller than x and an empty sum is understood as zero.

The asymptotic distribution of the intermediate sum $I_n(a,b)$ for fixed $0 < a < b$ has been thoroughly investigated in [3]. We found necessary and sufficient conditions for the existence of constants $A_n > 0$ and $C_n \in \mathbb{R}$ such that $A_n^{-1}(I_n(a,b)-C_n)$ converges in distribution along subsequences of the positive integers $\{n\}$ to non-degenerate limits and completely described the possible subsequential limiting distributions. Exactly the same programme has been carried out previously in [2] for the extreme sums $E_n(1)$. The aim of the present paper is to investigate the weak convergence of the suitably centered and normalized processes $I_n(a,\cdot)$ and $E_n(\cdot)$ in the supremum norm on $[a,b]$ and $[0,b]$, respectively.

Consider the inverse or quantile function of F defined as

$$Q(s) = \inf\{x : F(x) \geq s\}, \quad 0 < s \leq 1,$$

and introduce the associated left-continuous non-decreasing function

$$H(s) = -Q((1-s)-), \quad 0 \leq s < 1. \tag{1.4}$$

Consider the centering functions

$$\mu_n(a,t) = \mu_n(a,t;k_n) = -n \int_{\lceil ak_n \rceil/n}^{\lceil tk_n \rceil/n} H(s)ds, \quad 0 \leq a < t. \tag{1.5}$$

We say that a sequence $\{\xi_n(t) : a \leq t \leq b\}_{n=1}^{\infty}$ of stochastic processes has a distributionally equivalent version $\{\eta_n(t) : a \leq t \leq b\}_{n=1}^{\infty}$ if the distributional equality $\{\xi_n(t) : a \leq t \leq b\} =_{\mathcal{D}} \{\eta_n(t) : a \leq t \leq b\}$ holds for each $n \geq 1$, that is, all finite-dimensional distributions of $\xi_n(\cdot)$ and $\eta_n(\cdot)$ are the same on $[a,b]$ for each $n \geq 1$.

Theorem 1. *Let $\{k_n\}$ be a sequence as in (1.1) and fix $0 < a_o < b_o < \infty$. Suppose that there exist a subsequence $\{n'\}$ of the positive integers and positive numbers $B_{n'}$ along it such that for a function φ continuous on $[a_o,b_o]$, necessarily non-negative, non-decreasing and satisfying $\varphi(a_o) = 0$, we have*

$$\varphi_{n'}(a_o;x) := \int_{a_o}^{x} dH\left(\frac{sk_{n'}}{n'}\right)/B_{n'} \to \varphi(x) \text{ at each } x \in [a_o,b_o] \text{ as } n' \to \infty. \tag{1.6}$$

Then on a suitable probability space, for any choice of $a_o < a < b < b_o$, there exist a sequence $\{\tilde{I}_n(a,t) : a \leq t \leq b\}_{n=1}^{\infty}$ of distributionally equivalent versions of the sequence $\{I_n(a,t) : a \leq t \leq b\}_{n=1}^{\infty}$ and a standard Wiener process $W(t)$, $t \geq 0$, such that

$$\sup_{a \leq t \leq b} \left| \frac{1}{\sqrt{k_{n'}}\, B_{n'}} \{\tilde{I}_{n'}(a,t) - \mu_{n'}(a,t)\} - \int_a^t W(s)d\varphi(s) \right| \to 0 \text{ a.s.} \tag{1.7}$$

as $n' \to \infty$.

We note that by Theorem 1 in [3] for convergence in distribution of the process

$$Y_n(a,t) := \{I_n(a,t) - \mu_n(a,t)\}/\{\sqrt{k_n}\, B_n\}, \tag{1.8}$$

at a fixed point $a \le t \le b$ with $a_o < a < b < b_o$ along a subsequence of $\{n\}$ we can always choose

$$B_n = \Delta_n(a_o,b_o) := \max(H(b_o k_n/n) - H(a_o k_n/n),\ 1) > 0.$$

Then, with this choice of B_n, for the non-decreasing, left-continuous functions $\varphi_n(a_o;x)$ we have $0 \le \varphi_n(a_o;x) \le 1$ on $[a_o,b_o]$. Hence by a Helly selection one can always find a subsequence $\{n'\} \subset \{n\}$ and a non-negative, non-decreasing, left-continuous function φ on (a_o,b_o) such that $\varphi_{n'}(a_o;x)$ converges to $\varphi(x)$ as $n' \to \infty$ at any continuity point x of φ. Theorem 1 in [3] shows that one can hope for weak convergence of $I_{n'}(\cdot)$ in the supremum norm on $[a,b]$ only in the case when φ is continuous on some interval containing $[a,b]$. This is the underlying reason for condition (1.6).

Now we turn to the weak-convergence problem of the extreme-sum process $E_n(t) = I_n(0,t)$, $0 \le t \le b$. Even though the problem is now the behavior in the vicinity of zero, we still need a reference point $a_o > 0$ as in (1.6), which can in principle be chosen to be b_o.

Theorem 2. *Let $\{k_n\}$ be a sequence as in (1.1) and fix $0 < a_o \le b_o < \infty$. Suppose there exist a subsequence $\{n'\}$ of the positive integers and positive numbers B_n, such that for a function φ continuous on $(0,b_o]$, necessarily non-decreasing and satisfying $\varphi(a_o) = 0$, we have*

$$\varphi_{n'}(a_o;x) = \int_{a_o}^{x} dH\left[\frac{sk_{n'}}{n'}\right]/B_{n'} \to \varphi(x) \text{ at each } x \in (0,b_o] \tag{1.9}$$

as $n' \to \infty$ and

$$\lim_{a\downarrow 0}\ \limsup_{n'\to\infty} \int_0^a \sqrt{x}\, d\varphi_{n'}(a_o,x) = 0 \quad \text{and} \quad \lim_{a\downarrow 0} \int_0^a \sqrt{x}\, d\varphi(x) = 0. \tag{1.10}$$

Then on a suitable probability space, for any choice of $0 < b < b_o$, *there exist a sequence* $\{\tilde{E}_n(t) : 0 \le t \le b\}_{n=1}^{\infty}$ *of distributionally equivalent versions of the sequence* $\{E_n(t) : 0 \le t \le b\}_{n=1}^{\infty}$ *and a standard Wiener process* $W(t)$, $t \ge 0$, *such that*

$$\sup_{0\le t\le b} \left| \frac{1}{\sqrt{k_{n'}}\, B_{n'}} \left\{ \tilde{E}_{n'}(t) - \mu_{n'}(0,t) \right\} - \int_0^t W(s)d\varphi(s) \right| \to_P 0$$

as $n' \to \infty$.

We note that it is easy to see using integration by parts that if (1.9) holds and there exists a constant $\beta > -1/2$ such that $|\varphi_{n'}(a_o,x)| < x^{\beta}$ for all n' large enough and all $x > 0$ small enough, then condition (1.10) is also satisfied.

Now we formulate a corollary to Theorems 1 and 2 under the classical condition of extreme value theory. We say that F is in the domain of attraction of an extreme value distribution if $(X_{n,n}-c_n)/a_n$ converges in distribution to a non-degenerate random variable Y, where $a_n > 0$ and $c_n \in \mathbb{R}$ are some constants. As pointed out in [2], with earlier references, this happens if and only if there exists a constant $\gamma \in \mathbb{R}$ such that

$$\lim_{s\downarrow 0} \frac{H(sx)-H(sy)}{H(su)-H(sv)} = \begin{cases} (x^{-\gamma}-y^{-\gamma})/(u^{-\gamma}-v^{-\gamma}), & \text{if } \gamma \ne 0, \\ \log(x/y)/\log(u/v), & \text{if } \gamma = 0, \end{cases} \tag{1.11}$$

for all distinct $0 < x,y,u,v < \infty$. In this case we write $F \in \Lambda_{\gamma}$, where, with suitable choices of a_n and c_n,

$$\Lambda_{\gamma}(y) = P\{Y \le y\} = \begin{cases} \exp(-y^{1/\gamma}), \quad y > 0; & \text{if } \gamma > 0, \\ \exp(-\exp(-y)), \quad y \in \mathbb{R}; & \text{if } \gamma = 0, \\ \exp(-(-y)^{1/\gamma}), \quad y < 0; & \text{if } \gamma < 0. \end{cases}$$

For any $\gamma \in \mathbb{R}$, set

$$u_\gamma = \begin{cases} 1 & , \text{ if } \gamma > 0, \\ e & , \text{ if } \gamma = 0, \\ (1-\gamma)^{-1/\gamma} & , \text{ if } \gamma < 0, \end{cases} \quad \text{and} \quad v_\gamma = \begin{cases} (1+\gamma)^{-1/\gamma} & , \text{ if } \gamma > 0, \\ 1 & , \text{ if } \gamma = 0, \\ 1 & , \text{ if } \gamma < 0, \end{cases}$$

so that $u_\gamma^{-\gamma} - v_\gamma^{-\gamma} = -\gamma$ if $\gamma \neq 0$ and $\log(u_o/v_o) = 1$. For a sequence $\{k_n\}$ satisfying (1.1) we define

$$\Delta_n(\gamma) = H(u_\gamma k_n/n) - H(v_\gamma k_n/n) \ .$$

Choosing the reference point of Theorem 2 as $a_o = 1$, introduce now

$$\varphi_n(x) = \varphi_n(1,x) = \int_1^x dH\left(\frac{sk_n}{n}\right) / \Delta_n(\gamma) = \left\{H\left(\frac{xk_n}{n}\right) - H\left(\frac{k_n}{n}\right)\right\} / \Delta_n(\gamma), \tag{1.12}$$

which is well-defined for $0 < x < n/k_n$. Then, if $F \in \Lambda_\gamma$ for some $\gamma \in \mathbb{R}$, we obtain from (1.11) that

$$\varphi_n(x) \to \varphi_\gamma(x) := \begin{cases} (1-x^{-\gamma})/\gamma \ , & \text{if } \gamma \neq 0, \\ \log x & , \text{if } \gamma = 0, \end{cases} \quad \text{for any } x > 0 \tag{1.13}$$

as $n \to \infty$, that is, we have (1.9) with the continuous function $\varphi = \varphi_\gamma$ along the whole $\{n\}$ and for any $b_o > 0$, or, what is the same, (1.6) for any $0 < a_o < b_o$. Hence the first statement of Corollary 1 below follows from Theorem 1 and the second statement will follow from Theorem 2 after proving (1.10) for the present φ_n and $\varphi = \varphi_\gamma$.

Corollary 1. *If* $F \in \Lambda_\gamma$ *for some* $\gamma \in \mathbb{R}$ *and* $\{k_n\}$ *satisfies (1.1), then on a suitable probability space, for any choice of* $b > 0$, *there exist a sequence* $\{\tilde{E}_n(t) : 0 \le t \le b\}_{n=1}^{\infty}$ *of distributionally equivalent versions of the sequence* $\{E_n(t) : 0 \le t \le b\}_{n=1}^{\infty}$ *and a standard Wiener process* $W(t)$, $t \ge 0$, *such that for the sequence* $\{\tilde{I}_n(a,t) = \tilde{E}_n(t) - \tilde{E}_n(a) : a \le t \le b\}_{n=1}^{\infty}$, *being a distributionally equivalent version of the sequence* $\{I_n(a,t) : a \le t \le b\}_{n=1}^{\infty}$, *and for any* $0 < a < b$,

$$\sup_{a\le t\le b}\left|\frac{1}{\sqrt{k_n}\,A_n(\gamma)}\left\{\tilde{I}_n(a,t) - n\int_{[ak_n]/n}^{[tk_n]/n} Q(1-s)ds\right\} - \int_a^t W(s)s^{-1-\gamma}\,ds\right| \to 0$$

almost surely as $n \to \infty$. *Furthermore, if* $\gamma < 1/2$, *then for any* $b > 0$,

$$\sup_{0\le t\le b}\left|\frac{1}{\sqrt{k_n}\,A_n(\gamma)}\left\{\tilde{E}_n(t) - n\int_{0}^{[tk_n]/n} Q(1-s)ds\right\} - \int_0^t W(s)s^{-1-\gamma}\,ds\right| \to_P 0$$

as $n \to \infty$.

Notice that if $F \in A_\gamma$ and $\gamma \ge 1/2$, then we have (1.13) but condition (1.10) is not satisfied by the limiting function $\varphi = \varphi_\gamma$. In this case, according to Corollary 2 in [2], the appropriately centered extreme sums $E_n(t) - \mu_n(0+,t)$ require a norming sequence $A_n > 0$ to converge in distribution (denoted by $\to_{\mathcal{D}}$) to a non-degenerate random variable V that is heavier than the one needed by the centered intermediate sums. Namely, it follows from Corollary 2 in [2] that if $F \in A_\gamma$ for some $\gamma \ge 1/2$, then there is a sequence $A_n = A_n(k_n) > 0$, completely specified in [2], such that

$$\{I_n(a,b) - \mu_n(a,b)\}/A_n \to_P 0 \qquad \text{for all } 0 < a < b$$

and

$$\{E_n(t) - \mu_n(0+,t)\}/A_n \to_{\mathcal{D}} V \qquad \text{for all } t > 0,$$

where if $\gamma = 1/2$, then V is the *same* standard normal random variable for all $t > 0$, and if $\gamma > 1/2$, then V is the *same* stable random variable with index $1/\gamma$ for all $t > 0$.

Our next corollary discloses a curious Darling-Erdős type behavior for the extreme-sum process. Whenever $F \in A_\gamma$ for some $\gamma < 1/2$ and $\{k_n\}$ satisfies (1.1) write

$$e_n(t) := \sigma_\gamma\, t^{\gamma-1/2}\,\frac{E_n(t) - n\int_0^{[tk_n]/n} Q(1-s)ds}{\sqrt{k_n}\,A_n(\gamma)},$$

where

$$\sigma_\gamma = ((1-\gamma)(1-2\gamma)/2)^{1/2},$$

and for $T > 0$ and $\gamma < 1/2$ set

$$A(T) = (2 \log \max(T,e))^{1/2} \tag{1.14}$$

and

$$B_\gamma(T) = A(T) + (\log(\sqrt{\lambda_\gamma}/2\pi))^{1/2}/A(T) \tag{1.15}$$

where

$$\lambda_\gamma = (1-2\gamma)/4. \tag{1.16}$$

This behavior will be a consequence of the weak convergence of a time-changed variant of $e_n(\cdot)$ to the stochastic process

$$V_\gamma(x) := \sigma_\gamma\, e^{(\frac{1}{2}-\gamma)x} \int_0^{e^{-x}} W(u)u^{-1-\gamma}\, du, \quad 0 \le x < \infty.$$

It is readily verified that $V_\gamma(\cdot)$ is a sample-continuous mean zero stationary Gaussian process with covariance function given, for $x \ge 0$ and $h \ge 0$, by

$$r_\gamma(h) = E\, V_\gamma(x+h)V_\gamma(x)$$

$$= \begin{cases} \dfrac{1-2\gamma}{\gamma}\left\{\dfrac{\exp((\gamma-\frac{1}{2})h)}{1-2\gamma} - \exp\left[-\dfrac{h}{2}\right]\right\}, & \gamma < \frac{1}{2} \text{ and } \gamma \ne 0, \\ \frac{1}{2}(2+h)\exp\left[-\dfrac{h}{2}\right] & , \quad \gamma = 0. \end{cases}$$

Corollary 2. *Assume that* $F \in \Lambda_\gamma$ *with* $\gamma < 1/2$ *and let* $\{k_n\}$ *be a sequence satisfying (1.1). Then for any fixed* $0 < c < 1$ *the sequence* $\{e_n(e^{-x}) : 0 \le x \le \log(1/c)\}$ *of processes converges weakly in the Skorohod space* $D[0, \log(1/c)]$ *to the process* $\{V_\gamma(x) : 0 \le x \le \log(1/c)\}$. *Furthermore,*

$$\lim_{c\downarrow 0}\lim_{n\to\infty} P\left\{A(\log(1/c))\{\sup_{c\le t\le 1} e_n(t) - B_\gamma(\log(1/c))\} \le x\right\} = \exp(-e^{-x})$$

for all $x \in \mathbb{R}$.

Finally we would like to connect Theorem 1 to the classical theory of domains of partial attraction for the whole sums $X_1 + \dots + X_n$ and thereby show that Theorem 1 is not empty for any choice of $0 < a < b$ and non-negative, non-decreasing continuous function φ on $[a,b]$. In particular, we claim the following: Let $0 < a < b < \infty$ be arbitrary and let φ be any non-negative, non-decreasing, continuous function on $[a,b]$. Then there exist a distribution function F, a subsequence $\{n'\} = \{n'_j\}_{j=1}^{\infty}$, and a sequence $k'_j = k_{n'_j}$ satisfying $k'_j \to \infty$ and $k'_j/n'_j \to 0$ as $j \to \infty$ such that for the versions $\tilde{I}_{n'}$ of the intermediate sums $I_{n'}$ pertaining to F in Theorem 1 we have (1.7) as $n' \to \infty$. In fact, there is a universal F that does the job for all $0 < a < b$ and all functions φ on $[a,b]$ with the described properties.

Indeed, let $0 < a < b$ be arbitrary and φ be any continuous, non-negative, non-decreasing function on $[a,b]$. Choose $0 < a_o < a < b < b_o < \infty$ and extend the definition of φ so that the extended φ is continuous and non-decreasing on $[a_o,b_o]$ and $\varphi(a_o) = 0$. Now define

$$\psi(s) = \begin{cases} \varphi(a_o) - \varphi(b_o), & 0 < s \le a_o, \\ \varphi(s) - \varphi(b_o), & a_o < s \le b_o, \\ 0, & s > b_o, \end{cases}$$

and introduce $R(x) = -\inf\{s > 0 : \psi(s) \ge -x\}$, $x > 0$. Consider the spectrally right-sided infinitely divisible distribution without a normal component, the right Lévy measure of which is $R(x)$, $x > 0$. Then by the classical theorem of Khinchin ([4], p. 184) there is an F in the domain of partial attraction of this infinitely divisible law. Using Theorems 3 and 5 in [1], this means, in particular, that there exist a subsequence $\{n_j\}_{j=1}^{\infty}$ of the positive integers such that if Q denotes the quantile function belonging to F then we have

$$-Q((1 - \frac{s}{n_j})-)/B'_j \to \psi(s), \quad s > 0, \tag{1.17}$$

as $j \to \infty$, where $B'_j > 0$ are some constants. Now define $n'_j = \lceil n_j^{3/2} \rceil$ and $k'_j = \lceil n_j^{3/2} \rceil / n_j$, $j = 1,2,\dots$. Then $k'_j \to \infty$ and $k'_j/n'_j \to 0$ as $j \to \infty$, and it follows from (1.17) that

$$\left\{ H\left[\frac{sk'_j}{n'_j} \right] - H\left[\frac{a_o k'_j}{n'_j} \right] \right\} / B'_j \to \psi(s) - \psi(a_o) = \varphi(s)$$

for each $a_o \le s \le b_o$. This means that condition (1.6) is satisfied and hence by Theorem 1 we obtain (1.7) along $\{n'\} = \{n'_j\}_{j=1}^{\infty}$, with $B_{n'_j}$ replaced by the present B'_j. A universal F is obtained by using the distribution function F of any of the universal laws of Doeblin ([4], p. 189).

In order to give a flavor of the content of Theorems 1 and 2, we close this section by an illustrative example. Set

$$Q(1-s) = \{\beta + \sin(\log s)\}s^{-\gamma}, \quad 0 < s \le 1,$$

where $\gamma > 0$ and $\beta > (1+\gamma)/\gamma$. Differentiation shows that Q is an actual quantile function. For $j = 1,2,\dots,$ set

$$n'_j = \lceil \exp(4\pi_j) \rceil \quad \text{and} \quad k'_j = k_{n'_j} = n'_j \exp(-2\pi j),$$

so that $k'_j/n'_j = \exp(-2\pi j)$, $j = 1,2,\dots$. Also let

$$B'_j = B_{n'_j} = \exp(2\pi\gamma j), \quad j = 1,2,\dots,$$

and choose $a_o > 0$ arbitrarily. Then for any $a_o \le x < \infty$ and all j large enough,

$$\varphi_{n'_j}(a_o;x) = \left\{ Q\left[1 - x \frac{k'_j}{n'_j}\right] - Q\left[1 - a_o \frac{k'_j}{n'_j}\right] \right\} / B'_j$$

$$= \{\beta + \sin(\log s)\} x^{-\gamma} - \{\beta + \sin(\log a_o)\} a_o^{-\gamma}$$

$$=: \varphi(x).$$

We see that Theorem 1 applies along $\{n'\} = \{n_j'\}_{j=1}^{\infty}$ for all $\gamma > 0$ and all $a_o < a < b < \infty$ and, moreover, it is easily checked that Theorem 2 is also applicable when $0 < \gamma < 1/2$. Notice that by (1.11) the distribution function corresponding to such a Q is not in the domain of attraction of Λ_γ for any γ.

2. Proofs

Let $U_1, U_2, \ldots$ be independent random variables uniformly distributed on (0,1) with corresponding order statisics $U_{1,n} \leq \ldots \leq U_{n,n}$. Consider the uniform empirical and quantile processes $\alpha_n(t) = \sqrt{n}\,(G_n(t)-t)$ and $\beta_n(t) = \sqrt{n}\,(t-U_n(t))$, $0 \leq t \leq 1$, where $G_n(t) = n^{-1}\,\#\{1 \leq k \leq n : U_k \leq t\}$, $0 \leq t \leq 1$, and $U_n(t) = \inf\{0 \leq s \leq 1 : G_n(s) \geq t\}$, $0 < t \leq 1$, $U_n(0) = U_{1,n}$, so that $U_n(t) = U_{k,n}$ if $(k-1)/n < t \leq k/n$, $k=1,\ldots,n$. The tail empirical and quantile processes pertaining to the given sequence $\{k_n\}$ satisfying (1.1) are defined as

$$w_n(s) = (n/k_n)^{1/2}\,\alpha_n(sk_n/n) \quad \text{and} \quad v_n(s) = (n/k_n)^{1/2}\,\beta_n(sk_n/n), \quad 0 \leq s \leq n/k_n.$$

As pointed out in [6], $w_n(\cdot)$ converges weakly in the Skorohod space D[0,T], for any $T > 0$, to a standard Wiener process. Then by a Skorohod construction and the left-continuous version of Lemma 1 of Vervaat [8] (both also in [7]) we see that the sequences $\{w_n(\cdot)\}_{n=1}^{\infty}$ and $\{v_n(\cdot)\}_{n=1}^{\infty}$ have distributionally equivalent versions $\{\tilde{w}_n(\cdot)\}_{n=1}^{\infty}$ and $\{\tilde{v}_n(\cdot)\}_{n=1}^{\infty}$ on a rich enough probability space that carries a standard Wiener process W such that

$$\sup_{0\leq s\leq T} |\tilde{w}_n(s) - W(s)| \to 0 \quad \text{and} \quad \sup_{0\leq s\leq T} |\tilde{v}_n(s) - W(s)| \to 0 \quad \text{a.s.} \tag{2.1}$$

as $n \to \infty$.

In order to obtain the distributionally equivalent copies $\tilde{I}_n$ of Theorem

1, we first note that from (1.4),

$$(X_{1,n},\dots,X_{n,n}) =_{\mathcal{D}} (-H(U_{n,n}),\dots,-H(U_{1,n})) \quad \text{for each } n \geq 1.$$

Define $B_n > 0$ arbitrarily for an n which is not a member of $\{n'\}$ in (1.6). Then, using the notations in (1.2) and (1.5), starting out from the equality

$$\{I_n(a,t) - \mu_n(a,t) : a \leq t \leq b\} =_{\mathcal{D}} \left\{-\int_{U_n(ak_n/n)}^{U_n(tk_n/n)} n\, H(u)\, d\, G_n(u) + \int_{\lceil ak_n\rceil/n}^{\lceil tk_n\rceil/n} H(u)\, du : a \leq t \leq b\right\},\ n \geq 1,$$

and then integrating by parts, for the processes $Y_n(a,t)$ in (1.8) and for

$$Y_n^*(a,t) := M_n^*(a,t) - R_n^*(a) + R_n^*(t), \quad \text{where}$$

$$M_n^*(a,t) = \frac{n}{\sqrt{k_n}\, B_n} \int_{\lceil ak_n\rceil/n}^{\lceil tk_n\rceil/n} (G_n(u)-u)dH(u)$$

and

$$R_n^*(t) = \frac{n}{\sqrt{k_n}\, B_n} \int_{\lceil tk_n\rceil/n}^{U_n(tk_n/n)} (G_n(u)-u)dH(u)$$

we obtain

$$\{Y_n(a,t) : a\leq t\leq b\} =_{\mathcal{D}} \{Y_n^*(a,t) = M_n^*(a,t)-R_n^*(a)+R_n^*(t) : a\leq t\leq b\},\ n \geq 1. \quad (2.2)$$

Substituting now $u = sk_n/n$ and transferring to the probability space of the versions $\tilde{w}_n$ and $\tilde{v}_n$ in (2.1), we get

$$\{Y_n(a,t) : a\leq t\leq b\} =_{\mathcal{D}} \{\tilde{Y}_n(a,t) := \tilde{M}_n(a,t)-\tilde{R}_n(a)+\tilde{R}_n(t) : a\leq t\leq b\},\ n \geq 1. \quad (2.3)$$

where, with the obvious extension of the definition of φ_n, for an arbitrary n,

$$\tilde{M}_n(a,t) = \int_{\lceil ak_n \rceil/k_n}^{\lceil tk_n \rceil/k_n} \tilde{w}_n(s)d\,\varphi_n(a_o;s)$$

and

$$\tilde{R}_n(t) = \int_{\frac{\lceil tk_n \rceil}{n}}^{-\frac{1}{\sqrt{k_n}}\tilde{v}_n\left(\frac{\lceil tk_n \rceil}{k_n}\right) + \frac{\lceil tk_n \rceil}{k_n}} \left\{\tilde{w}_n(s) + s\sqrt{k_n} - \frac{\lceil tk_n \rceil}{\sqrt{k_n}}\right\} d\frac{H\left(\frac{sk_n}{n}\right)}{B_n}$$

$$= \int_0^{-\tilde{v}_n(\lceil tk_n \rceil/k_n)} \left\{\tilde{w}_n\left[\frac{\lceil tk_n \rceil}{k_n} + \frac{x}{\sqrt{k_n}}\right] + x\right\} d\,\frac{H\left[\frac{\lceil tk_n \rceil}{n} + x\frac{\sqrt{k_n}}{n}\right]}{B_n} .$$

Proof of Theorem 1. Relations (1.8) and (2.3) show the existence of versions $\{\tilde{I}_n(a,t) : 1 \le t \le b\}$ of $\{I_n(a,t) : 1 \le t \le b\}$ as claimed in the statement in the theorem once we prove that

$$\sup_{a \le t \le b} |\tilde{R}_{n'}(t)| \to 0 \quad \text{a.s.} \quad \text{as } n' \to \infty \tag{2.4}$$

and

$$\sup_{a \le t \le b} |\tilde{M}_{n'}(a,t) - \int_a^t W(s)d\varphi(s)| \to 0 \ \text{ a.s.} \ \text{ as } n' \to \infty. \tag{2.5}$$

By (2.1) and the fact that $|W(\cdot)|$ is bounded on any finite interval with probability 1, there exists an almost surely finite random variable $K > 0$ such that for all n' large enough

$$\sup_{s \le t \le b} |\tilde{R}_{n'}(t)| \le K \sup_{a \le t \le b} \int_{-K}^{K} d\,\frac{H\left[\frac{\lceil tk_{n'} \rceil}{n'} + x\frac{\sqrt{k_{n'}}}{n'}\right]}{B_{n'}}$$

$$= K \sup_{a \le t \le b} \frac{H\left[\frac{\lceil tk_{n'} \rceil}{n'} + K\frac{\sqrt{k_{n'}}}{n'}\right] - H\left[\frac{\lceil tk_{n'} \rceil}{n'} - K\frac{\sqrt{k_{n'}}}{n'}\right]}{B_{n'}}$$

$$= K \sup_{a\le t\le b} \left\{\varphi_{n'}\left(a_o, \frac{\lceil tk_{n'}\rceil}{k_{n'}} + \frac{K}{\sqrt{k_{n'}}}\right) - \varphi_{n'}\left(a_o, \frac{\lceil tk_{n'}\rceil}{k_{n'}} - \frac{K}{\sqrt{k_{n'}}}\right)\right\}$$

almost surely. Since $\varphi_{n'}(a_o,\cdot)$ is non-decreasing, by condition (1.6) and the continuity of φ this bound goes to zero as $n' \to \infty$ and hence we have (2.4).

To prove (2.5), first notice that (2.1) and (1.6) easily imply that

$$\sup_{a\le t\le b} \left|\tilde{M}_{n'}(a,t) - \int_{\lceil ak_{n'}\rceil/k_{n'}}^{\lceil tk_{n'}\rceil/k_{n'}} W(s)d\varphi_{n'}(a_o,s)\right| \to 0 \quad \text{a.s. as } n' \to \infty.$$

Next, for all n' large enough,

$$\sup_{a\le t\le b} \left|\int_{\lceil ak_{n'}\rceil/k_{n'}}^{\lceil tk_{n'}\rceil/k_{n'}} W(s)\, d\varphi_{n'}(a_o,s) - \int_a^t W(s)d\varphi_{n'}(a_o,s)\right|$$

$$\le 2 \sup_{a_o\le s\le b_o} |W(s)| \sup_{a\le t\le b} \left|\varphi_{n'}\left(a_o, \frac{\lceil tk_{n'}\rceil}{k_{n'}}\right) - \varphi_{n'}(a_o,t)\right|,$$

and this bound goes to zero again by (1.6) and the continuity of φ as $n' \to \infty$. Finally, (2.5) and hence the theorem will follow from these relations if we show that

$$\sup_{a\le t\le b} \left|\int_a^t W(s)d\varphi_{n'}(a_o,s) - \int_a^t W(s)d\varphi(s)\right| \to 0 \quad \text{a.s.} \qquad (2.6)$$

as $n' \to \infty$.

To verify this, notice that with probability 1 for any given $\epsilon > 0$ there exists a (random) partition $a = t_o < t_1 < \dots < t_m < t_{m+1} = b$ such that

$$\left|\int_a^{t_{i+1}} W(s)d\varphi_{n'}(a_o,s) - \int_a^{t_{i+1}} W(s)d\varphi(s)\right| < \frac{\epsilon}{2}$$

and

$$\int_{t_i}^{t_{i+1}} |W(s)| d\varphi_{n'}(a_o, s) + \int_{t_i}^{t_{i+1}} |W(s)|\, d\varphi(s) < \frac{\epsilon}{2}$$

for all i=0,...,m and n' large enough, where m does not depend on n'. Thus for any $t_i \le t \le t_{i+1}$ and i=0,...,m,

$$\left| \int_a^t W(s) d\varphi_{n'}(a_o, s) - \int_a^t W(s) d\varphi(s) \right| < \epsilon$$

almost surely for all n' large enough, proving (2.6). □

Proof of Theorem 2. First we note that it is easily checked that (1.10) implies the finiteness of $\mu_n(0,b)$ for all n large enough, so that the representation (2.2) holds true for a = 0. Hence, again defining $B_n > 0$ arbitrarily for an $n \notin \{n'\}$,

$$\left\{ \frac{E_n(t) - \mu_n(0,t)}{\sqrt{k_n}\, B_n} : 0 \le t \le b \right\} =_{\mathfrak{D}} \Big\{ M_n^*(0,a) + M_n^*(a,t) - R_n^*(0) + R_n^*(t) : 0 \le a \le t \le b \Big\}$$

for each $n \ge 1$. Furthermore, it follows from the derivation of (2.2) and (2.3) that for each $n \ge 1$,

$$\{(M_n^*(0,a), M_n^*(a,t), R_n^*(0), R_n^*(t)) : 0 \le a \le t \le b\}$$
$$=_{\mathfrak{D}} \{(\tilde{M}_n(0,a), \tilde{M}_n(a,t), \tilde{R}_n(0), \tilde{R}_n(t)) : 0 \le a \le t \le b\}.$$

Hence, in view of the fact that now we have (2.4) and (2.5) for any fixed $0 < a < b < b_o$, it suffices to prove

$$\lim_{a \downarrow 0} \limsup_{n' \to \infty} P\left\{ \sup_{0 \le t \le a} |M_{n'}^*(0,t)| > \epsilon \right\} = 0, \tag{2.7}$$

$$\lim_{a\downarrow 0} P\left\{\sup_{0\le t\le a}\left|\int_0^t W(s)d\varphi(s)\right| > \epsilon\right\} = 0, \tag{2.8}$$

and

$$\lim_{a\downarrow 0}\limsup_{n'\to\infty} P\left\{\sup_{0\le t\le a}|R^*_{n'}(t)| > \epsilon\right\} = 0, \tag{2.9}$$

where $\epsilon > 0$ is arbitrary.

We have

$$E\left[\sup_{0\le t\le a}|M^*_{n'}(0,t)|\right] \le \frac{1}{\sqrt{k_{n'}}\,B_{n'}} E\int_0^{[ak_{n'}]/n'} n'|G_{n'}(u)-u|dH(u)$$

$$\le \frac{\sqrt{n'}}{\sqrt{k_{n'}}\,B_{n'}}\int_0^{[ak_{n'}]/k_{n'}} \sqrt{u}\, dH(u)$$

$$= \int_0^{[ak_{n'}]/k_{n'}} \sqrt{s}\, d\varphi_{n'}(a_o,s)$$

$$\le \int_0^{2a} \sqrt{s}\, d\varphi_{n'}(a_o,s),$$

where the last inequality holds for all n' large enough. Hence, using condition (1.10), by the Markov inequality we obtain (2.7).

Also,

$$E\left[\sup_{0\le t\le a}\left|\int_0^t W(s)d\varphi(s)\right|\right] \le E\int_0^a |W(s)|d\varphi(s)$$

$$\le \int_0^a \sqrt{s}\, d\varphi(s),$$

and hence condition (1.10) and the Markov inequality again imply (2.8).

Finally, using the fact that for any $a > 0$,

$$\frac{n}{k_n} U_n\left(\frac{ak_n}{n}\right) \to_P a \quad \text{as} \quad n\to\infty,$$

which follows for example from (2.1), we obtain that for each a > 0,

$$\sup_{0\le t\le a} |R^*_{n'}(t)| \le \frac{1}{\sqrt{k_{n'}}\, B_{n'}} \int_0^{U_{n'}(ak_{n'}/n')} n'|G_{n'}(u) - u|dH(u)$$

$$= \frac{1}{\sqrt{k_{n'}}} \int_0^{n'U_{n'}(ak_{n'}/n')/k_{n'}} n'\left|G_{n'}\left(\frac{sk_{n'}}{n'}\right) - \frac{sk_{n'}}{n'}\right| d\varphi_{n'}(a_o,s)$$

$$\le \frac{1}{\sqrt{k_{n'}}} \int_0^{2a} n'\left|G_{n'}\left(\frac{sk_{n'}}{n'}\right) - \frac{sk_{n'}}{n'}\right| d\varphi_{n'}(a_o,s) + o_P(1)$$

as $n' \to \infty$. But the expectation of the first term here is not greater than

$$\int_0^{2a} \sqrt{s}\, d\, \varphi_{n'}(a_o,s),$$

and hence we obtain (2.9) as above. □

Proof of Corollary 1. Since

$$\int_0^a \sqrt{s}\, d\varphi_\gamma(s) = \int_0^a s^{-\gamma-1/2}\, ds \to 0 \quad \text{as } a \downarrow 0$$

whenever $\gamma < 1/2$, we only have to prove that

$$\lim_{a\downarrow 0} \limsup_{n\to\infty} \int_0^a \sqrt{s}\, d\varphi_n(s) = 0 \qquad \text{if } \gamma < 1/2, \tag{2.10}$$

where φ_n is given in (1.12).

Fix $0 < a < 1$. Then we have

$$\int_0^a \sqrt{s}\, d\varphi_n(s) = \sum_{i=0}^{\infty} \int_{a/2^{i+1}}^{a/2^i} \sqrt{s}\, d\varphi_n(s) \le \sqrt{a} \sum_{i=0}^{\infty} r_i(n,a)$$

where, using (1.11),

$$r_i(n,a) = 2^{-i/2}\{\varphi_n(a/2^i) - \varphi_n(a/2^{i+1})\}$$

$$= 2^{-i/2} \frac{H(2^{-(i+1)}ak_n/n)-H(2^{-i}ak_n/n)}{H(u_\gamma k_n/n) - H(v_\gamma k_n/n)}$$

$$= 2^{-i/2} \frac{H(2^{-1}ak_n/n)-H(ak_n/n)}{H(u_\gamma k_n/n)-H(v_\gamma k_n/n)} \prod_{m=1}^{i} \frac{H(2^{-(m+1)}ak_n/n)-H(2^{-m}ak_n/n)}{H(2^{-m}ak_n/n)-H(2^{-(m-1)}ak_n/n)}$$

$$\leq 2^{-i/2}\left\{ a^{-\gamma} \frac{|(1/2)^{-\gamma}-1|}{|\gamma|} + a^{1/2} \right\} \left\{2^{\gamma} + a\right\}^{i}$$

$$= \left\{ a^{-\gamma} \frac{|2^{\gamma}-1|}{|\gamma|} + a^{1/2} \right\} \left\{ 2^{\gamma-1/2} + a\, 2^{-1/2} \right\}^{i}$$

for all n large enough, where we use the convention that $|((1/2)^{-\gamma}-1)/\gamma| =$ $\log 2$ if $\gamma = 0$. Hence for all n large enough and all $a > 0$ small enough,

$$\int_0^a \sqrt{s}\, d\varphi_n(s) \leq \left[a + a^{1/2-\gamma} \frac{|2^{\gamma}-1|}{|\gamma|} \right] \left[1-2^{\gamma-1/2} - a\, 2^{-1/2}\right]^{-1} .$$

Since this bound goes to zero as $a \downarrow 0$, (2.10) follows. □

Proof of Corollary 2. The first statement follows directly from Corollary 1.

Calculation shows that for the covariance function $r_\gamma(\cdot)$ of the process $V_\gamma(\cdot)$ we have

$$r_\gamma(h) = 1 - \frac{\lambda_\gamma h^2}{2} + o(h^2) \quad \text{as} \quad h \to \infty,$$

where λ_γ is as in (1.16). Applying now Theorem 8.2.6 in Leadbetter, Lindgren and Rootzén [5], we get that for all $x \in \mathbb{R}$,

$$\lim_{T\to\infty} P\left\{A(T)\{ \sup_{0\leq x\leq T} V_\gamma(x) - B_\gamma(T)\} \leq x\right\} = \exp(-e^{-x}),$$

where $A(T)$ and $B_\gamma(T)$ are given in (1.15) and (1.16). This and the first statement now easily imply the second statement after a time change. □

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