# A Two-Player Zero-Sum Electromagnetic Differential Game with Uncertainty

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**Abstract**

We consider dynamic electromagnetic evasion-interrogation games where the evader can use ferroelectric material coatings to attempt to avoid detection while the interrogator can manipulate the interrogating frequencies to enhance detection. The resulting problem is formulated as a two-player zero-sum dynamic differential game in which the cost functional is based on the expected value of the intensity of the reflected signal. We show that there exists a saddle point for the relaxed form of this dynamic differential game in which the relaxed controls appear linearly in the dynamics governed by a partial differential equation.

# 1 Introduction

In an electromagnetic evasion-interrogation game, the evader wishes to minimize the intensity of the reflected signal to remain undetected in carrying out his mission while the interrogator wishes to maximize the intensity of reflected signal to detect the attacker. The results in [5] demonstrated that it is possible to design ferroelectric materials with appropriate dielectric permittivity and magnetic permeability to significantly attenuate reflections of electromagnetic interrogation signals from highly conductive targets such as airfoils and missiles. In addition, the results in [6] showed that if the evader employed a simple counter interrogation design based on a fixed set (assumed known) of interrogating frequencies, then by a rather simple counter-counter interrogation strategy (use of an interrogating frequency little more than 10% different from the assumed design frequencies), the interrogator can easily defeat the evader's material coatings counter interrogation strategy to obtain strong reflected signals. Thus, we can readily conclude from these two results that the evader and the interrogator must each try to confuse the other by introducing significant uncertainty in their design and interrogating strategies, respectively.

Based on this consideration, a static electromagnetic evasion-interrogation game (in the spirit of mixed strategies introduced by von Neumann [25]) was considered in [2], where the problem is mathematically formulated as a minimax game over sets of probability measures. In this formulation, the evader does not choose a single coating, but rather has a set of possibilities available for choice and only chooses the probabilities with which he will employ the materials on a target. By choosing his coatings randomly (according to a best strategy to be determined in a minmax game), he prevents adversaries from discovering which coating he will use – indeed, even he does not know which coating will be chosen for a given target. The interrogator, in a similar approach, determines best probabilities for choices of frequency and angle in the interrogating signals. A more realistic dynamic modeling is considered in [3] by introducing time dynamics into the problem, wherein the evader is allowed to make dynamic changes to his strategies in response to the dynamic input with uncertainty from the interrogator. In this paper, we consider a two player zero-sum differential game, which is formulated as a minimax problem over the sets of relaxed controls. In this formulation, both evader and interrogator choose a probability measure at each time $t$ in the presence of material uncertainty which is modeled as a stochastic process.

In Section 2 we present a description of our problem formulation and then show that there exists a saddle point for the resulting dynamic game in Section 3. Some summary remarks and proposed future research efforts are given in Section 4.

# 2 Problem Formulation

The minimax cost functional is based on the intensity of reflected signals from an object such as an airfoil or missile coated by a radar absorbent material of constant thickness. There are two ways to study the electromagnetic scattering [5, 6]. One way is to employ the far field pattern for reflected waves computed directly using Maxwell's equations. In two dimension, for a reflecting body with a given coating layer with an interrogating plane wave $E^{(i)}$, the scattered field $E^{(s)}$ satisfies the Helmholtz equation [11] as detailed in [2]. An alternative and much less computationally expensive one (as well as equally accurate in this setting – see [5, 6]) is to calculate the reflection coefficient based on a simple planar geometry (e.g., see Fig. 1) using Fresnel's formula for a perfectly conducting half plane.

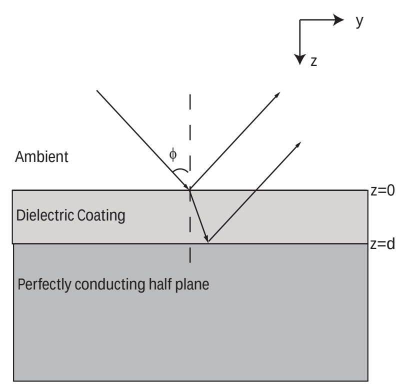


Figure 1: Interrogating high frequency wave impinging (angle of incidence $\phi$) on coated (thickness $d$) perfectly conducting surface

We will use the reflection coefficient to measure the strength of backscattering. In addition, we assume that a normally incident electromagnetic wave with the angular frequency $\omega$ is assumed to impinge the half plane. Then the corresponding wave length in the air is $2\pi c/\omega$, where the speed of light is $c = 3 \times 10^8$. Thus, the reflection coefficient $R$ for a wave impinging on a coating layer of thickness $d$ with relative dielectric permittivity $\epsilon$ and relative magnetic permeability $\mu$ is given by

$$R(\mu, \epsilon, \omega, d) = \frac{r_1 + r_2}{1 + r_1 r_2}, \tag{2.1}$$

where

$$r_1 = \frac{\epsilon - \sqrt{\epsilon\mu}}{\epsilon + \sqrt{\epsilon\mu}} \qquad \text{and} \qquad r_2 = \exp\left(2i\sqrt{\epsilon\mu}\omega d/c\right). \tag{2.2}$$

This expression can be derived directly from Maxwell's equation by considering the ratio of reflected to incident waves for example in the case of parallel polarized ($TE_x$) incident wave (e.g., see [5, 16]).

To incorporate some uncertainty in the reflected signal, we assume the real part $x$ of the magnetic permeability $\mu = x + i\mu_i$ of the coating has uncertainty described by an Itô diffusion process satisfying the stochastic differential equation

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t. \tag{2.3}$$

Here both $b$ and $\sigma$ are non-random functions that are assumed to be Lipschitz continuous, and $B_t$ denotes the standard Brownian motion. In addition, we assume that the interrogator has control of the frequency $\omega$ of the interrogating electromagnetic signals, and the evader has control of $\varepsilon$, the real part of the dielectric permittivity of surface coatings. At each time $t \in [0, T]$, the interrogator chooses parameters $\omega$ from a compact admissible set $\Omega \subset \mathbb{R}_+$, and the evader chooses parameters $\varepsilon$ from a compact admissible set $\mathcal{E} \subset \mathbb{R}_+$ in a measurable way. (Here $\mathbb{R}_+$ denotes the set of non-negative real numbers.) We can readily formulate our problem as a minimax problem, where the cost functional is dependent on the expected value of the intensity of the reflected signal.

## 2.1 Evolution of Expected Value of Intensity of Reflected Signal and Dynamic Differential Game

Let $\chi(x, \varepsilon, \omega) = |R(x + i\mu_i, \varepsilon + i\epsilon_i, \omega, d)|$, where $\mu_i$ and $\epsilon_i$ denote the imaginary parts of $\mu$ and $\epsilon$, respectively, which are assumed fixed in this paper. We then define

$$\tilde{v}(t, x) = \mathbb{E}^x \left[ \int_0^t \lambda e^{\lambda s} \chi(X_s, \varepsilon(s), \omega(s))\, ds + v_0(X_t) \right]$$

where $\mathbb{E}^x[\,\cdot\,]$ denotes the expectation with respect to the probability law of $\{X_t : t \geq 0\}$ when its initial value is $X_0 = x$, $\lambda > 0$ is a discount parameter, and $v_0$ is a nonnegative function that is used to denote the initial intensity of reflected signal. Following a standard technique for treating integrals (see section 10.3 of [21]), we next define

$$Z_t = \int_0^t \lambda e^{\lambda s} \chi(X_s, \varepsilon(s), \omega(s))\, ds.$$

Then the process $Y_t = (X_t, Z_t)^T$ satisfies

$$d \begin{pmatrix} X_t \\ Z_t \end{pmatrix} = \begin{pmatrix} b(X_t) \\ \lambda e^{\lambda t} \chi(X_t, \varepsilon(t), \omega(t)) \end{pmatrix} dt + \begin{pmatrix} \sigma(X_t) \\ 0 \end{pmatrix} dB_t.$$

Let $g(t,(x,z)) = \mathbb{E}[Z_t + v_0(X_t) \mid Y_0 = (x,z)^T]$, where $\mathbb{E}[\,\cdot \mid \cdot\,]$ denotes the conditional expectation. Then we have

$$\tilde{v}(t,x) = g(t,(x,0)).$$

Here the generator of the Itô diffusion process $\{Y_t : t \geq 0\}$ is

$$\mathcal{L}\phi(x,z) = b(x)\,\frac{\partial}{\partial x}\phi(x,z) + \frac{1}{2}\sigma^2(x)\,\frac{\partial^2}{\partial x^2}\phi(x,z) + \lambda e^{\lambda t}\chi(x,\varepsilon(t),\omega(t))\frac{\partial}{\partial z}\phi(x,z).$$

It then follows from Section 8.1 in [21] that $g$ satisfies the *backward Kolmogorov* equation

$$\frac{\partial}{\partial t}g = \mathcal{L}g, \quad g(0,(x,z)) = z + v_0(x). \tag{2.4}$$

A discussion of the relationship between this state and the semigroup generated by $\mathcal{L}$ can be found in [13]. Since $g = \tilde{v} + z$ is the solution to (2.4), it follows that $\tilde{v}$ satisfies

$$\begin{aligned}
&\frac{\partial}{\partial t}\tilde{v}(t,x) = b(x)\,\frac{\partial}{\partial x}\tilde{v}(t,x) + \frac{1}{2}\sigma^2(x)\,\frac{\partial^2}{\partial x^2}\tilde{v}(t,x) + \lambda e^{\lambda t}\chi(x,\varepsilon(t),\omega(t)),\\
&\tilde{v}(0,x) = v_0(x).
\end{aligned}$$

Now let $v(t,x) = e^{-\lambda t}\tilde{v}(t,x)$. It is easy to show that $v$ satisfies

$$\begin{aligned}
\frac{\partial}{\partial t}v(t,x) &= \mathcal{A}v(t,x) + \lambda\chi(x,\varepsilon(t),\omega(t)),\\
v(0,x) &= v_0(x),
\end{aligned}$$

where

$$\mathcal{A}v(t,x) = b(x)\,\frac{\partial}{\partial x}v(t,x) + \frac{1}{2}\sigma^2(x)\,\frac{\partial^2}{\partial x^2}v(t,x) - \lambda v(t,x).$$

We note that the state $v$ in this formulation is

$$v(t,x) = \mathbb{E}^x\left[\int_0^t \lambda e^{-\lambda(t-s)}\chi(X_s,\varepsilon(s),\omega(s))\,ds + e^{-\lambda t}v_0(X_t)\right],$$

the *expected value* of a measure of the reflected intensity.

We restrict $x$ to be in a finite interval $[\underline{x},\bar{x}]$, and set the boundary conditions to be zero. That is, we will consider the following state equation

$$\begin{aligned}
&\frac{\partial}{\partial t}v(t,x) = \mathcal{A}v(t,x) + \lambda\,\chi(x,\varepsilon(t),\omega(t)),\\
&v(t,\underline{x}) = 0,\; v(t,\bar{x}) = 0,\\
&v(0,x) = v_0(x),
\end{aligned} \tag{2.5}$$

The objective of the game for the evader is to choose a strategy such that the intensity of the reflected signal is as small as possible while the objective for the interrogator is to choose a strategy so that the intensity of the reflected signal is as large as possible. Hence, the cost functional for a minmax game with uncertainty can be formulated by

$$J(\varepsilon, \omega) = \int_0^T \int_{\underline{x}}^{\bar{x}} v(t, x; \varepsilon, \omega) dx dt. \tag{2.6}$$

It is well known that in general such problems may not have a saddle point over the pure strategies (e.g., see [7, 15]). A common approach that is used to circumvent this difficulty is to enlarge the class of controls to include relaxed controls (e.g., see [12, 22, 29]). Hence we will consider the game in a corresponding relaxed form.

## 2.2 Relaxed Form of Dynamic Differential Game

The notion of relaxed control, or generalized curve, was introduced into the calculus of variations (in the 40's) and optimal control (in the 60's) by a number of distinguished contributors such as Young [30, 31], McShane [18, 19, 20], Filippov [14] and Warga [26, 27, 28]. Since then, it has been studied by many other researchers (e.g., see [1, 9, 10, 17]).

Before we give the relaxed forms for (2.5) and (2.6), we will introduce needed theoretical background information on relaxed controls (e.g., see [12, 27, 28]). Let $C(\Omega)$ and $C(\mathcal{E})$ denote the spaces of continuous functions equipped with usual supremum norm, and $C^*(\Omega)$ and $C^*(\mathcal{E})$ be their corresponding topological dual spaces taken with the weak star topology which is equivalent to the Prohorov metric topology [8, 23] used in the static games in [2]. We define the spaces $\mathcal{P}(\Omega)$ and $\mathcal{P}(\mathcal{E})$ as the spaces of all regular probability measures defined on the Borel subsets of $\Omega$ and $\mathcal{E}$, respectively. Then with the Prohorov metric, $\mathcal{P}(\Omega)$ and $\mathcal{P}(\mathcal{E})$ are compact and convex subsets of $C^*(\Omega)$ and $C^*(\mathcal{E})$, respectively. In addition, as noted above convergence in the Prohorov metric is equivalent to weak star convergence. For more information on Prohorov metric, the interested readers can refer to [8, 23].

Let $L^1(0, T; C(\Omega))$ be the Banach space of Lebesgue integrable functions from $[0, T]$ to $C(\Omega)$ with the norm

$$\|g_\omega\|_{L^1(0,T;C(\Omega))} = \int_0^T \|g_\omega(t)\|_{C(\Omega)} dt.$$

The Banach space $L^1(0, T; C(\mathcal{E}))$ and its norm is similarly defined. It is known that both $L^1(0, T; C(\Omega))$ and $L^1(0, T; C(\mathcal{E}))$ are separable. We denote the topological dual of $L^1(0, T; C(\Omega))$ and $L^1(0, T; C(\mathcal{E}))$ by $L^1(0, T; C(\Omega))^*$ and $L^1(0, T; C(\mathcal{E}))^*$, respectively. By the Dunford-Pettis theorem (e.g., see [28, Theorem IV.1.8]), we know that

$$L^1(0, T; C(\Omega))^* \cong L^\infty(0, T; C^*(\Omega))$$

and

$$L^1(0, T; C(\mathcal{E}))^* \cong L^\infty(0, T; C^*(\mathcal{E})).$$

Here $L^\infty(0, T; C^*(\Omega))$ is a Banach space of essentially bounded measurable functions from $[0, T]$ to $C^*(\Omega)$ with the norm

$$\|\psi_\omega\|_{L^\infty(0,T;C^*(\Omega))} = \operatorname*{ess\,sup}_{t\in[0,T]} |\psi_\omega(t)|(\Omega).$$

The Banach space $L^\infty(0,T;C^*(\mathcal{E})))$ and its norm is similarly defined. However, in this paper we shall consider $L^\infty(0,T;C^*(\Omega))$ and $L^\infty(0,T;C^*(\mathcal{E}))$ taken with the weak star topology. A sequence $\{\psi_{\omega,j}\}$ in $L^\infty(0,T;C^*(\Omega))$ is said to be convergent in this topology if there exists a point $\psi_\omega \in L^\infty(0,T;C^*(\Omega))$ such that for any $g_\omega \in L^1(0,T;C(\Omega))$ we have

$$\lim_{j\to\infty}\int_0^T\int_\Omega g_\omega(t,\omega)\psi_{\omega,j}(t)(d\omega)dt = \int_0^T\int_\Omega g_\omega(t,\omega)\psi_\omega(t)(d\omega)dt.$$

The convergence of a sequence in $L^\infty(0,T;C^*(\mathcal{E}))$ with the weak star topology is similarly defined.

The relaxed control for the interrogator is a mapping $\psi_\omega : [0,T] \to \mathcal{P}(\Omega)$, and this mapping is measurable (respectively, continuous) if $\int_\Omega h_\omega(\omega)\psi_\omega(t)(d\omega)$ is measurable (respectively, continuous) function of $t \in [0,T]$ for every continuous real-valued function $h_\omega$ on $\Omega$. A relaxed control for the evader $\psi_\varepsilon : [0,T] \to \mathcal{P}(\mathcal{E})$ is defined similarly. We shall identify these controls which differ only on a set of measure zero. Let

$$\mathscr{R}(\Omega) = \{\psi_\omega \mid \psi_\omega : [0,T] \to \mathcal{P}(\Omega) \text{ is measurable}\}.$$

and

$$\mathscr{R}(\mathcal{E}) = \{\psi_\varepsilon \mid \psi_\varepsilon : [0,T] \to \mathcal{P}(\mathcal{E}) \text{ is measurable}\}.$$

Let $\mathscr{B}_\Omega$ and $\mathscr{B}_\mathcal{E}$ denote the unit ball of $L^\infty(0,T;C^*(\Omega))$ and $L^\infty(0,T;C^*(\mathcal{E}))$, respectively. That is,

$$\mathscr{B}_\Omega = \{\psi \in L^\infty(0,T;C^*(\Omega)) \mid \|\psi\|_{L^\infty(0,T;C^*(\Omega))} \leq 1\}$$

and

$$\mathscr{B}_\mathcal{E} = \{\psi \in L^\infty(0,T;C^*(\mathcal{E})) \mid \|\psi\|_{L^\infty(0,T;C^*(\mathcal{E}))} \leq 1\}.$$

Then the weak norm topology and weak star topology of $\mathscr{B}_\Omega$ (respectively, $\mathscr{B}_\mathcal{E}$) coincide, and with this topology $\mathscr{B}_\Omega$ (respectively, $\mathscr{B}_\mathcal{E}$) is a compact metric space (see [28, Theorem I.3.11 and Theorem I.3.12]). Note that for any $\psi_\omega \in \mathscr{R}(\Omega)$ and $\psi_\varepsilon \in \mathscr{R}(\mathcal{E})$ we have $\psi_\omega(t)(\Omega) = 1$ and $\psi_\varepsilon(t)(\mathcal{E}) = 1$. Hence, $\mathscr{R}(\Omega) \subset \mathscr{B}_\Omega$ and $\mathscr{R}(\mathcal{E}) \subset \mathscr{B}_\mathcal{E}$. In addition, we have the following important results.

**Theorem 2.1.** *(See [28, IV.2.1] or [12, Theorem 3.9]) The sets $\mathscr{R}(\Omega)$ and $\mathscr{R}(\mathcal{E})$ can be considered as closed convex subsets of the unit ball of $L^\infty(0,T;C^*(\Omega))$ and $L^\infty(0,T;C^*(\mathcal{E}))$, respectively, so with the weak star topology both $\mathscr{R}(\Omega)$ and $\mathscr{R}(\mathcal{E})$ are compact.*

Let $\psi_\omega \in \mathscr{R}(\Omega)$ and $\psi_\varepsilon \in \mathscr{R}(\mathcal{E})$. Then by Lemma 3.13 in [12] we know that $\psi_\varepsilon \times \psi_\omega$ is a measurable relaxed control on $\mathcal{E} \times \Omega$, and $\psi_\varepsilon \times \psi_\omega$ can be considered to belong to the unit sphere of the topological dual $L^\infty(0,T;C^*(\mathcal{E}\times\Omega))$ of $L^1(0,T;C(\mathcal{E}\times\Omega))$. With this background information on relaxed controls, we can now put the state equation (2.5) into relaxed form

$$\begin{aligned}
&\frac{\partial}{\partial t}v(t,x) = \mathcal{A}v(t,x) + f(t,x),\\
&v(t,\underline{x}) = 0,\ v(t,\bar{x}) = 0, \qquad (2.7)\\
&v(0,x) = v_0(x),
\end{aligned}$$

where

$$f(t,x) = \lambda \int_\Omega \int_\mathcal{E} \chi(x,\varepsilon,\omega)\psi_\varepsilon(t)(d\varepsilon)\ \psi_\omega(t)(d\omega). \tag{2.8}$$

The cost functional corresponding to the relaxed controls $\psi_\varepsilon$ and $\psi_\omega$ is defined by

$$J(\psi_\varepsilon,\psi_\omega) = \int_0^T \int_{\underline{x}}^{\bar{x}} v(t,x;\psi_\varepsilon,\psi_\omega)dxdt. \tag{2.9}$$

Hence, for this relaxed formulation (2.9) with (2.7), the evader does not choose a single coating at each time $t$, but rather has a set of possibilities available for choices. The interrogator, in a similar approach, determines best probabilities for choices of frequency in the interrogating signals at each time $t$.

**Remark 2.2.** *By* (2.1)*, it is easy to see that $\chi$ is continuous in $[\underline{x},\bar{x}]\times\mathcal{E}\times\Omega$. By assumption both $\mathcal{E}$ and $\Omega$ are compact. Hence, $\chi$ is bounded. Let*

$$f_\varepsilon(t,x,\omega) = \int_\mathcal{E} \chi(x,\varepsilon,\omega)\psi_\varepsilon(t)(d\varepsilon).$$

*Then by Lebesgue dominated convergence theorem we know that $f_\varepsilon(t,\cdot,\cdot)$ is continuous in $[\underline{x},\bar{x}]\times\Omega$ for fixed $t$, and by the definition of relaxed controls we know $f_\varepsilon(\cdot,x,\omega)$ is measurable for fixed $(x,\omega)$. In addition, we have*

$$\begin{aligned} |f_\varepsilon(t,x,\omega)| &\leq \|\chi\|_{C([\underline{x},\bar{x}]\times\mathcal{E}\times\Omega)}\psi_\varepsilon(t)(\mathcal{E}) \\ &= \|\chi\|_{C([\underline{x},\bar{x}]\times\mathcal{E}\times\Omega)}. \end{aligned} \tag{2.10}$$

*Thus, $f_\varepsilon \in L^\infty(0,T;C([\underline{x},\bar{x}]\times\Omega))$, which implies that $f_\varepsilon \in L^1(0,T;C([\underline{x},\bar{x}]\times\Omega))$. Note that*

$$f(t,x) = \lambda \int_\Omega f_\varepsilon(t,x,\omega)\psi_\omega(t)(d\omega).$$

*Hence, $f(t,\cdot)$ is continuous in $[\underline{x},\bar{x}]$ for fixed $t$, and $f(\cdot,x)$ is measurable for fixed $x$. Similarly, we find*

$$|f(t,x)| \leq \lambda\|\chi\|_{C([\underline{x},\bar{x}]\times\mathcal{E}\times\Omega)}. \tag{2.11}$$

*Thus, $f \in L^\infty(0,T;C([\underline{x},\bar{x}]))$. In addition, by Fubini's theorem we can exchange the order of integration in* (2.8)*.*

# 3 Existence of Saddle Points

In this section we will show that the relaxed form of the dynamic differential game (2.9) subject to (2.7) has a saddle point. We assume that there exists a positive constant $\sigma_{\inf}$ such that $\sigma(x) \geq \sigma_{\inf}$ for any $x \in [\underline{x},\bar{x}]$. Let $H = L^2(\underline{x},\bar{x})$, $V = H_0^1(\underline{x},\bar{x})$, and denote the topological dual space $V^*$ by $V^* = H^{-1}(\underline{x},\bar{x})$. If we identify $H$ with its topological dual $H^*$ then $V \hookrightarrow H = H^* \hookrightarrow V^*$ is a Gelfand triple. Throughout this presentation $\|\cdot\|_H$ and $\|\cdot\|_V$ and $\|\cdot\|_{V^*}$ are used to denote the norms in $H$, $V$ and $V^*$, respectively, $\langle\cdot,\cdot\rangle$ denotes the inner product in $H$, and $\langle\cdot,\cdot\rangle_{V^*,V}$ represents the duality paring between $V^*$ and $V$. Following standard conventions, we use an over dot ( $\dot{}$ ) to denote the derivative with respect to the time variable $t$, and use prime ( $'$ ) to represent the derivative with respective to the space variable $x$. In addition, for convenience we may use $\|\cdot\|_\infty$ to denote both the norms in $L^\infty(\underline{x},\bar{x})$ and $C([\underline{x},\bar{x}])$.

Define the sesquilinear form $a$ on $V \times V$ by

$$a(\phi,\varphi) = -\langle b\phi',\varphi\rangle + \frac{1}{2}\langle \phi', (\sigma^2\varphi)'\rangle + \lambda\langle\phi,\varphi\rangle.$$

By Remark 2.2, we know that $f(t) \in C([\underline{x},\bar{x}])$. Hence, we may now rewrite (2.7) in the weak form

$$\begin{aligned} &\langle \dot{v}(t),\varphi\rangle_{V^*,V} + a(v(t),\varphi) = \langle f(t),\varphi\rangle, \\ &v(0) = v_0 \end{aligned} \tag{3.1}$$

for any $\varphi \in V$. Here and elsewhere $v(t)$ and $f(t)$ denote the functions $v(t,\cdot)$ and $f(t,\cdot)$, respectively.

**Theorem 3.1.** *Given $v_0 \in H$. Then there exists a unique solution $v$ for (3.1) with $v \in H^1(0,T;V^*) \cap L^2(0,T;V)$. In addition, there exists a positive constant $\kappa$ such that for any $t \in [0,T]$*

$$\|v(t)\|_H^2 \le \kappa\left(\|v_0\|_H^2 + \int_0^t \|f(s)\|_{V^*}^2 ds\right), \tag{3.2}$$

*and*

$$\int_0^T \|v(t)\|_V^2 ds \le \kappa\left(\|v_0\|_H^2 + \int_0^T \|f(t)\|_{V^*}^2 ds\right). \tag{3.3}$$

*Furthermore, we have $v \in C(0,T;H)$.*

*Proof.* Note that $V$ is continuously imbedded in $H$, and $H$ is continuously imbedded in $V^*$. Hence, there exists a constant $\gamma > 0$ such that

$$\|\varphi\|_H \le \gamma\|\varphi\|_V, \quad \text{for any } \varphi \in V, \tag{3.4}$$

and

$$\|h\|_{V^*} \le \gamma\|h\|_H, \quad \text{for any } h \in H. \tag{3.5}$$

Since $\sigma$ is Lipshcitz continuous, $\sigma' \in L^\infty(\underline{x},\bar{x})$. Thus, by (3.4) and (3.5) we find that for any $\phi,\varphi \in V$ we have

$$\begin{aligned} |a(\phi,\varphi)| &\le \|b\|_\infty\|\phi'\|_H\|\varphi\|_H + \frac{1}{2}\|\sigma^2\|_\infty\|\phi'\|_H\|\varphi'\|_H \\ &\quad +\|\sigma\|_\infty\|\sigma'\|_\infty\|\phi'\|_H\|\varphi\|_H + \lambda\|\phi\|_H\|\varphi\|_H \\ &\le \gamma\|b\|_\infty\|\phi\|_V\|\varphi\|_V + \frac{1}{2}\|\sigma^2\|_\infty\|\phi\|_V\|\varphi\|_V \\ &\quad +\gamma\|\sigma\|_\infty\|\sigma'\|_\infty\|\phi\|_V\|\varphi\|_V + \gamma^2\lambda\|\phi\|_V\|\varphi\|_V. \end{aligned}$$

Let $\varrho = \gamma\|b\|_\infty + \frac{1}{2}\|\sigma^2\|_\infty + \gamma\|\sigma\|_\infty\|\sigma'\|_\infty + \gamma^2\lambda$. Then by the above inequality we have

$$|a(\phi,\varphi)| \le \varrho\|\phi\|_V\|\varphi\|_V, \qquad \text{for any } \phi,\varphi \in V. \tag{3.6}$$

For any $\varphi \in V$ we also obtain

$$a(\varphi,\varphi) \geq \left(\frac{1}{2}\sigma_{\text{inf}}^2 - 2\theta\right) \|\varphi\|_V^2 - \frac{\|b\|_\infty^2 + \|\sigma\|_\infty^2\|\sigma'\|_\infty^2}{4\theta}\|\varphi\|_H^2.$$

Setting $\theta = \frac{1}{8}\sigma_{\text{inf}}^2$, we have

$$a(\varphi,\varphi) + \alpha_H\|\varphi\|_H^2 \geq \alpha_V\|\varphi\|_V^2, \tag{3.7}$$

where $\alpha_V = \frac{1}{4}\sigma_{\text{inf}}^2$ and $\alpha_H = \frac{\|b\|_\infty^2 + \|\sigma\|_\infty^2\|\sigma'\|_\infty^2}{4\theta}$. By Remark 2.2, we know that $f \in L^\infty(0,T;C([\underline{x},\bar{x}]))$. Hence, $f \in L^2(0,T;V^*)$. Thus, by Theorem 2.1 in [4] we know that for any $v_0 \in H$ there exists a unique solution $v$ for (3.1) with $v \in H^1(0,T;V^*) \cap L^2(0,T;V)$, and (3.2) and (3.3) hold for some positive constant $\kappa$. Furthermore, $v \in C(0,T;H)$, and thus the initial condition in (3.1) is meaningful. □

**Remark 3.2.** *By Remark 2.2, we know that $f(t) \in C([\underline{x},\bar{x}])$, which implies that $f(t) \in H$. Thus, we can easily obtain from* (2.11)

$$\kappa_H \equiv \|f(t)\|_H^2 \leq (\bar{x} - \underline{x})\lambda^2\|\chi\|_{C([\underline{x},\bar{x}]\times\mathcal{E}\times\Omega)}^2. \tag{3.8}$$

*By* (3.2), (3.3), (3.5) *and* (3.8) *we find*

$$\|v(t)\|_H^2 \leq \kappa\left(\|v_0\|_H^2 + T\gamma^2\kappa_H\right), \tag{3.9}$$

*and*

$$\int_0^T \|v(t)\|_V^2 ds \leq \kappa\left(\|v_0\|_H^2 + T\gamma^2\kappa_H\right). \tag{3.10}$$

*From* (3.9) *and* (3.10), *we see that both $\|v(t)\|_H^2$ and $\int_0^T \|v(t)\|_V^2 ds$ are bounded by a positive constant which is independent of the choices of $\psi_\omega$ and $\psi_\varepsilon$.*

**Remark 3.3.** *Let $v(t,x)$ be the solution to* (3.1). *Then by* (3.4) *and* (3.6) *we find*

$$\begin{aligned} |\langle \dot{v}(t),\varphi\rangle_{V^*,V}| &= |-a(v(t),\varphi) + \langle f(t),\varphi\rangle| \\ &\leq \varrho\|v(t)\|_V\|\varphi\|_V + \gamma\|f(t)\|_H\|\varphi\|_V, \end{aligned}$$

*which implies that*

$$\begin{aligned} \|\dot{v}(t)\|_{V^*} &= \sup_{\|\phi\|_V\leq 1}\{|\langle \dot{v}(t),\phi\rangle_{V^*,V}| \mid \phi \in V\} \\ &\leq \varrho\|v(t)\|_V + \gamma\|f(t)\|_H. \end{aligned}$$

*By* (3.8) *and the above equation, we obtain*

$$\|\dot{v}(t)\|_{V^*}^2 \leq 2\varrho^2\|v(t)\|_V^2 + 2\gamma^2\kappa_H.$$

*Thus, by* (3.10) *and integrating the above equation we have*

$$\int_0^T \|\dot{v}(t)\|_{V^*}^2 dt \leq 2\varrho^2\kappa\|v_0\|_H^2 + 2T\gamma^2\kappa_H(\varrho^2\kappa + 1). \tag{3.11}$$

*From the above equation we see that* $\int_0^T \|\dot{v}(t)\|_{V^*}^2 dt$ *is bounded by a positive constant that is independent of the choices of* $\psi_\omega$ *and* $\psi_\varepsilon$.

By the definition for $J$ defined in (2.9), we know that to show $J$ is separately continuous in each of its variables, it suffice to show that for given $\psi_\varepsilon \in \mathscr{R}(\mathcal{E})$ and a sequence $\{\psi_{\omega,j}\} \subset \mathscr{R}(\Omega)$ converging to $\psi_\omega$ in $\mathscr{R}(\Omega)$ we have

$$\lim_{j\to\infty} \int_0^T \int_{\underline{x}}^{\bar{x}} v(t,x;\psi_\varepsilon,\psi_{\omega,j})dxdt = \int_0^T \int_{\underline{x}}^{\bar{x}} v(t,x;\psi_\varepsilon,\psi_\omega)dxdt, \tag{3.12}$$

and for given $\psi_\omega \in \mathscr{R}(\Omega)$ and a sequence $\{\psi_{\varepsilon,j}\} \subset \mathscr{R}(\mathcal{E})$ converging to $\psi_\varepsilon$ in $\mathscr{R}(\mathcal{E})$ we have

$$\lim_{j\to\infty} \int_0^T \int_{\underline{x}}^{\bar{x}} v(t,x;\psi_{\varepsilon,j},\psi_\omega)dxdt = \int_0^T \int_{\underline{x}}^{\bar{x}} v(t,x;\psi_\varepsilon,\psi_\omega)dxdt. \tag{3.13}$$

Actually by using (3.9), (3.10) and (3.11) and similar arguments as in [10, Lemma 2.1], we can show that (3.12) and (3.13) both hold, that is, $J$ is separately continuous in each variable. For convenience of the reader, we will take (3.12) as a example to show that it holds in the following lemma.

**Lemma 3.4.** *Given* $\psi_\varepsilon \in \mathscr{R}(\mathcal{E})$, *and assume that the sequence* $\{\psi_{\omega,j}\} \subset \mathscr{R}(\Omega)$ *is convergent to* $\psi_\omega$ *in* $\mathscr{R}(\Omega)$. *Then we have* (3.12).

*Proof.* For notation convenience, we let $f_j = f(\cdot,\cdot\,;\,\psi_\varepsilon,\psi_{\omega,j})$ and $v_j = v(\cdot,\cdot\,;\,\psi_\varepsilon,\psi_{\omega,j})$. By (3.9), (3.10) and (3.11) , we know that $\{v_j\}$ is bounded in $C(0,T;H)$ and also in $L^2(0,T;V)$, and $\{\dot{v}_j\}$ is bounded in $L^2(0,T;V^*)$. Thus, there exists a subsequence - again denoted by $v_j$ - such that

$$v_j \to \hat{v} \text{ weakly in } L^2(0,T;V),$$
$$\dot{v}_j \to \dot{\hat{v}} \text{ weakly in } L^2(0,T;V^*).$$

Observe that $V$ is also compactly imbedded in $H$. Hence, by Theorem 2.1 in [24] we have

$$v_j \to \hat{v} \text{ strong in } L^2(0,T;H). \tag{3.14}$$

Note that $L^2(0,T;H)$ is continuously imbedded in $L^1(0,T;L^1(\underline{x},\bar{x}))$. Hence, by (3.14) we know that $v_j$ is strongly convergent to $\hat{v}$ in $L^1(0,T;L^1(\underline{x},\bar{x}))$, that is,

$$\lim_{j\to\infty} \int_0^T \int_{\underline{x}}^{\bar{x}} v(t,x;\psi_\varepsilon,\psi_{\omega,j})dxdt = \int_0^T \int_{\underline{x}}^{\bar{x}} \hat{v}(t,x)dxdt.$$

Thus, to complete the proof we only need to show that $\hat{v} = v(\cdot,\cdot\,;\,\psi_\varepsilon,\psi_\omega)$.

Let $g(t,x) = \eta(t)\varphi(x)$, where $\varphi \in V$, and $\eta \in C^1(0,T)$ with $\eta(0) = 0$ and $\eta(T) = 0$. We set $v = v_j$ in (3.1), and then multiply (3.1) by $\eta(t)$ and integrate to find

$$\int_0^T \langle \dot{v}_j(t),\varphi\rangle_{V^*,V}\,\eta(t)dt + \int_0^T a(v_j(t),\varphi)\eta(t)dt = \int_0^T \langle f_j(t),\varphi\rangle\eta(t)dt.$$

Integrating by parts for the first term of the above equation, we have

$$-\int_0^T \langle v_j(t),\varphi\rangle \dot{\eta}(t)dt + \int_0^T a(v_j(t),\varphi)\eta(t)dt = \int_0^T \langle f_j(t),\varphi\rangle \eta(t)dt. \tag{3.15}$$

By Fubini's theorem, the right-hand side of (3.15) can be written as

$$\begin{aligned}
&\int_0^T \langle f_j(t),\varphi\rangle \eta(t)dt \\
&= \int_0^T \left[\int_{\underline{x}}^{\bar{x}} \lambda\varphi(x)\left(\int_\Omega f_\varepsilon(t,x,\omega)\psi_{\omega,j}(t)(d\omega)\right)dx\right]\eta(t)dt \\
&= \int_{\underline{x}}^{\bar{x}} \lambda\varphi(x)\left[\int_0^T\int_\Omega \eta(t)f_\varepsilon(t,x,\omega)\psi_{\omega,j}(t)(d\omega)dt\right]dx.
\end{aligned} \tag{3.16}$$

By Remark 2.2, we know that $f_\varepsilon \in L^\infty(0,T;C([\underline{x},\bar{x}]\times\Omega))$. Since $\eta \in C^1(0,T)$, we have $\eta f_\varepsilon \in L^\infty(0,T;C([\underline{x},\bar{x}]\times\Omega))$, which implies $\eta f_\varepsilon \in L^1(0,T;C([\underline{x},\bar{x}]\times\Omega))$. Since $\psi_{\omega,j}$ is convergent to $\psi_\omega$ in $\mathscr{R}(\Omega)$, letting $j\to\infty$, passing to the limit in (3.16) and using Fubini's theorem we find

$$\lim_{j\to\infty}\int_0^T \langle f_j(t),\varphi\rangle\eta(t)dt = \int_0^T \langle f(t),\varphi\rangle\eta(t)dt.$$

Now we let $j\to\infty$ and pass the limit through term by term for (3.15) to obtain

$$-\int_0^T \langle \hat{v}(t),\varphi\rangle \dot{\eta}(t)dt + \int_0^T a(\hat{v}(t),\varphi)\eta(t)dt = \int_0^T \langle f(t),\varphi\rangle \eta(t)dt.$$

Integrating by parts for the first term in the above equation, we find

$$\int_0^T \left(\langle \dot{\hat{v}}(t),\varphi\rangle_{V^*,V} + a(\hat{v}(t),\varphi)\right)\eta(t)dt = \int_0^T \langle f(t),\varphi\rangle\eta(t)dt. \tag{3.17}$$

Note that the class of $\eta$'s for which the above holds are dense in $L^2(0,T)$. Hence, we have (3.17) holding for all $\eta\in L^2(0,T)$. Thus, we have $\hat{v}$ satisfies the first equation of (3.1). To obtain $\hat{v}(0)=v_0$, we may use the same arguments with arbitrary $\eta\in C^1(0,T)$ with $\eta(T)=0$ but $\eta(0)\neq 0$. Therefore, by the uniqueness of the solution for (3.1) we have $\hat{v}=v$. ☐

**Remark 3.5.** *As the example given in [12] shows that the identity mapping from $\mathscr{R}(\mathcal{E})\times\mathscr{R}(\Omega)\to\mathscr{R}(\mathcal{E}\times\Omega)$ is not jointly continuous, the cost functional $J$ defined by (2.9) is not jointly continuous over the space $\mathscr{R}(\mathcal{E})\times\mathscr{R}(\Omega)$.*

**Theorem 3.6.** *(See [32, Corollary 3.2]) Let $\mathbb{X}$ be a nonempty compact and convex subset of a Hausdorff topological vector space, and let $\mathbb{Y}$ be a nonempty convex subset of a Hausdorff topological space, respectively. Suppose that $\Phi:\mathbb{X}\times\mathbb{Y}\to\mathbb{R}$ satisfies (i) for each fixed $x\in\mathbb{X}$, $y\longmapsto\Phi(x,y)$ is lower semicontinuous and quasiconvex; (ii) for each fixed $y\in\mathbb{Y}$, $x\longmapsto\Phi(x,y)$ is upper semicontinuouos and quasiconcave. Then we have*

$$\max_{x\in\mathbb{X}}\min_{y\in\mathbb{Y}}\Phi(x,y) = \min_{y\in\mathbb{Y}}\max_{x\in\mathbb{X}}\Phi(x,y).$$

*Moreover, if $\mathbb{Y}$ is compact, then $\Phi$ has a saddle point in $\mathbb{X}\times\mathbb{Y}$.*

Note that $J$ is continuous and linear in each variable. Thus, by Theorems 2.1 and 3.6 we know that $J$ has a saddle point, which is summarized in the following theorem.

**Theorem 3.7.** *There exists a pair of relaxed controls $\psi_\omega^* \in \mathscr{R}(\Omega)$ and $\psi_\varepsilon^* \in \mathscr{R}(\mathcal{E})$ such that*

$$J(\psi_\varepsilon^*, \psi_\omega) \leq J(\psi_\varepsilon^*, \psi_\omega^*) \leq J(\psi_\varepsilon, \psi_\omega^*)$$

*for any $\psi_\varepsilon \in \mathscr{R}(\mathcal{E})$ and $\psi_\omega \in \mathscr{R}(\Omega)$.*

# 4 Concluding Remarks

In this paper a two-player zero-sum dynamic differential game is considered in the context of electromagnetic pursuit-evasion. The problem is formulated as a minimax game over the sets of relaxed controls, where the cost functional is based on the expected value of the intensity of reflected signal. We have established that this game has a saddle point.

From Remark 3.5, we know that the cost functional $J$ is not jointly continuous, which implies that there are challenges in carrying out standard numerical approximations in the domain $\mathscr{R}(\mathcal{E}) \times \mathscr{R}(\Omega)$. Hence, an immediate effort is to identify a subset of this domain that is still compact and on which $J$ is jointly continuous. One can then develop a computational framework similar in spirit to that employed in the static case in [2].

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