Well-Posedness for a Nonsmooth Acoustic System

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February 18, 2003

Abstract

We consider an acoustic wave system with discontinuous coefficients and nonsmooth inputs. Existence, uniqueness and continuous dependence on input data of weak solutions are established.

1 Introduction

In this note we treat a particular wave equation that arises in the investigation of an electromagnetic interrogation technique relying on electromagnetic/acoustic interaction. This technique depends on the known fact that pulsed acoustic waves can act as "mirrors" for pulsed electromagnetic waves and that the resulting electromagnetic wave reflections can be used to characterize or identify target materials [1], [2], [6]. In order to accurately describe the full system dynamics underlying the technique, one must be able to model each system individually. The system studied here describes the behavior of the acoustic system for propagation in time and space. This acoustic system cannot be expected to yield solutions that are smooth in time or space since the inputs are pulsed and the waves are propagated across material interfaces. To facilitate both theoretical and computational [3] considerations, it is useful to write the equation in variational or weak form. If a solution is to possess less smooth time derivatives, such as second derivatives that are H^{-1} functions of time, the corresponding system needs to be written in variational or distributional form in both time and space. The traditional arguments for well-posedness of a wave equation in variational form in space alone cannot be applied directly to such an alternate formulation. Thus, another approach is required and this note outlines one which can be used to establish well-posedness of wave equations expressed in variational form in both time and space.

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2 Problem formulation

We consider a one-dimensional wave equation in the context of acoustic propagation. This equation describes the propagation of an acoustic pressure wave through a material consisting of two homogeneous layers. In the left layer of the material the wave propagates with one constant wave speed, in the right layer the wave travels at a different speed. The boundary conditions are given by the input of windowed sine wave at $z = 1$ and a no reflection, or total absorbing, condition at $z = 0$. A schematic of the geometry is given in Figure 1. The system is initially at rest. Then the equations that govern this system are given by

$$\ddot{w} - c^2(z)w'' = 0$$

$$\begin{array}{ll} w(0,z) = 0 & w(t,1) = f(t) \\ \dot{w}(0,z) = 0 & \dot{w}(t,0) - c(0)w'(t,0) = 0 \\ & \\ w(t,z_2-) = w(t,z_2+) & c^2(z_2-)w'(t,z_2-) = c^2(z_2+)w'(t,z_2+) \end{array}$$

where

$$c(z) = \begin{cases} c_2 & 0 \le z < z_2 \\ c_1 & z_2 \le z \le 1, \end{cases} \tag{1}$$

for $0 \le z_2 < 1$ and $f \in H_0^1(0,T)$.

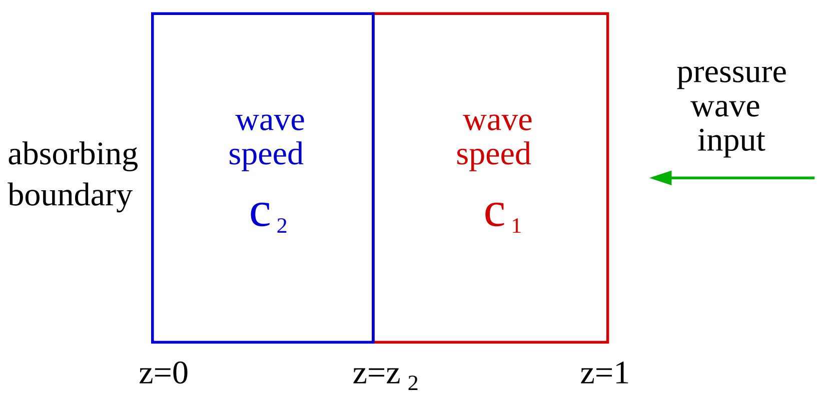

Figure 1 Schematic diagram of geometry

We note that there is a nonhomogeneous time-dependent Dirichlet boundary condition at $z = 1$ which accounts for the introduction of the acoustic wave pulse. To treat this condition, we make a change of variables which also facilitates finite element approximations. Thus we introduce a new state variable p defined by

$$p(t,z) = w(t,z) - zf(t).$$

In this new state our system is

$$\ddot{p} - c^2(z)p'' + z\ddot{f}(t) = 0$$

$$\begin{aligned} &p(0,z) = 0 \qquad && p(t,1) = 0 \\ &\dot{p}(0,z) = 0 \qquad && \dot{p}(t,0) - c(0)p'(t,0) - c(0)f(t) = 0 \end{aligned} \tag{2}$$

$$p(t, z_2-) = p(t, z_2+)$$

$$c^2(z_2-)p'(t,z_2-) + c^2(z_2-)f(t) = c^2(z_2+)p'(t,z_2+) + c^2(z_2+)f(t),$$

where $c(z)$ is defined above. It is clear that this system has a homogeneous boundary condition at $z = 1$. We note that for $f \in H_0^1(0,T)$, the term $\ddot{f}$ in the equation must be interpreted in some type of weak or distributional form.

3 Theoretical results

Following the ideas in [3], [6], we derive a full variational form of the system. We assume that $f \in H_0^1(0,T)$ and look for solutions $p \in H_L^1(0,T; H_R^1(0,1))$ such that

$$\begin{aligned} &\int_0^1 \int_0^T \left(\dot{p}\phi\dot{\psi} - c^2(z)p'\phi'\psi + z\dot{f}(t)\phi\dot{\psi} \right) \, dt \, dz \\ &+ \int_0^T -c_2\dot{p}(t,0)\phi(0)\psi + (c_1^2 - c_2^2)f\phi(z_2)\psi \, dt = 0 \end{aligned} \tag{3}$$

holds for all $\phi \in H_R^1(0,1)$, $\psi \in H_R^1(0,T)$. Here $H_R^1(a,b)$ (respectively, $H_L^1(a,b)$) are the usual $H^1(a,b)$ functions subject to the boundary condition $\phi(b) = 0$ (respectively, $\phi(a) = 0$).

In [3] it is shown that (3) embodies (weakly) the natural boundary and initial conditions of (2) whereas the essential conditions $p(0,z) = 0, p(t,1) = 0$ and $p(t,z_2-) = p(t,z_2+)$ must be imposed via the requirement $p \in H_L^1(0,T; H_R^1(0,1))$; recall [7] that for $\Omega \subset R^2, H^1(\Omega)$ embeds continuously in $C(\Omega)$.

Since f is an $H_0^1(0,T)$ function, we anticipate that p has only the smoothness $H_L^1(0,T; H_R^1(0,1))$. We thus cannot expect to have an analytic solution p for (2) with derivatives that exist strongly in $[0,T] \times [0,1]$. However as we show here, we can write down an analytic solution p that satisfies (2) in a distributional sense $[0,T] \times [0,1]$. That is, this solution will satisfy (3) with $p(0,z) = 0,\ p(t,1) = 0$, and $p(t,z_2-) = p(t,z_2+)$.

Theorem 1 *Suppose that the wave speed c is of the form (1) and the forcing function f is in $H_0^1(0,T)$. Then the system (3) has a unique solution, given by (7) below, and the solution p depends continuously on the forcing function f from $f \in L^2(0,T)$ to $p \in L^2(0,T; L^2(0,1))$.*

Our arguments rely on the fact that, given a smooth enough forcing function, the wave equation has a strong analytic solution. We use the approximation theory (i.e., mollifiers) detailed in [5] (Section 5.3) to find a sequence of $C^\infty(0,T)$ approximations $\{f_m\}$ to f such that

$$f_m \to f \text{ in } H^1(0,T). \tag{4}$$

For each m, we now consider the system

$$\ddot{p}_m - c^2 p_m'' + z\ddot{f}_m = 0$$

$$\begin{aligned} p_m(0,z) &= 0 \qquad & p_m(t,1) &= 0 \\ \dot{p}_m(0,z) &= 0 \qquad & \dot{p}_m(t,0) - c_1 p_m'(t,0) &= 0 \end{aligned} \tag{5}$$

$$\begin{aligned} p_m(t,z_2-) &= p_m(t,z_2+) \\ c_2^2 p_m'(t,z_2-) + c_2^2 f(t) &= c_1^2 p_m'(t,z_2+) + c_1^2 f(t) \end{aligned}$$

defined for $(t,z) \in [0,T] \times [0,1]$. We assume first that $T \leq \frac{2}{c_1}(1-z_2)$. This permits us to consider the system without possible reflections at $z=1$ from reflections at the interface z_2.
Since $f_m \in C^\infty(0,T)$ this system has an analytic solution

$$p_m(t,z) = \begin{cases} K_2 f_m(y_0(t,z)) - z f_m(t) & 0 \leq z < z_2 \\ f_m(y_1(t,z)) - K_1 f_m(y_2(t)) - z f_m(t) & z_2 \leq z \leq 1, \end{cases} \tag{6}$$

where

$$\begin{aligned} y_0(t,z) &= t - \frac{1}{c_2}(z_2 - z) - \frac{1}{c_1}(1-z_2) \\ y_1(t,z) &= t - \frac{1}{c_1}(1-z) \\ y_2(t,z) &= t + \frac{1}{c_1}(1-z) - \frac{2}{c_1}(1-z_2). \end{aligned}$$

The coefficients

$$\begin{aligned} K_1 &= \frac{c_2 - c_1}{c_1 + c_2} \\ K_2 &= \frac{2c_1}{c_1 + c_2} \end{aligned}$$

are derived from the interface conditions. We note that p_m satisfies (5), and hence also $p = p_m$ is in $H_L^1(0,T;H_R^1(0,1))$ and a solution to (3) with $f = f_m$.
Motivated by the form of p_m in (6), we define the $H_L^1(0,T;H_R^1(0,1))$ function p for $(t,z) \in [0,T] \times [0,1]$ as follows

$$p(t,z) = \begin{cases} K_2 f(y_0(t,z)) - z f(t) & 0 \leq z < z_2 \\ f(y_1(t,z)) - K_1 f(y_2(t,z)) - z f(t) & z_2 \leq z \leq 1. \end{cases} \tag{7}$$

We note that in equations (6) and (7) the functions f_m and f are evaluated at points $y < 0$; to account for this, we assume that $f_m(y) = 0$ and $f(y) = 0$ for any $y < 0$.
Equations (4)-(7) imply that for almost every $z \in (0,1)$, $\dot{p}_m(\cdot,z) \to \dot{p}(\cdot,z)$ and $p_m'(\cdot,z) \to p'(\cdot,z)$ in $L^2(0,T)$, while $p_m(\cdot,z) \to p(\cdot,z)$ in $H_L^1(0,T)$. In addition, $\dot{p}_m(\cdot,0) \to \dot{p}(\cdot,0)$ in $L^2(0,T)$. Moreover, for almost every $t \in (0,T)$, $\dot{p}_m(t,\cdot) \to \dot{p}(t,\cdot)$, and $p_m'(t,\cdot) \to p'(t,\cdot)$ in $L^2(0,1)$, hence $p_m(t,\cdot) \to p(t,\cdot)$ in $H_R^1(0,1)$.

To establish that p satisfies (3) we use the fact that p_m satisfies

$$\int_0^1 \int_0^T \left(\dot{p}\phi\dot{\psi} - c^2(z)p'\phi'\psi + z\dot{f}(t)\phi\dot{\psi} \right) dt\ dz + \int_0^T -c_2\dot{p}(t,0)\phi(0)\psi + (c_1^2 - c_2^2)f\phi(z_2)\psi\ dt = 0$$

with $f = f_m,\ p(0,z) = 0,\ p(t,1) = 0$, and $p(t, z_2-) = p(t, z_2+)$ for all $\phi \in H_R^1(0,1)$, $\psi \in H_R^1(0,T)$. Then for any $\phi \in H_R^1(0,1)$, $\psi \in H_R^1(0,T)$

$$\int_0^1 \int_0^T \dot{p}\phi\dot{\psi} - c^2p'\phi'\psi + z\dot{f}\phi\dot{\psi}\ dt\ dz + \int_0^T -c_2\dot{p}(t,0)\phi(0)\psi + (c_1^2 - c_2^2)f\phi(z_2)\psi\ dt =$$

$$\int_0^1 \int_0^T (\dot{p} - \dot{p}_m)\phi\dot{\psi}\ dt\ dz - \int_0^1 \int_0^T c^2(p' - p_m')\phi'\psi\ dt\ dz + \int_0^1 \int_0^T z(\dot{f} - \dot{f}_m)\phi\dot{\psi}\ dt\ dz$$

$$+ \int_0^T -c_2(\dot{p}(t,0) - \dot{p}_m(t,0))\phi(0)\psi + (c_1^2 - c_2^2)(f - f_m)\phi(z_2)\psi\ dt =$$

$$\int_0^1 \phi \int_0^T (\dot{p} - \dot{p}_m)\dot{\psi}\ dt\ dz - \int_0^1 c^2\phi' \int_0^T (p' - p_m')\psi\ dt\ dz$$

$$+ \int_0^1 z\phi \int_0^T (\dot{f} - \dot{f}_m)\dot{\psi}\ dt\ dz \tag{8}$$

$$-c_2 \int_0^T (\dot{p}(t,0) - \dot{p}_m(t,0))\phi(0)\psi\ dt + \int_0^T (c_1^2 - c_2^2)(f - f_m)\phi(z_2)\psi\ dt.$$

We note that

$$\int_0^T (\dot{p} - \dot{p}_m)\dot{\psi}\ dt \to 0, \quad \int_0^T (p' - p_m')\psi\ dt \to 0, \quad \text{and} \quad \int_0^T (\dot{f} - \dot{f}_m)\dot{\psi}\ dt \to 0$$

almost everywhere in $[0,1]$ as m goes to infinity and moreover each is bounded almost everywhere in $[0,1]$. Passing to the limit in (8) and using the fact that the last two terms converge to zero, we obtain

$$\int_0^1 \int_0^T \dot{p}\phi\dot{\psi} - c^2p'\phi'\psi + z\dot{f}\phi\dot{\psi}\ dt\ dz + \int_0^T -c_2\dot{p}(t,0)\phi(0)\psi + (c_1^2 - c_2^2)f\phi(z_2)\ dt = 0.$$

We may thus conclude that (3) has a solution $p \in H_L^1(0,T; H_R^1(0,1))$, given by (7), which holds for all $\phi \in H_R^1(0,1)$, $\psi \in H_R^1(0,T)$.
We next show that this solution is unique. This is equivalent to showing that the only solution $p \in H_L^1(0,T; H_R^1(0,1))$ that satisfies

$$\int_0^1 \int_0^T \left(\dot{p}\phi\dot{\psi} - c^2p'\phi'\psi \right) dt\ dz - \int_0^T c_2\dot{p}(t,0)\phi(0)\psi\ dt = 0 \tag{9}$$

for all $\phi \in H_R^1(0,1)$, $\psi \in H_R^1(0,T)$ is $p = 0$. (Equation (9) is equation (3) with $f = 0$.) Thus our goal here is to show that the solution to (9) must satisfy $|p(s,\cdot)|_{L^2(0,1)} = 0$ for any s in $[0,T]$.
Following ideas in [4], [7], we begin by defining the test function ϕ_s for any $s \in [0,T]$

$$\phi_s(t,z) = \begin{cases} \int_t^s -p(r,z)\ dr, & t < s \\ 0, & t \geq s. \end{cases}$$

We note that $\dot{\phi}_s(t,z) = p(t,z)$ for $0 \leq t < s$ and that $\phi_s(t,z) = \dot{\phi}_s(t,z) = \phi_s'(t,z) = 0$ for $s \leq t \leq T$.
We may then rewrite (9) with the test function ϕ_s in the following way

$$\int_0^1 \int_0^T \left(\dot{p}\dot{\phi}_s - c^2 p' \phi_s'\right)\ dt\ dz - \int_0^T c_2 \dot{p}(t,0)\phi_s(t,0)\ dt =$$

$$\int_0^1 \int_0^s \left(\dot{p}\dot{\phi}_s - c^2 p' \phi_s'\right)\ dt\ dz - \int_0^s c_2 \dot{p}(t,0)\phi_s(t,0)\ dt =$$

$$\int_0^1 \int_0^s \left(\dot{p}p - c^2 \dot{\phi}_s{}' \phi_s'\right)\ dt\ dz + \int_0^s c_2 p(t,0)\dot{\phi}_s(t,0)\ dt =$$

$$\int_0^1 \int_0^s \frac{1}{2}\frac{d}{dt}(p^2) - c^2 \frac{1}{2}\frac{d}{dt}(\phi_s'{}^2)\ dt\ dz + \int_0^s c_2 p^2(t,0)\ dt =$$

$$\frac{1}{2}\int_0^1 (p^2(s,z) + c^2 \phi_s'{}^2(0,z))\ dz + \int_0^s c_2 p^2(t,0)\ dt = 0.$$

Thus we may conclude that for any solution p of (9), $|p(s,\cdot)|_{L^2(0,1)} = 0$ for any $s \in [0,T]$. It follows that (3) has a unique solution.
Finally, we show that p, the unique solution p to (3) defined by (7), depends continuously on the choice of forcing function f. Using (7), we have

$$\int_0^1 p^2(t,z)\ dz = \int_{z_2}^1 (f(y_1) - K_1 f(y_2) - z f(t))^2\ dz + \int_0^{z_2} (K_2 f(y_0) - z f(t))^2\ dz.$$

We expand the squared terms, use the Cauchy-Schwartz inequality, and combine like terms to obtain

$$\int_0^1 p^2(t,z)\ dz \leq \int_{z_2}^1 3f^2(y_1) + 3K_1^2 f^2(y_2) + 3z^2 f^2(t)\ dz + \int_0^{z_2} 2K_2^2 f^2(y_0) + 2z^2 f^2(t)\ dz.$$

Next we use the definitions of y_0, y_1, and y_2 and change the variable of integration to obtain that

$$\int_0^1 p^2(t,z)\ dz \leq$$

$$3c_1 \int_{t-\frac{1}{c_1}(1-z_2)}^t f^2(s)\ ds - 3K_1^2 c_1 \int_{t-\frac{1}{c_1}(1-z_2)}^{t-\frac{2}{c_1}(1-z_2)} f^2(s)\ ds + 3\int_{z_2}^1 z^2 f^2(t)\ dz +$$

$$2K_2^2 c_2 \int_{t-\frac{1}{c_2}z_2-\frac{1}{c_1}(1-z_2)}^{t-\frac{1}{c_1}(1-z_2)} f^2(s)\ ds + 2\int_0^{z_2} z^2 f^2(t)\ dz.$$

We now recall that $f(t) = 0$ for $t < 0$ and that we are only considering $t \leq T$, so we may write

$$\int_0^1 p^2(t,z)\ dz \leq$$

$$3c_1 \int_0^T f^2(s)\ ds + 3K_1^2 c_1 \int_0^T f^2(s)\ ds + 2K_2^2 c_2 \int_0^T f^2(s)\ ds +$$

$$3\int_{z_2}^1 z^2 f^2(t)\ dz + 2\int_0^{z_2} z^2 f^2(t)\ dz =$$

$$(3c_1 + 3K_1^2 c_1 + 2K_2^2 c_2) \int_0^T f^2(s)\ ds + 3\int_{z_2}^1 z^2 f^2(t)\ dz + 2\int_0^{z_2} z^2 f^2(t)\ dz.$$

We thus have

$$\int_0^T \int_0^1 p^2(t,z)\ dz\ dt \leq$$

$$(3c_1 + 3K_1^2 c_1 + 2K_2^2 c_2) \int_0^T \int_0^T f^2(s)\ ds\ dt + 3\int_0^T f^2(t)\ dt \leq$$

$$(T(3c_1 + 3K_1^2 c_1 + 2K_2^2 c_2) + 3) \int_0^T f^2(t)\ dt.$$

Therefore,

$$|p|^2_{L^2(0,T;L^2(0,1))} \leq K|f|^2_{L^2(0,T)},$$

so that p depends continuously on f in the desired sense.
The above arguments establish existence, uniqueness and continuous dependence in the case $T \leq \frac{2}{c_1}(1 - z_2)$. For the given geometry and wave speeds, $\frac{2}{c_1}(1 - z_2)$ is the travel time for input waves at $z = 1$ which are reflected from $z = z_2$ to reach the "hard" or supraconductive boundary at $z = 1$ where they are again partially reflected. To extend the ideas above to finite intervals $[0, T]$ for $T > \frac{2}{c_1}(1 - z_2)$, one must include any and all subsequent reflections from $z = 1$ in the input (in addition to f) in defining the analytic solutions in (6), (7) and the associated weak solution in (3). The detailed arguments are tedious but rather straightforward and can be readily carried out to establish the results of the stated theorem for finite intervals $[0, T]$ with $T > \frac{2}{c_1}(1 - z_2)$. We omit details while at the same time noting that in the electromagnetic applications mentioned in the introduction and detailed in [1], [2], one usually does not require extremely long time intervals for the acoustic waves since the coupling electromagnetic waves are significantly more rapid and these wave reflections are the ones used in the related inverse problems.

4 Concluding remarks

In this paper, we have demonstrated the well-posedness for a wave equation in variational form in both time and space. The solution of the system is shown to exist, be

unique, and depend continuously on the forcing function. Due to the full variational form of the system, the approach taken is somewhat nonstandard.
Although the system considered here is rather specialized and chosen based on its utility in a particular application, the techniques can be extended with suitable modifications for other wave equations. In particular, the ideas can be extended to describe an acoustic wave traveling in a medium with more than two layers. However the increase in material interfaces causes an increased number of wave reflections and significantly adds to the tedium of constructing an analytical solution.

5 Acknowledgements

This research was supported in part by the Air Force Office of Scientific Research under grant AFOSR-F49620-01-1-0026 and in part through a Department of Education GAANN Fellowship to J.K. Raye under Grant P200A70707.

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