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CONTROL VARIATE MODELS FOR SENSITIVITY ESTIMATES OF REPAIRABLE ITEM SYSTEMS

Mohamed A. Ahmed
Donald Gross
Douglas R. Miller
Department of Operations Research
School of Engineering and Applied Science
George Washington University
Washington, DC 20052, U.S.A.

ABSTRACT

We previously developed a new modeling idea for comparing infinite-source, ample-server models (∞/∞) and finite-source, finite-server models (f/f). The comparison provides an efficient estimate of the error when approximating an f/f system with an ∞/∞ system and allows the analytical solution of the ∞/∞ model to be used as a control variate. We show that using ∞/∞ models as control variates for f/f systems can be an effective variance reduction technique for sensitivity estimates (gradients, Hessians). We also compare this to a method using concomitant control variates for the sensitivity estimates. Using these sensitivity estimates we will be able to determine more efficiently when ∞/∞ models are good approximations for f/f systems.

1. INTRODUCTION

Ahmed and Miller (1986) presented a method for efficiently simulating the difference between the behavior of finite-source, finite-server (f/f) and infinite-source, ample-server (∞/∞) models. An example of f/f is the machine repair model and an example of ∞/∞ is the M/G/∞ queue. Our purpose is to develop general guidelines to determine when the ∞/∞ solutions are acceptable approximations for the f/f models, and to correct them when they are not. We want to learn how the various parameters and particular structural assumptions affect the difference between f/f and ∞/∞ models.

A method to estimate simultaneously the performance measure and all its sensitivities (derivatives with respect to various parameters) was recently introduced by Rubinstein (1986) and Reiman and Weiss (1986). Although the point estimate of the derivative is strongly consistent, its variance is quite large compared to that of the point estimate of the original expectation. In light of this it is desirable to have some means of reducing the variance of the derivative estimate. We show that the composite model that incorporates the behavior of both f/f and ∞/∞ models is an efficient way to estimate the sensitivities of the difference between f/f and ∞/∞ models.

In Section 2 we briefly describe the repairable item system performance. Section 3 gives the main results for sensitivity analysis by Rubinstein (1986) and Reiman and Weiss (1986). Section 4 gives the statistical framework of control variates as a variance reduction technique. In Section 5 we review the composite model and show that it is an efficient way to estimate the sensitivities with respect to the arrival and service rates of machine repair systems. Section 6 gives some concluding remarks.

2. REPAIRABLE ITEM SYSTEM PERFORMANCE

A simple case of a repairable item system is the machine repair model shown in Figure 1. The situation modeled has a population consisting of M items which we desire to be operational at all times and Y spares that support the system. There are C repair channels. If more than C items require repair, a queue forms at the repair facility. Operating times until failure are exponentially distributed random variables with the mean time to failure of any item denoted by $1/\lambda$. Repair is generally distributed with mean time to repair denoted by $1/\nu$ [Cooper (1981), Gross and Harris (1985), Kleinrock (1975)].

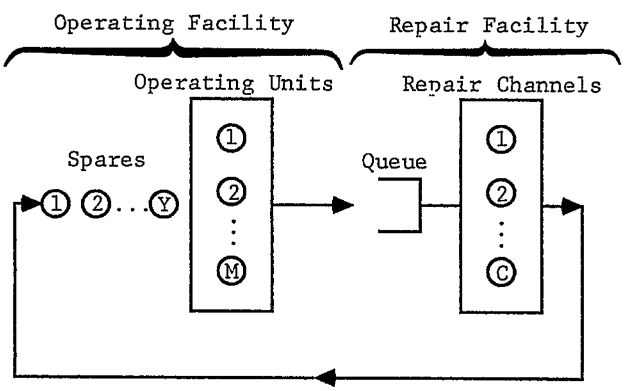

Figure 1: Schematic of a Machine Repair Model

This system has finite repair capacity and finite source (calling population). No analytical solution exists, except for some special cases with exponential service times [Gross and Miller (1984); Gross, Miller, and Soland (1983,1985)]. An approximate model with the ∞/∞ simplifying assumptions for the machine repair system would be an $M/G/\infty$ queue. This model is attractive because it can be solved analytically.

Infinite-source, ample-server models are much more tractable than finite-source, finite-repair-capacity models; however, they are only approximations for most repairable item systems and hence they should only be used when they are "good" approximations, in which case they are clearly the model of choice. This leads to the problem of computing or estimating the difference in performance between infinite-source, ample-server models (∞/∞) and finite-source, finite-server models (f/f).

Once obtaining an efficient estimation procedure for the error when approximating the f/f with an ∞/∞ system, we want to search the parameter space to identify regions of acceptable approximation error. We wish to learn how the estimated errors change if some parameters are changed. In short, we need to efficiently estimate the derivatives of the estimated errors with respect to various parameters. We apply the main results from Rubinstein (1986) and Reiman and Weiss (1986) for sensitivity analysis to the composite model that incorporates the behavior of both f/f and ∞/∞ models.

3. SENSITIVITY ANALYSIS VIA LIKELIHOOD RATIOS (SCORE FUNCTION)

Rubinstein (1986), and Reiman and Weiss (1986) showed that while simulating a single sample path from the underlying system, one can estimate simultaneously both the performance measure and its sensitivities (gradients, Hessian, etc.). For example, if λ is the rate of a Poisson process that occurs somewhere in the system, N is the number of Poisson events in (0,T], and ψ is a performance measure with expectation over (0,T] denoted by $E_\lambda[\psi]$ to emphasize the dependence of the expectation upon the parameter λ, then subject to some technical conditions, Reiman and Weiss (1986) showed that

$$\frac{d}{d\lambda} E_\lambda[\psi] = E_\lambda[\{\frac{N}{\lambda} - T\}\cdot\psi]$$

and

$$\frac{d^2}{d\lambda^2} E_\lambda[\psi] = E_\lambda[\{\frac{N(N-1)}{\lambda^2} - 2\,\frac{N}{\lambda}\,T + T^2\}\cdot\psi].$$

It is important to note that, since $E[(N/\lambda)-T] = 0$, $dE_\lambda[\psi]/d\lambda$ can be written as

$$\frac{d}{d\lambda} E_\lambda[\psi] = \mathrm{Cov}[\frac{N}{\lambda} - T,\ \psi].$$

To obtain derivatives of expectations with respect to parameters of continuous probability distributions, Rubinstein (1986) showed that

$$\frac{\partial}{\partial\nu} E_\nu[\psi] = E_\nu[\psi\cdot\{\frac{\partial}{\partial\nu} \ln \prod_{i=1}^{n} f(x_i,\nu)\}]$$

and

$$\frac{\partial^2}{\partial\nu^2} E_\nu[\psi]$$

$$= E_\nu[\psi\cdot\{\frac{\partial^2}{\partial\nu} \ln \prod_{i=1}^{n} f(x_i,\nu) + (\frac{\partial}{\partial\nu} \ln \prod_{i=1}^{n} f(x_i,\nu))^2\}].$$

Here n is the number of instances of the random variable X which has density $f(x,\nu)$ depending on a parameter ν.

For the sensitivity estimates of transient performance measures of a stochastic system, we now consider a modification of the Rubinstein results. In such situations the data are said to be Type I (or "time") censored, while Rubinstein's results assume that we have a complete sample available [see Lawless (1982)].

__Proposition__. Let $X_1,\ldots,X_n$ be i.i.d. random variables with p.d.f. $f(x,\nu)$ depending on a parameter ν, and survivor function $\bar{F}(x,\nu)$. Suppose that associated with each X_i, $i = 1,\ldots,n$, there is a fixed censoring time L_i, then

$$\frac{\partial}{\partial\nu} E_\nu[\psi]$$

$$= E_\nu[\psi\cdot\frac{\partial}{\partial\nu} \ln \prod_{i=1}^{n} f(t_i,\nu)^{\delta_i}\bar{F}(L_i,\nu)^{1-\delta_i}] \qquad (2.1)$$

where $t_i = \min(X_i,L_i)$ and $\delta_i = 1$ if $X_i \leq L_i$, and 0 otherwise.

__Proof__. The joint p.d.f. of t_i and δ_i is

$$f(t_i,\nu)^{\delta_i} \bar{F}(L_i,\nu)^{1-\delta_i}$$

To see this, note that the distribution of (t_i,δ_i) has components

$$\Pr(t_i=L_i,\delta_i=0) = \Pr(\delta_i=0)$$
$$= \Pr(T_i > L_i) = \bar{F}(L_i,\nu)$$

$$\Pr(t_i,\delta_i=1)$$
$$= \Pr(t_i|\delta_i=1)\Pr(\delta_i=1), \quad t_i < L_i$$
$$= \Pr(t_i|t_i < L_i)\Pr(\delta_i=1)$$
$$= [f(t_i)/(1-\bar{F}(L_i))][1-\bar{F}(L_i)]$$
$$= f(t_i).$$

These expressions can be combined into the single expression

$$\Pr(t_i,\delta_i) = f(t_i,\nu)^{\delta_i}\,\bar{F}(L_i,\nu)^{1-\delta_i},$$

and if pairs (t_i,δ_i) are independent, the likelihood is

$$\prod_{i=1}^{n} f(t_i,\nu)^{\delta_i}\bar{F}(L_i,\nu)^{1-\delta_i}.$$

__Example__. Let $f(t) = \lambda e^{-\lambda t}$ for $t > 0$ and let T_i be the time between events i and i+1. Then the likelihood is

$$L = \prod_{i=1}^{n} (\lambda e^{-\lambda t_i})^{\delta_i} e^{-\lambda L_i(1-\delta_i)}$$
$$= \lambda^r e^{-\lambda T}$$

and $\ln L = r\ln\lambda - \lambda T$ so that

$$\frac{\partial \ln L}{\partial\lambda} = \frac{r}{\lambda} - T$$

where r is the observed number of arrivals in (0,T]. Hence from (2.1),

$$\frac{\partial}{\partial\lambda} E_\lambda(\psi) = E_\lambda[(\frac{r}{\lambda} - T)\cdot\psi],$$

which is the result of Reiman and Weiss for the Poisson process.

It was pointed out by Rubinstein (1986) and Reiman and Weiss (1986) that the variance of the estimator for the derivative can be prohibitively large. To obtain more accurate estimators, one can use efficient variance reduction techniques. We will use the control variate technique for variance reduction.

4. CONTROL VARIATES

A random variable C is a control variate for Y if its expectation, μ_C, is known and if it is correlated with Y. For any fixed value of the control coefficient, a, the controlled estimator

$$Y(a) = Y - a(C-\mu_C)$$

is an unbiased estimator of μ_Y. Furthermore,

$$\mathrm{Var}[Y(a)] = \mathrm{Var}[Y] + a^2\mathrm{Var}[C] - 2a\ \mathrm{Cov}[Y,C]$$

so that Y(a) has a smaller variance than Y if and only if $2a\ \mathrm{Cov}[Y,C] > a^2\ \mathrm{Var}[C]$.

The control variate methods with which we are concerned involve the simultaneous use of simulation with an analytically tractable model that approximates the system under study, and uses the corresponding output C from the simulation of this second model as a control variate. By "analytically tractable" we mean specifically that E[C] can be computed exactly, and we hope that the "similarity" of the simpler model to the original one will induce positive correlation between C and Y. This method of obtaining control variates is referred to as an external control variate technique. We also apply the method of concomitant control variates to the sensitivity estimates, using the likelihood ratio, and compare it to the external control variate technique.

We now turn to the question of how the control coefficient, a, should be specified. The special case a = 1 (if it is believed that Cov[Y,C] > 0) requires that Cov[Y,C] > Var[C]/2 for variance reduction; by allowing other values for a, we can do better. The optimum control coefficient, a*, is given by (see Wilson 1984)

$$a^* = \frac{\mathrm{Cov}[Y,C]}{\mathrm{Var}[C]} \tag{4.1}$$

and $\mathrm{Var}[Y(a^*)] = \mathrm{Var}[Y](1 - \rho^2_{YC})$, where ρ_{YC} is the correlation coefficient between Y and C. From this, it is clear that the greater the correlation between the variable to be estimated and the control variate, the greater the variance reduction

5. COMPUTATIONAL EXPERIENCE

5.1 Review of the Composite Model

The composite model is an efficient way to estimate the difference in behavior between finite-source, finite-server models (f/f) and infinite-source, ample-server models (∞/∞). The idea is to simulate an open queuing network which incorporates the behavior of both f/f and ∞/∞ models. The composite model simulates the common behavior of the two systems once and the special behavior of each system once. For more details on the composite model, see Ahmed and Miller (1986).

To clarify these ideas, suppose we are interested in estimating one of the performance measures of the machine repair system, $E[\psi_F]$. We select an M/G/∞ queue as an approximate model for which it is possible to calculate analytically its performance measure $E[\psi_\infty]$. We then use the composite model to estimate the difference between these two processes, $E[\psi_D]$. We may estimate $E[\psi_F]$ as follows:

$$\hat{E}[\psi_F] = E[\psi_\infty] + \hat{E}[\psi_D].$$

It is easy to verify that a reduction in variance over a straightforward simulation has been achieved if

$$Var[\psi_\infty] < 2\ Cov[\psi_\infty, \psi_F].$$

In Ahmed and Miller (1986) the control coefficient was chosen to be one. An extension was made to those experiments using the optimal control coefficient given by (4.1) and better results were obtained (see Table 2). We are interested in four different performance measures of the machine repair system. They are:

η_1 = avg. no. items in or awaiting repair
η_2 = avg. no. busy repair channels
η_3 = avg. no. operating machines
η_4 = probability that M machines operating.

We estimate these performance measures for a transient system at t = 30. Some of the cases considered are shown in Table 1. The repair times are gamma with mean $1/\nu$ and shape parameter 2. The results for the test cases of Table 1 are given in Table 2.

Table 1: Some Test Cases

Case No.	M	Y	C	Traffic Intensity[†]	λ	ν
1	24	6	7	.27	0.1	1.25
2	24	6	4	.48	0.1	1.25
3	24	3	4	.48	0.1	1.25
4	24	0	4	.48	0.1	1.25
5	24	6	3	.64	0.1	1.25
6	24	3	3	.64	0.1	1.25
7	24	0	3	.64	0.1	1.25
8	24	6	2	.96	0.1	1.25
9	24	3	2	.96	0.1	1.25
10	24	0	2	.96	0.1	1.25

[†]We define traffic intensity as $M\lambda/C\nu$.

5.2 Sensitivity Estimates

We consider the average number of items in or awaiting repair as one of the performance measures of the machine repair system. We estimate the sensitivity of this performance measure with respect to the arrival rate as well as the service rate for a transient system at time t = 30 (Poisson arrival process with rate $M\lambda$, the service times are gamma with mean $1/\nu$ and shape parameter 2). We use three approaches. The first is simply to simulate an f/f (machine repair) model of the system. The second is to simulate the composite model, which uses an ∞/∞ (M/G/∞ queue) model of the system as a control variate for the f/f model. In order to estimate the sensitivity of the performance of the f/f system using the composite simulator, we must compute the sensitivity of the behavior of the ∞/∞ system analytically. This can be done using the basic properties of the transient M/G/∞ queue; see the appendix.

The third approach is to simulate the f/f model using the method of concomitant (internal) control variates. The control variates that we use arise naturally from likelihood ratios. We suggest here a class of random variables with 0 mean that can be used as internal control variates. For the derivatives with respect to the Poisson rates, we choose

$$C_1 = [\frac{N}{\lambda} - T]$$

as a control variate. For the derivatives with respect to the service rates, we choose, as a control variate,

$$C_2 = \frac{\partial}{\partial\nu} \ell n \prod_{i=1}^{n} f(t_i,\nu)^{\delta_i} \bar{F}(L_i,\nu)^{1-\delta_i}$$

which appears in equation (2.1).

The efficiency measure of a procedure is taken as inversely proportional to the product of the variance of the estimator and the CPU time required to execute the procedure:

$$\text{Efficiency} = \frac{1}{\text{variance} \bullet \text{CPU time}}.$$

Table 2: Comparisons of Composite Model and Straight Simulation of Machine Repair Systems for a = 1 and Optimal a (a*)

Case No.	Ratios for Estimator of Variance							
	η_1		η_2		η_3		η_4	
	a=1	a*	a=1	a*	a=1	a*	a=1	a*
1	.000	.000	.000	.000	.000	.000	.000	.000
2	.058	.056	.133	.101	.500	.440	.462	.462
3	.055	.055	.128	.094	.156	.156	.098	.097
4	.107	.096	.191	.131	.107	.096	.149	.147
5	.327	.326	.668	.355	.853	.837	.800	.800
6	.279	.279	.629	.327	.496	.493	.404	.398
7	.237	.233	.552	.288	.237	.233	.273	.252
8	.873	.871	8.408	.872	.993	.993	.991	.987
9	.812	.808	6.047	.820	.895	.894	.988	.872
10	.728	.718	3.808	.735	.728	.718	.850	.540

Note: Entries correspond to Var(composite)/Var(finite/finite), and a is the control coefficient.

Table 3: Results of Composite and Concomitant Variance Reduction Procedures and Direct Simulation for Sensitivity Estimates

Case No.	Straightforward Simulation				Composite Approach				Concomitant Control			
	$\hat{d}_\lambda$	$\hat{d}_\nu$	$\sigma^2_{d_\lambda}$	$\sigma^2_{d_\nu}$	$\hat{d}_\lambda$	$\hat{d}_\nu$	$\sigma^2_{d_\lambda}$	$\sigma^2_{d_\nu}$	$\hat{d}_\lambda$	$\hat{d}_\nu$	$\sigma^2_{d_\lambda}$	$\sigma^2_{d_\nu}$
1	.404	- .952	.070	.492	.800	- 1.534	.000	.000	.800	- 1.272	.024	.182
2	.576	-1.089	.082	.574	.999	- 1.709	.002	.014	.999	- 1.431	.030	.234
3	.512	- .960	.077	.506	.921	- 1.541	.002	.014	.926	- 1.300	.027	.193
4	.449	- .857	.063	.386	.819	- 1.342	.002	.043	.824	- 1.277	.022	.144
5	.923	-1.398	.117	.716	1.399	- 2.655	.016	.095	1.420	- 1.916	.045	.279
6	.774	-1.142	.105	.632	1.228	- 1.752	.013	.090	1.252	- 1.522	.038	.239
7	.644	-1.104	.082	.464	1.052	- 1.614	.007	.085	1.064	- 1.551	.030	.177
8	4.825	-9.555	.561	3.448	5.744	-10.677	.184	1.613	6.005	-10.643	.154	1.017
9	2.873	-6.311	.367	2.040	3.622	- 7.199	.116	.891	3.809	- 7.192	.111	.611
10	1.627	-4.301	.204	1.119	2.202	- 4.968	.057	.469	2.315	- 5.099	.066	.363

We estimate the efficiency by observing the CPU time for each execution and estimating the variance of the estimators of the gradient of the performance measures of interest [Hammersley and Handscomb (1964)].

We used various cases as tests to compare the different simulation approaches. Some of the cases considered are shown in Table 1. Each case was simulated for 1000 replicates. For each case we looked at the estimated mean number of items in or awaiting repair, η_1, and its variance; and the estimated derivative of $\hat{\eta}_1$ with respect to the arrival rate, $\hat{d}_\lambda$, and the service rate, $\hat{d}_\nu$, and their variances. The results for test cases of Table 1, showing point estimates and variances of the derivatives of the three methods, are given in Table 3.

Table 4 presents comparisons of variances of the estimated derivatives, CPU times, and efficiencies for three approaches: the composite model (C), direct simulation of the f/f model (F), and the internal (concomitant) control variate approach (I). The composite approach, when compared with direct simulation of the f/f model (C/F), produces a significant decrease in the variance of all estimators; but it also increases the CPU time. Nevertheless, the efficiency of the composite approach is superior to the straightforward approach of simulating the f/f system. Furthermore, the composite approach achieves a reduction in the variance of the performance estimate, as well as of its derivative, of up to 100%. For example, from Tables 2 and 4 we see for

Table 4: Comparisons of Composite Model vs. f/f, Internal Control Variable vs. f/f, and Composite vs. Internal Approaches

Case No.	Ratios for Estimators of Performance								
	Variance			Average CPU Time			Efficiency		
	C/F	I/F	C/I	C/F	I/F	C/I	C/F	I/F	C/I
	$\hat{d}_\lambda$ = derivative with respect to the arrival rate								
1	.000	.344	.000	1.019	1.010	1.009	∞	2.878	∞
2	.027	.364	.074	1.053	1.013	1.039	35.173	2.712	13.006
3	.024	.345	.070	1.042	1.011	1.031	39.987	2.867	13.856
4	.034	.352	.097	1.042	1.012	1.030	28.226	2.807	10.056
5	.136	.383	.355	1.075	1.010	1.064	6.840	2.585	2.646
6	.123	.363	.339	1.096	1.012	1.083	7.418	2.722	2.725
7	.091	.368	.247	1.095	1.015	1.079	10.036	2.692	3.748
8	.328	.273	1.201	1.145	1.014	1.129	2.663	3.612	.737
9	.317	.301	1.053	1.155	1.013	1.140	2.729	3.280	.820
10	.278	.322	.863	1.162	1.016	1.144	3.096	3.057	1.013
	$\hat{d}_\nu$ = derivative with respect to the service rate								
1	.000	.369	.000	1.019	1.010	1.009	∞	2.683	∞
2	.025	.408	.061	1.053	1.013	1.039	37.987	2.420	15.700
3	.027	.381	.071	1.042	1.011	1.031	35.544	2.596	13.691
4	.111	.373	.298	1.042	1.012	1.030	8.646	2.649	3.264
5	.133	.390	.341	1.075	1.010	1.064	6.994	2.539	2.755
6	.143	.378	.378	1.096	1.012	1.083	6.380	2.614	2.441
7	.185	.381	.486	1.095	1.015	1.079	4.936	2.586	1.909
8	.469	.295	1.590	1.145	1.014	1.129	1.862	3.343	.557
9	.438	.300	1.460	1.155	1.013	1.140	1.977	3.291	.601
10	.420	.324	1.296	1.162	1.016	1.144	2.049	3.038	.674

Note: Entries for X/Y correspond to Var(X)/Var(Y), CPU(X)/CPU(Y), and Efficiency(X)/Efficiency(Y).

the first test case that the variances of the performance estimate and their derivatives are 0.0. The first test case means that the ∞/∞ model is an exact solution for the f/f model. Now, considering C vs. I (C/I), for light and medium traffic intensity, the composite model is superior to the internal control variate approach. For heavy traffic, where there is little overlap between ∞/∞ and f/f, the internal control variate approach is better.

6. CONCLUSIONS

This preliminary study shows that the composite model is an efficient way to estimate the sensitivities of the behavior of finite-source, finite-server models. We plan to use this method to determine when ∞/∞ models are good approximations for f/f systems. An efficient estimator of the gradient will help us search the parameter space to identify regions of acceptable approximate error.

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APPENDIX: M/G/∞ QUEUE

Let $N(t)$ denote the number of items in repair at time t when arrivals occur according to a Poisson process with rate $M\lambda$. Then $N(t)$ has a Poisson distribution with mean [see Gross and Harris (1985)]

$$E[N(t)] = M\lambda \int_0^t \bar{F}(s,t)ds,$$

where

$$\bar{F}(s,t) = \Pr\{\text{a unit entering repair at time } s \text{ is still in repair at time } t\}.$$

When the service time distribution functions are gamma with mean $1/\nu$ and integer shape parameter 2, the expected number of items in repair is

$$E[N(t)]$$
$$= M\lambda \int_0^t e^{-2\nu x} \sum_{j=0}^{1} \frac{(2\nu x)^j}{j!} dx$$
$$= \frac{M\lambda}{\nu} - M\lambda(\frac{1}{\nu} + t)e^{-2\nu t}$$

and

$$\frac{\partial E[N(t)]}{\partial M\lambda} = \frac{1}{\nu} - (\frac{1}{\nu} + t)e^{-2\nu t}$$

$$\frac{\partial E[N(t)]}{\partial \nu} = -\frac{M\lambda}{\nu^2} + 2M\lambda t(\frac{1}{\nu} + t)e^{-2\nu t} + \frac{M\lambda}{\nu^2} e^{-2\nu t}.$$

REFERENCES

Ahmed, M. A. and Miller, D. R. (1986). Infinite-source, ample-server control variate models for finite-source, finite-server models. In: Proceedings of the 1986 Winter Simulation Conference. Institute of Electrical and Electronics Engineers, 316-325.

Cooper, R. B. (1981). Introduction to Queueing Theory, Second Edtion. North Holland.

Gross, D. and Harris, C. M. (1985). Fundamentals of Queueing Theory, Second Edtion. John Wiley & Sons, Inc., New York.

Gross, D. and Miller, D. R. (1984). Multi-echelon repairable item provisioning in a time-varying environment using the randomization technique. Naval Research Logistics Quarterly 31, 347-361.

Gross, D., Miller, D. R., and Plastiras, C. G. (1984). Simulation methodologies for transient Markov processes: A comparative study based on multi-echelon repairable item inventory systems. In: Proceedings of the Summer Computer Simulation Conference. Boston, MA, 37-43.

Gross, D., Miller, D. R., and Soland, R. M. (1983). A closed queueing network model for multi-echelon repairable item provisioning. IIE Transactions 15, 344-352.

Gross, D., Miller, D. R., and Soland, R. M. (1985). On some common interests among reliability, inventory, and queuing. IEEE Transactions 34, 204-211.

Hammersley, J. M. and Handscomb, J. M. (1964). Monte Carlo Methods. Methuen, London.

Hillestad, R. J. (1981). Dyna-METRIC: Dynamic multi-echelon technique for recoverable item control. WD-911-AF, Rand Corporation, Santa Monica, CA.

Kleinrock, L. (1975). Queueing Systems, Volume 1. John Wiley & Sons, Inc., New York.

Law, A. M. and Kelton, W. D. (1982). Simulation Modeling and Analysis. McGraw-Hill, Hightstown, NJ.

Lawless, J. F. (1982). Statistical Models and Methods for Lifetime Data. John Wiley & Sons, Inc., New York.

Reiman, M. I. and Weiss, A. (1986). Sensitivity analysis for simulations via likelihood ratios. In: Proceedings of the 1986 Winter Simulation Conference. Institute of Electrical and Electronics Engineers, 285-295.

Rubinstein, R. Y. (1986). The score function approach for sensitivity analysis of computer simulation models. Mathematics and Computers in Simulation 28, 1-29.

Sherbrooke, C. C. (1968). METRIC: A multi-echelon technique for recoverable item control. Operations Research 16, 122-141.

Wilson, J. R. (1984). Variance reduction techniques for digital simulation. American Journal of Mathematics and Management Science 4, 277-312.

AUTHORS' BIOGRAPHIES

MOHAMED A. AHMED is a Graduate Research Assistant in the Department of Operations Research at George Washington University. He has an M.S. in Operations Research from G.W.U. He is currently working on his D.Sc. in Operations Research.

Mohamed A. Ahmed
Department of Operations Research, SEAS
George Washington University
Washington, D.C. 20052, U.S.A.
(202) 994-8232

DONALD GROSS is Professor and Chairman, Department of Operations Research, School of Engineering and Applied Science, George Washington University. A member of the Operations Research Society of America, The Institute of Management Sciences, the Washington Operations Research/Management Science Council, and the Institute of Industrial Engineers, Dr. Gross has contributed numerous articles to such professional journals as IIE Transactions, Journal of Applied Probability, Management Science, Naval Research Logistics, Operations Research, and others. He is co-author (with Carl M. Harris) of Fundamentals of Queueing Theory, Second Edition, John Wiley & Sons, Inc., New York, 1985.

Donald Gross
Department of Operations Research, SEAS
George Washington University
Washington, D.C. 20052, U.S.A.
(202) 994-6084

DOUGLAS R. MILLER is Professor of Operations Research at George Washington University. He received his Ph.D. in Operations Research from Cornell, was Assistant and Associate Professor of Statistics at the University of Missouri-Columbia, and worked at NASA Langley Research Center before coming to G.W.U. He is a member of ORSA, ASA, and IEEE Computer Society. His research interests cover many topics in the realm of applied probability and statistics.

Douglas R. Miller
Department of Operations Research, SEAS
George Washington University
Washington, D.C. 20052, U.S.A.
(202) 994-7528