Proceedings of the 1995 Winter Simulation Conference
ed. C. Alexopoulos, K. Kang, W. R. Lilegdon, and D. Goldsman

VARIANCE REDUCTION FOR DISCRETE EVENT SYSTEMS DRIVEN BY POISSON PROCESSES

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ABSTRACT

In studies on conditions guaranteeing variance reduction for Common Random Numbers (CRN), there is often the implicit assumption that the timing inputs, i.e., the random variables of clock samples, are generated via the inverse transform method. Some recommend using only inverse transform method when using CRN, suggesting that this strategy yields the best result, i.e., the highest degree of variance reduction. In this paper, we derive conditions guaranteeing variance reduction for a special class of systems—generalized semi-Markov processes (GSMP) with exponential clock samples—when using CRN and when the clock samples are generated via a version of the acceptance-rejection method. Our preliminary experimental results show that the variance reduction under this construction may be higher, sometimes significantly, than when inverse transform method is used.

1 INTRODUCTION

In studies on conditions guaranteeing variance reduction for Common Random Numbers (CRN), there is often the implicit assumption that the timing inputs, i.e., the random variables of clock samples, are generated via the inverse transform method. Some recommend using only inverse transform method when using CRN, suggesting that this strategy yields the best result, i.e., the highest degree of variance reduction (for an informative discussion on the use of inverse transform when using CRN, see Glasserman and Yao 1992b, Section 2.2). Far less analysis is available when the timing inputs are sampled via some version of the acceptance-rejection method (Glasserman and Vakili 1994 provide some results related to this case). In this paper, we derive conditions guaranteeing variance reduction for a special class of systems—generalized semi-Markov processes (GSMP) with exponential clock samples—when using CRN and when the clock samples are generated via a version of the acceptance-rejection method. Our preliminary experimental results show that the variance reduction under this construction may be higher, sometimes significantly, than when inverse transform method is used.

Consider two generalized semi-Markov processes. Let $L_i : \Omega \rightarrow \mathbf{R}$ represent the random variable of some performance index of system i ($i = 1, 2$). One would expect that to compare the two systems, it is more appropriate to compare their performances under the same set of inputs, i.e., it is more appropriate to compare $L_1(\omega)$ with $L_2(\omega)$. This intuition is validated if $Var(L_1 - L_2)$ is smaller under the common input approach when compared with independent sampling, or, equivalently, if L_1 and L_2 are positively correlated when common inputs are used.

The common approach to establishing variance reduction is to rely on two notions of (a) *association*—a strong form of positive correlation defined for probability measures on partially ordered sets—and (b) *monotonicity*. In this approach, it is shown that L_1 and L_2 are monotone functions of the clock readings. If the measure defined on the space of clock readings is associated, then variance reduction of the CRN follows from the fact that increasing functions of an associated measure are positively correlated, i.e., $\mathsf{Cov}(L_1, L_2) \geq 0$ (see, e.g., Heidelberger and Iglehart 1979, and Glasserman and Yao 1992b). It is often assumed that clock readings of the same event form i.i.d. sequences of random variables and that clock readings of different events are independent. Association of the resulting measure on the input space follows from established results immediately (Esary et al. 1967).

The input space for our particular construction of a GSMP with exponential clock readings corresponds to that of a finite number of Poisson processes. Samples of clock readings are generated by a well-known acceptance-rejection method from the inter-event times of the Poisson processes; in this sense, it may be said that the Poisson processes drive the GSMP. We show that the natural partial order for this construction is one that is different from the partial order used when clock samples are generated via inverse transform. Our variance reduction results follow from two basic results proved in the paper: (1) The Poisson measure defined on the input space is associated. (2) Condition **M**, shown by Glasserman and Yao (1992a) to imply monotonicity of the event epoch with respect to the commonly used partial order on the input space, also implies monotonicity with respect to the partial order defined on the input space we use in this paper.

The rest of the paper is organized as follows. The GSMP model and our particular construction is defined in Section 2. In Section 3 we show that the Poisson measure defined on the input space is associated. In Section 4 we prove that condition **M** implies monotonicity of the event epochs. Variance reduction and experimental results are given in Section 5. We conclude with a brief summary.

2 MODEL

We begin with the definition of a generalized semi-Markov *scheme* (GSMS). To simplify the presentation, we limit ourselves to deterministic schemes, i.e., those in which the triggering event of a transition and the current state of the system uniquely determine the next state of the system.

2.1 A Generalized Semi Markov Scheme

Let S, a finite or countably infinite set, be the state space of the system, and $A = \{\alpha_1, \cdots, \alpha_k\}$ be the set of events; for each state $s \in S$, let $\mathcal{E}(s)$ be the set of active events in s. Let $\mathcal{E} = \{\mathcal{E}(s); s \in S\}$. For each $s \in S$ and $\alpha \in \mathcal{E}(s)$, let $f_\alpha(s)$ be the unique next state of the system if the present state is s when α occurs. Let $S_\alpha = \{s; \alpha \in \mathcal{E}(s)\}$. In other words, S_α is the set of all states in which α is active. Therefore, $f_\alpha : S_\alpha \to S$. Let $\Psi = \{f_\alpha; \alpha \in A\}$. Given these definitions, a deterministic GSMS is defined by $\mathcal{G} = (S, A, \mathcal{E}, \Psi)$. A GSMS captures (defines) the structure of the system.

2.2 Timing Inputs

Let $\mathbf{R}_+$ be the set of non-negative real numbers, representing the time axis ($t \geq 0$). Let $\mathcal{B}$ be the Borel σ-field on $\mathbf{R}_+$. A *simple counting measure* m on $\mathbf{R}_+$ is a measure on $(\mathbf{R}_+, \mathcal{B})$ such that

1. $m(C) \in \{0, 1, 2, \cdots, \infty\}$ for all $C \in \mathcal{B}$,
2. $m([a, b]) < \infty$ for all $a, b \in \mathbf{R}_+$,
3. $m(\{x\}) \in \{0, 1\}$.

To each counting measure m there corresponds a unique sequence $\omega = \{t_n; n \geq 1\} \subset \mathbf{R}_+$ such that

1. $t_1 < t_2 < \cdots$,
2. $m(C) = \sum_{n \geq 1} I\{t_n \in C\}$.

where I is the indicator function.

Let M be the space of all simple counting measures. Let $M_{\alpha_1}, \cdots, M_{\alpha_k}$ be k copies of M. To simplify the presentation, and with a slight abuse of notation, we write $\omega_\alpha \in M_\alpha$; ω_α refers to the sequence of epochs associated with a counting measure in M_α. We define the space of timing inputs, Ω, as follows

$$\Omega = \prod_{i=1}^{k} M_{\alpha_i}.$$

Let $\omega \in \Omega$; then $\omega = (\omega_{\alpha_1}, \cdots, \omega_{\alpha_k})$. To make things more explicit, we write $\omega_{\alpha_i} = \{t_1(\alpha_i), t_2(\alpha_i), \cdots\}$. In words, ω_{α_i} is the sequence of epochs "reserved" for the occurrences of event α_i.

We denote the superposition of the components of ω by $\{(t_1, e_1), (t_2, e_2), \cdots\}$. Note that $\{t_1, t_2, \cdots\} = \cup_{i=1}^{k} \{t_1(\alpha_i), t_2(\alpha_i), \cdots\}$ and $e_n = \alpha_i$ if $t_n = t_{n_i}(\alpha_i)$, for some n_i.

2.3 The State Process

To define the sample path of the system corresponding to the input $\omega \in \Omega$, i.e., $\{X_t(\omega); t \geq 0\}$, we proceed as follows:

Fix an initial state $s_0 \in S$. (In the rest of the paper we assume the initial state of the system is fixed and is equal to s_0.) Define the discrete-time sequence $\{Y_n(\omega); n \geq 0\}$—the sequence of states of the system at instances $\{t_0 = 0, t_1, t_2, \cdots\}$—recursively, by $Y_0(\omega) = s_0$, and

$$Y_{n+1}(\omega) = \begin{cases} f_{e_{n+1}}(Y_n(\omega)), & \text{if } e_{n+1} \in \mathcal{E}(Y_n(\omega)); \\ Y_n(\omega), & \text{otherwise.} \end{cases}$$

In other words, if event e_{n+1} is active in state $Y_n(\omega)$, the reserved epoch for this event is used and a transition to a new state occurs; on the other hand, if e_{n+1} is not active in state $Y_n(\omega)$ this event is simply ignored (the reserved epoch is not used) and the time is advanced to the next epoch.

The state trajectory is defined by

$$X_t(\omega) = \sum_{n=0}^{\infty} Y_n(\omega) I\{t_n \leq t < t_{n+1}\}.$$

We now define a particular probability measure on the set of inputs.

2.4 Poisson Probability Measures

The Poisson probability measure on M is defined as follows: Let $\mathcal{M}$ be the σ-algebra generated by functions $m \to m(C)$. Let the probability measure $\mathcal{P}$, defined on $(M, \mathcal{M})$, be such that

1. If $C_1, \cdots, C_r \in \mathcal{B}$ are disjoint then $\mathcal{P}(m(C_1) = n_1, \cdots, m(C_r) = n_r) = \prod_{i=1}^{r} \mathcal{P}(m(C_i) = n_i)$, i.e., events $\{m(C_i) = n_i\}, i = 1, \cdots, r$ are independent;

2. For each $C \in \mathcal{B}$ and $n \geq 0$,

$$\mathcal{P}(m(C) = n) = e^{-\lambda\mu(C)} \frac{(\lambda\mu(C))^n}{n!},$$

where $\mu(C)$ is the Lebesque measure of the set C. A random element of $(M, \mathcal{M})$ with distribution $\mathcal{P}$ is a Poisson process with rate λ.

Let $(M_{\alpha_i}, \mathcal{M}_{\alpha_i}, P_{\alpha_i})$ be defined as above, where the rate of P_{α_i} is λ_{α_i}. Then the probability space of the inputs is defined by

$$(\Omega, \mathcal{F}, P) = (\prod_{i=1}^{k} M_{\alpha_i}, \prod_{i=1}^{k} \mathcal{M}_{\alpha_i}, \prod_{i=1}^{k} P_{\alpha_i}).$$

One of our main results, given in the next section, is that the probability measure P is associated.

3 ASSOCIATION OF POISSON PROBABILITY MEASURES

Association is a strong form of positive correlation and a property of random variables, random vectors, and, more generally, of probability measures on partially ordered set. Introduced by Esary et al. (1967) for sets of real-valued random variables, it is defined as follows:

A set of real-valued random variables $\{X_1, \ldots, X_n\}$ is said to be *associated* if, for any two increasing real-valued functions f and g,

$$\mathrm{Cov}[f(X_1, \ldots, X_n), g(X_1, \ldots, X_n)] \geq 0$$

when the covariance exists.

More generally, let $(\Omega, \leq)$ be a partially ordered set. A set A is called an *increasing set* or an *upper set* if

$$x \in A, x \leq y \Rightarrow y \in A.$$

For example, on the real line, $\mathbf{R}$ (with the usual order), increasing sets are sets of the form $[a, \infty)$, or (a, ∞), where $a \in \mathbf{R}$. For any partial order on Ω, the set of upper sets defines the partial order and vice versa. A function $f : \Omega \to \mathbf{R}$ is increasing iff the inverse image of upper sets in $\mathbf{R}$ are upper sets in Ω.

A probability measure μ on Ω is called *associated* if

$$\mu(A_1 \cap A_2) \geq \mu(A_1)\mu(A_2) \tag{1}$$

for all upper sets A_1, A_2. An equivalent definition is

$$\int fg d\mu \geq \int f d\mu \int g d\mu \tag{2}$$

for all bounded and increasing $f, g : \Omega \to \mathbf{R}$ (Lindqvist 1988). Given this definition, an n-dimensional vector of real-valued random variables is associated if its distribution is an associated measure on $\mathbf{R}^n$.

In what follows we use the following properties of associated sets of random variables (see Esary et al. 1967).

1. *A set consisting of a single random variable is associated.*

2. *Independent random variables are associated.*

3. *Subsets of associated random variables are associated.*

4. *Increasing functions of associated random variables are associated.*

Having stated these general definitions and results, we now turn to the input space, Ω, defined in the previous section. We define the following partial order on Ω:

We define partial orders on the sets M_{α_i} as follows. Let $\omega_{\alpha_i}, \omega'_{\alpha_i} \in M_{\alpha_i}$. Then

$$\omega_{\alpha_i} \leq \omega'_{\alpha_i} \Longleftrightarrow \omega_{\alpha_i} \subseteq \omega'_{\alpha_i}.$$

The above partial order, expressed in terms of the corresponding counting measures, becomes:

$$m_{\alpha_i} \leq m'_{\alpha_i} \Longleftrightarrow m_{\alpha_i}(C) \leq m'_{\alpha_i}(C), \quad \text{for all } C \in \mathcal{B}.$$

The partial order on the input set Ω is defined as the product or componentwise order; in other words,

$$\omega \leq \omega' \Longleftrightarrow \omega_{\alpha_i} \leq \omega'_{\alpha_i}, \quad 1 \leq i \leq k.$$

We are now prepared to state and prove the main results of this section.

Proposition 1 *The Poisson probability measure P_{α_i} defined on $(M_{\alpha_i}, \mathcal{M}_{\alpha_i})$ is associated.*

Proof. The upper sets on M_{α_i} are generated by the following sets

$$\{m \in M_{\alpha_i}; m(C) \geq k\},$$

where $k \in \mathbf{Z}^+$ and $C \in \mathcal{B}$. It is sufficient to show that condition (1) is verified for these sets.

Let $C_1, C_2 \in \mathcal{B}$. The sets $C_1 \cap C_2$, $C_1 \cap C_2'$ and $C_1' \cap C_2$ are disjoint (C' is the complement of C); hence, because of the properties of Poisson measures, $N_1 = m(C_1 \cap C_2)$, $N_2 = m(C_1 \cap C_2')$, and $N_3 = m(C_1' \cap C_2)$ are independent random variables. The random variable N_i is associated ($i = 1, 2, 3$) (property 1 above). Given the independence of N_i's, $\{N_1, N_2, N_3\}$ is an associated set of random variables (property 2 above). Also note that $m(C_1) = N_1 + N_2$, $m(C_2) = N_1 + N_3$, and that functions $f(x_1, x_2, x_3) = x_1 + x_2$ and $g(x_1, x_2, x_3) = x_1 + x_3$ are increasing. Therefore, $m(C_1)$ and $m(C_2)$ are associated random variables (property 4 above). Let $k_1, k_2 \in \mathbf{Z}^+$ and let $A_i = \{m; m(C_i) \geq k_i\}$ ($i = 1, 2$)be two upper sets. The inequality

$$P(A_1 \cap A_2) \geq P(A_1)P(A_2)$$

follows from the fact that $m(C_1)$ and $m(C_2)$ are associated. □

Theorem 1 *The probability measure P defined on $(\Omega, \mathcal{F})$ is associated.*

Proof . The proof follows from the above proposition and Theorem 3.3 of Lindqvist (1988) that states: If two probability measures on two sets are associated, then the product measure defined on the product space of the two sets is associated. □

We now turn our attention to a number of outputs of the GSMP defined in the previous section.

4 Monotonicity of Event Epochs

We consider the following outputs of the system defined in Section 2. Let $\{T_\alpha(n); n \geq 1\}, \alpha \in A$, be the sequences of event epochs, and $\{D_\alpha(t); t \geq 0\}, \alpha \in A$, the number of events of a type occurred prior to, or at time t. A number of performance indices of interest can be expressed as functions of these outputs. These quantities are defined as follows:

$T_\alpha(0)(\omega) = 0$ and for $n \geq 1$,

$$\begin{aligned} T_\alpha(n)(\omega) &= \min\{t_i, i \geq 1 : e_i = \alpha, \alpha \in \mathcal{E}(Y_{i-1}(\omega)), \\ &\qquad t_i > T_\alpha(n-1)(\omega)\}; \\ D_\alpha(t)(\omega) &= \max\{n \geq 0 : T_\alpha(n)(\omega) \leq t\}. \end{aligned}$$

4.1 Condition M

Glasserman and Yao (1994) identify a monotonicity condition (condition **M**) that guarantees monotonicity of event epochs with respect to clock samples in the usual construction of a GSMP from a GSMS. In this section we will show that the same condition **M** also guarantees monotonicity of event epochs in our construction; in our case with respect to the partial order defined on the space of inputs Ω in the previous section.

Before giving the basic result of this section, we need to describe the condition **M**. This requires giving a number of definitions and results. In this, we follow Glasserman and Yao (1994).

Condition **M** is a structural condition on the scheme. Given a GSMS $\mathcal{G}$ and an initial state s_0, a feasible *string* $\sigma = \beta_1 \cdots \beta_r$ is a finite sequence of events such that, $\beta_1 \in \mathcal{E}(s_0)$, $\beta_2 \in \mathcal{E}(f_{\beta_1}(s_0))$, and so on. Let $f_\sigma(s_0) = f_{\beta_r} \circ \cdots \circ f_{\beta_1}(s_0)$ and $\mathcal{E}(\sigma) = \mathcal{E}(f_\sigma(s_0))$. The set of feasible strings of $\mathcal{G}$, $\mathcal{L}$, is called the *language* generated by $\mathcal{G}$.

Consider a string σ in $\mathcal{L}$. Let $[\sigma]_{\alpha_i}$ be the number of instances of α_i in σ and define the *score* of σ, denoted by $[\sigma]$, by $[\sigma] = \{[\sigma]_{\alpha_i}; 1 \leq i \leq k\}$. The score space associated with the language $\mathcal{L}$ is the set consisting of all feasible scores and is defined by

$$\mathcal{N} = \{x \in Z_+^k : \exists \sigma \in \mathcal{L}, [\sigma] = x\}.$$

A scheme is called *noninterruptive* if

$$s \in S, \alpha, \beta \in \mathcal{E}(s), \alpha \neq \beta \Rightarrow \beta \in \mathcal{E}(f_\alpha(s)),$$

i.e., if the occurrence of an event does not de-activate an active event. A scheme is called *permutable*, if

$$\sigma_1, \sigma_2 \in \mathcal{L}, [\sigma_1] = [\sigma_2] \Rightarrow \mathcal{E}(\sigma_1) = \mathcal{E}(\sigma_2).$$

In other words, the numbers of events of each type in a string determine the set of active events associated with the string, independently of the order of the events in the string. For a permutable scheme, its *characteristic function*, $\chi : \mathcal{N} \to Z_+^k$, is defined by

$$\chi_\alpha(x) = x_\alpha + I\{\alpha \in \mathcal{E}(x)\}.$$

A characteristic function specifies the one-step behavior of the evolution of the strings. (Note that $\mathcal{E}(x) = \mathcal{E}(\sigma)$ for any σ such that $[\sigma] = x$; permutability makes this definition unambiguous.)

We are now prepared to define condition **M**. A scheme (equivalently the language generated by it) is said to satisfy condition **M** if for all $\sigma_1, \sigma_2 \in \mathcal{L}$,

$$[\sigma_1] \leq [\sigma_2] \Rightarrow \mathcal{E}(\sigma_1) \backslash A_{12} \subseteq \mathcal{E}(\sigma_2),$$

where $A_{12} := \{\alpha \in A, [\sigma_1]_\alpha < [\sigma_2]_\alpha\}$.

This condition specifies that if one string's score dominates the other's, this condition is preserved as the two strings evolve. Glasserman and Yao prove, among other things, that condition M is equivalent, on the one hand, to the scheme being noninterruptive and permutable, and on the other, to its characteristic function χ being increasing (Theorem 3.10 of Glasserman and Yao 1994). It is the latter that we use to establish the main result of this section.

Theorem 2 *If a GSMS satisfies condition* M *then $D_{\alpha_i}(t)$ is increasing in ω for all $t > 0$, and $i = 1, \cdots, k$, i.e.,*

$$\omega \leq \omega' \quad \Rightarrow \quad D_{\alpha_i}(t)(\omega) \leq D_{\alpha_i}(t)(\omega').$$

Proof. Let $\omega' = \{(t_1, e_1), (t_2, e_2), , \cdots\}$ and $\omega \leq \omega'$; therefore $\omega \subseteq \omega'$. $D_{\alpha_i}(t)(\omega)$ and $D_{\alpha_i}(t)(\omega')$ are constants on $t_n \leq t < t_{n+1}$ $(n = 1, \ldots)$, therefore it is sufficient to show that $D_{\alpha_i}(t_n)(\omega) \leq D_{\alpha_i}(t_n)(\omega')$ for all $n = 1, 2, \ldots$ We have

$$D_{\alpha_i}(t_{n+1})(\omega') = \begin{cases} \chi_{\alpha_i}(D_{\alpha_i}(t_n)(\omega')), & e_{n+1} = \alpha_i; \\ D_{\alpha_i}(t_n)(\omega'), & e_{n+1} \neq \alpha_i, \end{cases}$$

and

$$D_{\alpha_i}(t_{n+1})(\omega) = \begin{cases} \chi_{\alpha_i}(D_{\alpha_i}(t_n)(\omega), & e_{n+1} = \alpha_i, \\ & (t_{n+1}, e_{n+1}) \in \omega; \\ D_{\alpha_i}(t_n)(\omega), & \text{otherwise.} \end{cases}$$

The proof is by induction on n. Set $D_{\alpha_i}(t_0)(\omega) = D_{\alpha_i}(t_0)(\omega') = 0$ for all $\alpha_i \in A$. Assume $D_{\alpha_i}(t_n)(\omega) \leq D_{\alpha_i}(t_n)(\omega')$. Since condition M is in force, the characteristic function χ is increasing; hence, $\chi_e(D_{\alpha_i}(t_n)(\omega)) \leq \chi_e(D_{\alpha_i}(t_n)(\omega'))$. The above recursions show that under all conditions $D_{\alpha_i}(t_{n+1})(\omega) \leq D_{\alpha_i}(t_{n+1})(\omega')$ and the induction step is complete. □

Due to the relation between $\{D_{\alpha_i}(t); t \geq 0\}$ and the sequence $\{T_{\alpha_i}(n); n \geq 1\}$, the next corollary follows immediately.

Corollary 1. *If a GSMS satisfies condition* M *then $T_{\alpha_i}(n)$ is decreasing in ω for all $t > 0$, and $i = 1, \cdots, k$, i.e.,*

$$\omega \leq \omega' \quad \Rightarrow \quad T_{\alpha_i}(n)(\omega) \geq T_{\alpha_i}(n)(\omega').$$

5 VARIANCE REDUCTION

Our variance reduction results follow directly from the results of the previous sections. Specifically, a corollary of the results of the previous section is

Theorem 3 *Let $\mathcal{G}_1$, and $\mathcal{G}_2$ be two schemes that satisfy condition* M. *Let L_1 and L_2 be performance indices for the GSMPs defined by $\mathcal{G}_1$, and $\mathcal{G}_2$, respectively. Assume L_i is increasing (decreasing) with respect to the sequence of event epochs. Let W be a random element of the input space with distribution P, then*

$$\mathsf{Cov}(L_1(W), L_2(W)) \geq 0.$$

In other words, if common inputs are used then the performance indices are positively correlated.

An immediate consequence of the above theorem, using a routine argument, is that using common inputs reduces $\mathsf{Var}(L_1 - L_2)$ when compared to using independent inputs.

A question that naturally arises is how much variance reduction is obtained in this case, and how does the amount of variance reduction compare with that obtained by using the usual CRN. To address the second question, we performed some preliminary simulation experiments. The results of these experiments indicate that the effectiveness of the two methods depend on the models simulated. The difference in some cases may be significant.

5.1 Experimental Results

We simulated two systems: an $M/M/1$ queue under different traffic intensities, and a closed Jackson network with different population sizes. In both cases we studied the correlation induced on different performance indices when the systems were "far apart" (ρ = traffic intensity of $M/M/1$ queue = 0.2, 0.5 and 0.9, N = population size of the Jackson network = 30, 40, 50) and when they were "close" (ρ = 0.8, 0.85, 0.9, N = 46, 48, 50); we also estimated the induced correlation at different instances in time as the simulation evolved, in order to study the dependence on time. The experiments were performed, on the one hand, using the usual CRN approach, and on the other, using a particular implementation of using common Poisson inputs known as the Standard Clock technique (see Vakili 1992).

5.1.1 $M/M/1$ Experiments

An $M/M/1$ queue with λ = arrival rate =1 was simulated at different values of μ = service rate. From each simulation, the following performance indices where estimated: (a) the average number of customers in the system, (b) the average time spent in the queue, and (c) the probability that the time spent in the queue is greater than a fixed value.

Two sets of experiments were performed: (i) ρ = 0.2, 0.5, 0.9, and (ii) ρ = 0.8, 0.85, 0.9. The experiments were run using the usual CRN and the Standard Clock. The correlation of the performance indices across the alternatives were estimated from 100 independent replications. These values where estimated at t = 2000, 4000, 8000, 16000, 32000, 64000.

The following were observed:

1. *The correlation was higher for "closer" systems compared to those that were "far apart."*
2. *All performance indices showed the same qualitative behavior.*
3. *The correlation of CRN was consistently higher than that of the Standard Clock.*
4. *There was no significant dependence on time.*

The correlation induced by the usual CRN and by the Standard clock on *the average number of customers in the system* for $\rho = 0.2$ and $\rho = 0.9$ at $t = 2000, 4000, 8000, 16000, 32000, 64000$ are graphed in Figure 1. The same quantities for $\rho = 0.85$ and $\rho = 0.9$ are graphed in Figure 2.

5.1.2 Closed Jackson Network Experiments

A closed Jackson network with 5 servers was simulated. Let μ_i= rate of service at server i, and p_{ij} = the probability that the customer goes to server j after departing from server i ($i = 1, \cdots, 5$). The following values were used: $\mu_1 = 0.7, \mu_2 = 0.6, \mu_3 = 0.5, \mu_4 = 0.4, \mu_5 = 0.3$, and $p_{12} = 0.4, p_{15} = 0.6, p_{21} = 0.2, p_{24} = 0.8, p_{32} = 0.65, p_{35} = 0.35, p_{41} = 0.5, p_{43} = 0.5, p_{54} = 1$.

From each simulation, the following performance indices where estimated: (a) the average time spent at server 1, (b) the average time spent at server 3, (c) the average queue size at server 1, (d) the average queue size at servers 3, (e) the total number of departures from server 1, (f) the total number of departures from server 3. Server 3 was a heavily loaded server and server 1 a lightly loaded one.

Two sets of experiments were performed: (i) $N = 30, 40, 50$, and (ii) $N = 46, 48, 50$. The experiments were run using the usual CRN and the Standard Clock. The correlation of the performance indices across the alternatives were estimated from 100 independent replications. These values where estimated at $t = 2000, 4000, 8000, 16000, 32000, 64000$.

The following were observed:

1. *The correlation was higher for "closer" systems compared to those that were "far apart."*
2. *All performance indices showed the same qualitative behavior.*
3. *The correlation for the Standard Clock was consistently higher than that of CRN, in some cases significantly.*
4. *There was no significant dependence on time in the case of Standard Clock, whereas the correlation for the CRN was an increasing function of time, starting from values close to zero.*

The correlation induced by the usual CRN and by the Standard clock on *the average number of customers at server 1* for $N = 30$ and $N = 50$ at $t = 2000, 4000, 8000, 16000, 32000, 64000$ are graphed in Figure 3. The same quantities for $N = 48$ and $N = 50$ are graphed in Figure 4.

6 SUMMARY

We derived conditions that guarantee variance reduction for common random numbers (CRN) method when the clock samples of GSMPs with exponential clock samples are generated via an acceptance-rejection method. Our preliminary simulation experiments show that for some systems this approach may perform better than when CRN is used in conjunction with the inverse transform method for generating clock samples.

ACKNOWLEDGMENTS

The author would like to acknowledge James Vaughan for programming and performing the simulation experiments.

Research supported in part by the National Science Foundation under Grants DDM-9215368 and EID-9212122.

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AUTHOR BIOGRAPHY

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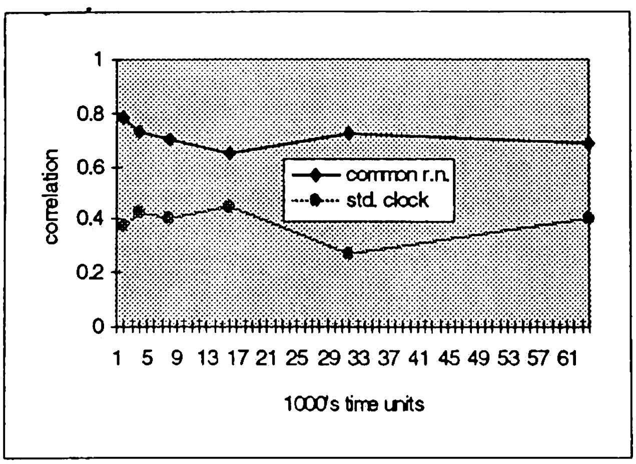

Figure 1: M/M/1 Queue (ρ=0.2, 0.9)

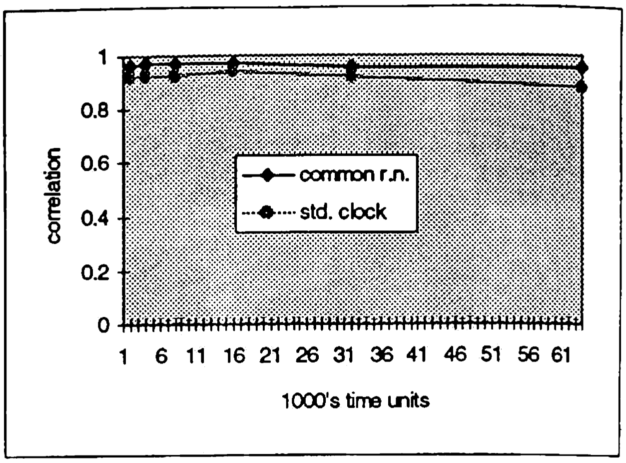

Figure 2: M/M/1 Queue (ρ=0.85, 0.9)

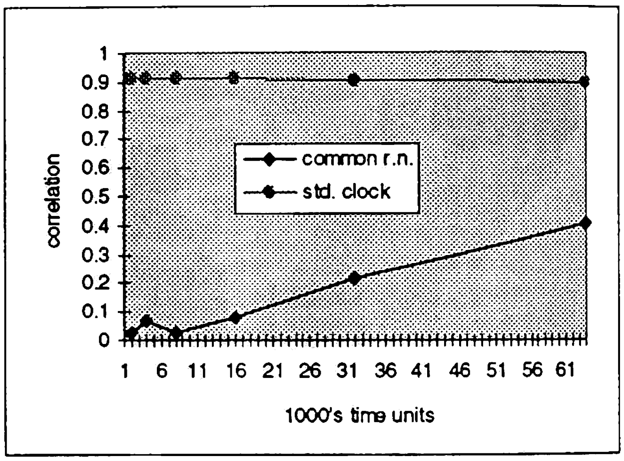

Figure 3: Closed Jackson Network (N= 30, 50)

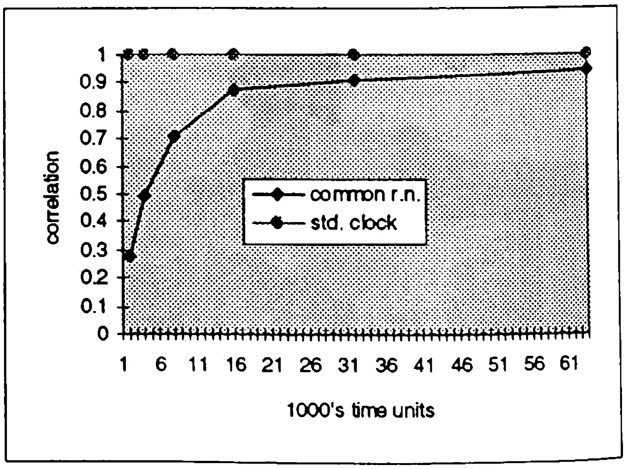

Figure 4: Closed Jackson Network (N= 48, 50)