A NORMAL SCALE MIXTURE REPRESENTATION OF THE LOGISTIC DISTRIBUTION

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ABSTRACT

In this paper it is shown that the logistic distribution can be represented as a scale mixture of the standard normal distribution where the mixing density is related to the Kolmogorov–Smirnov distribution. Two derivations of the theorem are presented that give rise to two different representations of the Kolmogorov–Smirnov distribution. The induced identity is of independent interest and is not widely published nor easily derived directly.

Key words and phrases. Kolmogorov-Smirnov distribution; logistic distribution; normal scale mixture.

1. INTRODUCTION

Recently I encountered a problem in which it was advantageous to approximate the logistic distribution, $F(t) = 1/(1+e^{-t})$, with discrete mixtures having the form

$$F_k(t) = \sum_{i=1}^{k} p_{k,i}\Phi(ts_{k,i}), \qquad (k = 1, 2, \ldots), \tag{1.1}$$

where Φ is the standard normal cumulative distribution function. These approximations provide a solution to a problem posed by Cox (1970, p. 110) and are useful in statistical models involving convolutions of a normal distribution with F, such as logistic regression measurement error models and random effects logistic regression models; see, for example, Carroll, Spiegelman, Bailey, Lan and Abbott (1984). Monahan and Stefanski (1989) discuss applications in detail and provide tables of $\{p_{k,i}, s_{k,i}\}_{i=1}^{k}$ that minimize

$$D_k^* = \sup_t \mid F(t) - F_k(t) \mid, \tag{1.2}$$

for $k = 1, 2, \ldots$. The approximation for $k = 3$ is very good and improves significantly with increasing k.

In this paper I prove the following theorem, providing explanation for the quality of the approximations and mathematical justification for the class of approximants in (1.1).

THEOREM. *Let F and Φ denote the standard logistic and normal cumulative distribution functions respectively. Then*

$$F(t) = \int_0^\infty \Phi(t/\sigma)q(\sigma)d\sigma, \tag{1.3}$$

where $q(\sigma) = (d/d\sigma)L(\sigma/2)$ and L is the Kolmogorov-Smirnov distribution,

$$L(\sigma) = 1 - 2\sum_{n=1}^{\infty}(-1)^{(n+1)}\exp(-2n^2\sigma^2). \tag{1.4}$$

In Section 2, two derivations of (1.3) are presented that give rise to two different representations of the Kolmogorov-Smirnov distribution. The induced identity is of independent interest and is not widely published nor easily derived directly.

2. THE LOGISTIC DISTRIBUTION AS A GAUSSIAN SCALE MIXTURE

Consider the integral equation

$$f(t) = \int_0^\infty \sigma^{-1}\phi(t/\sigma)q(\sigma)d\sigma, \tag{2.1}$$

when $f(t) = e^{-t}/(1+e^{-t})^2$ and ϕ is the standard normal density. A change-of-variables $v = 1/2\sigma^2$ in the integral and evaluation at $t = \sqrt{s},\quad s > 0$, shows that

$$f(\sqrt{s}) = \int_0^\infty e^{-sv}h(v)dv, \tag{2.2}$$

where

$$h(v) = q\big(1/\sqrt{2v}\big)/\sqrt{8\pi v^2}. \tag{2.3}$$

It follows from (2.2) that $f(\sqrt{s})$ is the Laplace transform of h.

Note that for $t > 0$, $F(t) = (1+e^{-t})^{-1} = \sum_{n=0}^{\infty}(-1)^n \exp(-nt)$. Upon differentiation and appeal to symmetry it follows that

$$f(t) = \sum_{n=1}^{\infty}(-1)^{(n+1)} n \exp\{-n \mid t \mid\}, \tag{2.4}$$

and thus

$$f(\sqrt{s}) = \sum_{n=1}^{\infty}(-1)^{(n+1)} n \exp\{-n\sqrt{s}\}. \tag{2.5}$$

Since $e^{-a\sqrt{s}}$ is the Laplace transform of $h_a(t) = a\exp\{-a^2/4t\}/\sqrt{4\pi t^3}$, it follows from (2.5) that

$$h(t) = \sum_{n=1}^{\infty}(-1)^{(n+1)} \frac{n^2}{2\sqrt{\pi t^3}} \exp\left\{-\frac{n^2}{4t}\right\},$$

and from (2.3) that

$$q(\sigma) = 2\sigma \sum_{n=1}^{\infty} (-1)^{(n+1)} n^2 \exp\left\{-\frac{n^2\sigma^2}{2}\right\}. \tag{2.6}$$

Note that in (2.6), $q(\sigma) = (d/d\sigma)L(\sigma/2)$ where L is given in (1.4). □

An alternative method of solving (2.1) leads to an interesting identity that is not easily derived directly. In terms of moment generating functions (2.1) becomes

$$\frac{t\pi}{\sin(t\pi)} = \int_0^\infty \exp\left(\frac{t^2\sigma^2}{2}\right) q(\sigma)d\sigma.$$

The change-of-variables $v = \sigma^2/2$, evaluation at $t = i\sqrt{s}$, and a geometric series expansion of $1/\{1 - \exp(-2\pi\sqrt{s})\}$ results in the identities

$$\int_0^\infty e^{-sv} g(v)dv = \frac{2\pi\sqrt{s}\exp(-\pi\sqrt{s})}{1 - \exp(-2\pi\sqrt{s})} = 2\pi \sum_{n=0}^{\infty} \sqrt{s}\exp\{-\pi(2n+1)\sqrt{s}\}, \tag{2.7}$$

where

$$g(v) = q(\sqrt{2v})/\sqrt{2v}. \tag{2.8}$$

Since $\sqrt{s}\exp(-a\sqrt{s})$ is the Laplace transform of

$$-\left(\frac{d}{da}\right) h_a(t) = -\left(\frac{d}{da}\right)\left[\frac{a\exp\{-a^2/4t\}}{\sqrt{4\pi t^3}}\right],$$

it follows from (2.7) that

$$g(t) = 2\pi \sum_{n=0}^{\infty} \frac{\pi^2(2n+1)^2 - 2t}{4\sqrt{\pi t^5}} \exp\left\{-\frac{\pi^2(2n+1)^2}{4t}\right\}.$$

Using (2.8) and integrating q term-by-term shows that $q(\sigma) = (d/d\sigma)L^*(\sigma/2)$ where

$$L^*(\sigma) = \frac{\sqrt{2\pi}}{\sigma} \sum_{n=0}^{\infty} \exp\left\{-\frac{\pi^2(2n+1)^2}{8\sigma^2}\right\}. \tag{2.9}$$

Of course L and L^* must be equal thus showing that the right-hand sides of (1.4) and (2.9) are equal. This identity is not easily established directly; see for example, Feller (1948), Smirnov (1948) and Monahan (1989). The alternative representation (2.9) is useful for computing $L(\sigma)$ for small σ, Monahan (1989).

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