ABSTRACT

WANG, TIANHENG. Hybrid Impulsive and Mean Field Control Systems. (Under the direction of Negash Medhin.)

This dissertation deals with hybrid impulsive optimal control problems and mean field game/control problems. In addition to continuous control which is well studied in optimal control theory, the system in our study is also influenced by impulsive control which changes the state of the system in discrete time. Hybrid impulsive control problems receive considerable attention for their wide applications in epidemiology, economy and sociology.

We will discuss impulsive optimal control in both deterministic version and stochastic version. Perturbation methods will be used to derive the Pontryagin Minimum Principle which characterizes the necessary conditions for the optimal control and trajectory. In stochastic problems the necessary conditions involve coupled forward and backward stochastic differential equations(FBSDE). The solution to the coupled forward backward stochastic differential equation with jump conditions will be provided. Dynamic programming is also discussed and the comparison between Hamilton-Jacobi-Bellman(HJB) Equation and Minimum Principle will be illustrated. A Multi-group SIR with Vaccination model will be studied as an example in detail and numerical results will be given at the end of the chapters.

Mean field game/control is an extension of stochastic optimal control which deals with a control problem involving a large number of interacting agents. Because the evolution of the system satisfies a measure valued SDE, the Minimum Principle or HJB equation characterizing the optimal control will be coupled with the Fokker-Planck equation which describes the evolution of the probability distribution of the state process. We will discuss an interesting problem of mean field game with a dominating player, where the dominating player make decisions based on the behavior of a representative player rather than individual minor players. At the end we will discuss impulsive mean field control problems.

Hybrid Impulsive and Mean Field Control Systems

by
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BIOGRAPHY

Tianheng Wang was born and raised in Wenzhou, a small city located at the east coast of China. In the year of 2008, he went to Zhejiang University and started his study of mathematics. In the year of 2012, he came to United States to continue his relationship with mathematics at NC State University. He worked with Doctor Medhin on optimal control problems.

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Chapter 1

Deterministic Impulsive Control Problems

In this chapter we study deterministic impulsive optimal control problems. In many control problems changes in the dynamics occur unexpectedly or are applied by a controller as needed. The times at which changes occur are not necessarily known a priori, or they are probabilistic. In manufacturing systems and flight operations changes in the control system may be automatically implemented as needed in response to possibly unexpected external factors affecting the operations of the system. In health-care it is necessary to launch a reasonably effective and timely policy to deal with infectious disease epidemics long-term and short-term([2], [3], [12], [15], [18], [19], [25]). Thus, impulsive control problems have received considerable attention for their wide applications in engineering, epidemiology, economics and sociology.

Multi-group SIR model is important in studying the spreading of diseases. Taking migration into consideration, the rate of level of interactions between groups and within groups are important features of the model. Stability of the dynamic system and reproduction number are different with the single-group SIR model([16], [26], [31]). The positive preserve of the state variables needs justification.

In Section 1.1, we introduce a multi-group SIR model and discuss its stability properties. We will introduce a vaccination strategy and formulate the SIR model as a combined continuous and discrete control problem. In Section 1.2 we give detailed statement of the general impulsive optimal control problem and obtain the necessary conditions characterizing the optimal control and trajectory. In Section 1.3, we give numerical results for the impulsive SIR model.

1.1 Deterministic Multi-group SIR Model

Let us begin this chapter by considering a situation where a disease is spreading in the Triangle region, i.e. Raleigh, Durham and Chapel Hill. One could get infected by contacting infectious people at work, at school, at restaurants, at grocery stores and all public places. Taking into account the large group of commuters, a person from Raleigh cannot ignore the possibility of getting infected by infectious people from the other two cities.

We denote by S_k, I_k and R_k the size of the susceptible, infectious and recovered population of city k, respectively. Let β_{kj} be the transmission rate between city k and city j, and $\beta_{kj}S_kI_j$ represent the new infections in city k caused by coming into contact with infectious people of city j. We let Λ_k, d_k, γ_k denote the birth rate, death rate and recovery rate of city k, respectively. Thus, we could describe the spread of the disease by the following SIR model:

$$\begin{cases} \dot{S}_k = \Lambda_k - \sum_{j=1}^n \beta_{kj}S_kI_j - d_kS_k, \\ \dot{I}_k = \sum_{j=1}^n \beta_{kj}S_kI_j - (d_k+\gamma_k)I_k, \\ \dot{R}_k = \gamma_kI_k - d_kR_k. \end{cases} \tag{1.1}$$

To simplify the model, we assume that once an infected individual recovers, he or she will be immune to this disease. Under this assumption the size of the susceptible and infectious population will not be affected by the size of the recovered population. Thus, we will focus on the reduced SIR model:

$$\begin{cases} \dot{S}_k = \Lambda_k - \sum_{j=1}^n \beta_{kj}S_kI_j - d_kS_k, \\ \dot{I}_k = \sum_{j=1}^n \beta_{kj}S_kI_j - (d_k+\gamma_k)I_k, \end{cases} \tag{1.2}$$

in the rest of the thesis.

1.1.1 Stability

In this section we will study the stability of the SIR system (1.2). It is clear that the system always has a disease free solution $\{S_k^0 = \frac{\Lambda_k}{d_k}, I_k^0 = 0\}_{k=1}^n$. We call $E^0 = [S_1^0, 0, S_2^0, 0, \ldots, S_n^0, 0]$ the disease free equilibrium.

In the study of epidemiology, the reproduction number represents the number of cases generated by one infected person over the course of his infectious period. The reproduction number $\mathcal{R}_0$ of system (1.2), which is defined to be the greatest eigenvalue of the matrix B, where

$$B_{kj} = \frac{\beta_{kj}\Lambda_k}{d_k(d_k+\gamma_k)}, \tag{1.3}$$

plays a role of threshold in the long term qualitative behavior of the SIR system. When $\mathcal{R}_0 > 1$, Guo, Li and Shuai [16] proved that there exists an endemic equilibrium $E^* = [S_1^*, I_1^*, \ldots, S_n^*, I_n^*]$ other than the disease free equilibrium, where $\{S_k^* > 0, I_k^* > 0\}_{k=1}^n$ satisfies

$$\Lambda_k - \sum_{j=1}^n \beta_{kj} S_k^* I_j^* - d_k S_k^* = 0, \tag{1.4}$$

$$\sum_{j=1}^n \beta_{kj} S_k^* I_j^* - (d_k+\gamma_k) I_k^* = 0. \tag{1.5}$$

The stability regarding the reproduction number is described in the following

Theorem 1.1.1.

1. *If $\mathcal{R}_0 < 1$, then the disease free equilibrium E^0 of the system (1.2) is globally stable.*
2. *If $\mathcal{R}_0 > 1$, then the endemic equilibrium E^* of the system (1.2) is globally stable.*

We will use the following lemmas to prove the theorem.

Lemma 1.1.2. *Let A be a matrix where each column sums to zero. Then*

1. *the cofactors C_{ij} of a given column are equal, i.e.,*

$$C_{ij} = C_{kj}, \quad 1 \le i, j, k \le n, \tag{1.6}$$

2. *if the off-diagonal entries of A are all negative, then there exists a vector $\boldsymbol{w}$ with all positive entries which solves $A\boldsymbol{w} = 0$.*

Lemma 1.1.3. *The system (1.2) will have a nonnegative solution $\{S_k(t), I_k(t)\}_k$, $t \in (0, \infty)$.*

Proof of Lemma 1.1.2.

1. We will prove the result for the first column, and the same applies to the other columns. Denote by $\tilde{A}_{ij}$ the submatrix of A with ith row and jth column removed. It is obvious that $\tilde{A}_{11}$ differs from $\tilde{A}_{i1}$ in the top $i-1$ rows. We have

$$C_{i1} = (-1)^{-1-i}\det(\tilde{A}_{i1}) = (-1)^{-1-i}\det\left(\begin{bmatrix} a_{12} & a_{13} & \dots & a_{1n} \\ a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i-1,2} & a_{i-1,3} & \dots & a_{i-1,n} \\ a_{i+1,2} & a_{i+1,3} & \dots & a_{i+1,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix}\right)$$

$$= (-1)^{-1-i}\det\left(\begin{bmatrix} -\sum_{k=2}^{n} a_{k2} & -\sum_{k=2}^{n} a_{k3} & \dots & -\sum_{k=2}^{n} a_{kn} \\ a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i-1,2} & a_{i-1,3} & \dots & a_{i-1,n} \\ a_{i+1,2} & a_{i+1,3} & \dots & a_{i+1,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix}\right)$$

$$= (-1)^{-i}\det\left(\begin{bmatrix} a_{i2} & a_{i3} & \dots & a_{in} \\ a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i-1,2} & a_{i-1,3} & \dots & a_{i-1,n} \\ a_{i+1,2} & a_{i+1,3} & \dots & a_{i+1,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i-1,2} & a_{i-1,3} & \dots & a_{i-1,n} \\ a_{i2} & a_{i3} & \dots & a_{in} \\ a_{i+1,2} & a_{i+1,3} & \dots & a_{i+1,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix}\right) = \det(\tilde{A}_{11}) = C_{11}. \tag{1.7}$$

Using the same reason we will have $C_{1j} = C_{ij}$, for every column j.

2. for any $i = 1, 2, \dots, n$, we have

$$\sum_{j=1}^{n} a_{ij}C_{jj} = \sum_{j=1}^{n} a_{ij}C_{ij} = \det(A) = 0, \tag{1.8}$$

where the second equation is the cofactor expansion for $\det(A)$ in terms of ith row. So the vector $\mathbf{w} = [C_{11}\ C_{22}\ \ldots\ C_{nn}]^T$ solves $A\mathbf{w} = 0$. If all the off diagonal entries of A are negative, then each diagonal entry of A equals the sum of absolute value of all the other entries of the column. We could see that $\tilde{A}_{kk}^T$ is diagonally dominant matrix, then

$$C_{kk} = (-1)^{k+k}\det(\tilde{A}_{kk}) = \det(\tilde{A}_{kk}^T) > 0. \tag{1.9}$$

$\square$

Proof of Lemma 1.1.3. Let $\tau_e = \inf\{t : \|S(t)\| = \infty \text{ or } \|I(t)\| = \infty\}$ denote the explosion time, and we know that the system (1.2) has unique continuous solution $\{S_k(t), I_k(t)\}$ on $t \in (0, \tau_e)$. We define

$$\tau_m = \inf\{t : \min\{S_k(t), I_k(t), k = 1, ..., n\} \leq m^{-1} \text{ or } \max\{S_k(t), I_k(t), k = 1, ...n\} \geq m\}. \tag{1.10}$$

We know that $m \leq n$ implies $\tau_m \leq \tau_n \leq \tau_e$ for sufficiently large m and n. Now we claim that $\lim\limits_{m\to\infty} \tau_m = \infty$. Otherwise, we assume that $\lim\limits_{m\to\infty} \tau_m = T < \infty$. Clearly it is true that $\tau_m \leq T$, $\forall m > 0$. Using the fact that $g(x) = x - 1 - \ln x \geq 0$, we define

$$V(t) = \sum_k a_k g(\frac{S_k}{a_k}) + g(I_k) = \sum_k (S_k - a_k - a_k \ln \frac{S_k}{a_k}) + (I_k - 1 - \ln I_k), \tag{1.11}$$

where a_k's are nonnegative coefficients to be determined. Differentiating $V(t)$ with respect to time, we have

$$
\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}t} &= \sum_k \Big((1-\frac{a_k}{S_k})(\Lambda_k - \sum_j \beta_{kj} S_k I_j - d_k S_k) + (1-\frac{1}{I_k})(\sum_j \beta_{kj} S_k I_j - (d_k+\gamma_k) I_k) \Big) \\
&= \sum_k \Big(\Lambda_k - d_k S_k - \frac{a_k \Lambda_k}{S_k} + a_k \sum_j \beta_{kj} I_j + a_k d_k - (d_k+\gamma_k) I_k - \sum_j \frac{\beta_{kj} S_k I_j}{I_k} + d_k + \gamma_k \Big)
\end{aligned} \tag{1.12}
$$

Now we choose a_k satisfying $\sum_k a_k \beta_{kj} \leq d_j + \gamma_j$, yielding

$$
\sum_k a_k \sum_j \beta_{kj} I_j - \sum_k (d_k+\gamma_k) I_k = \sum_j \Big(\sum_k a_k \beta_{kj} - d_j - \gamma_j \Big) I_j \leq 0, \text{ if } I_j \geq 0.
$$

We integrate the equation (1.12) from 0 to τ_m, then we have

$$
\begin{aligned}
V(\tau_m) = V(0) + \int_0^{\tau_m} &\sum_k \Big(\Lambda_k - d_k S_k - \frac{a_k \Lambda_k}{S_k} + a_k \sum_j \beta_{kj} I_j \\
&+ a_k d_k - (d_k+\gamma_k) I_k - \sum_j \frac{\beta_{kj} S_k I_j}{I_k} + d_k + \gamma_k \Big) \mathrm{d}t \\
\leq V(0) + \int_0^{\tau_m} &\sum_k (\Lambda_k + a_k d_k + d_k + \gamma_k) \mathrm{d}t \\
= V(0) + \tau_m &\sum_k (\Lambda_k + a_k d_k + d_k + \gamma_k) \\
\leq V(0) + T &\sum_k (\Lambda_k + a_k d_k + d_k + \gamma_k).
\end{aligned}
$$

At time τ_m, we know at least one of $\{S_k, I_k\}_k$ has the value m or m^{-1}, so we will have

$$
V(\tau_m) \geq \max\{m - a_k - \ln \frac{m}{a_k}, m^{-1} - a_k - \ln m a_k, m - 1 - \ln m, m^{-1} - 1 + \ln m\}. \tag{1.13}
$$

The right hand side of (1.13) could be arbitrarily large if $m \to \infty$. Since $V(\tau_m) \leq V(0) + T \sum_k (\Lambda_k + a_k d_k + d_k + \gamma_k)$, we will have a contradiction unless $\lim_{m\to\infty} \tau_m = T = \infty$. Thus the solution $\{S_k(t), I_k(t)\}$ to system (1.2) is nonnegative on $t \in (0, \infty)$.

□

Proof of Theorem 1.1.1.

1. First let us consider the system

$$\begin{cases} \dot{X}_k = \Lambda_k - d_k X_k, \\ X_k(0) = S_k(0) \end{cases} \tag{1.14}$$

By the comparison principle we have $S_k(t) \leq X_k(t)$. Then, we consider the Lyapunov function $V(t)$ as follows

$$V(t) = \sum_{k=1}^{n} \frac{v_k}{d_k + \gamma_k} I_k(t), \tag{1.15}$$

where $\mathbf{v} = [v_1, \ldots, v_n]^T$ is the left eigenvector of the matrix defined in (1.3). We need to justify that $\{v_k\}_{k=1}^n$ are positive numbers such that the function defined in (1.15) is a valid Lyapunov function. Perron Frobenius Theorem states that a matrix with all positive components has a positive eigenvalue, and the eigenvector corresponding to that eigenvalue has all positive components. Thus $V(t)$ is a nonnegative function. Differentiating $V(t)$ along the solution of (1.2), we have that

$$
\begin{aligned}
\dot{V}(t) &= \sum_{k=1}^{n} \frac{v_k}{d_k+\gamma_k}\Big(\sum_{j=1}^{n}\beta_{kj}S_k(t)I_j(t) - (d_k+\gamma_k)I_k(t)\Big) \\
&\le \sum_{k=1}^{n}\sum_{j=1}^{n}\frac{v_k}{d_k+\gamma_k}\beta_{kj}X_k(t)I_j(t) - \sum_{k=1}^{n} v_k I_k(t) \\
&= \sum_{j=1}^{n}\sum_{k=1}^{n} v_k\frac{\beta_{kj}\Lambda_k}{d_k(d_k+\gamma_k)}I_j(t) + \sum_{j=1}^{n}\sum_{k=1}^{n} v_k\frac{\beta_{kj}}{d_k+\gamma_k}I_j(t)\Big(X_k(t)-\frac{\Lambda_k}{d_k}\Big) - \sum_{k=1}^{n} v_k I_k(t) \\
&= \sum_{j=1}^{n}\mathcal{R}_0 v_j I_j(t) + \sum_{j=1}^{n}\sum_{k=1}^{n} v_k\frac{\beta_{kj}}{d_k+\gamma_k}I_j(t)\Big(X_k(t)-\frac{\Lambda_k}{d_k}\Big) - \sum_{k=1}^{n} v_k I_k(t) \\
&= \sum_{j=1}^{n}\Big[\sum_{k=1}^{n} v_k\frac{\beta_{kj}}{d_k+\gamma_k}\Big(X_k(t)-\frac{\Lambda_k}{d_k}\Big) + (\mathcal{R}_0-1)v_j\Big]I_j(t).
\end{aligned}
$$

We know that $\lim_{t\to\infty} X_k(t) = \dfrac{\Lambda_k}{d_k}$. With the assumption that $\mathcal{R}_0 < 1$, we will have $\dot{V}(t) \le 0$ when t is sufficiently large, and the equality holds only if $I_j = 0$. Thus the disease free equilibrium E^0 is globally stable.

2. Now we suppose that $\mathcal{R}_0 > 1$, in which case there exists an endemic equilibrium $E^* = [S_1^*, S_2^*, \ldots, S_n^*, I_1^*, I_2^*, \ldots, I_n^*]$.

Define $\bar{\beta}_{kj} = \beta_{kj}S_k^* I_j^*$, and

$$
\bar{B} = \begin{bmatrix}
\sum_{j\neq 1}\bar{\beta}_{1j} & -\bar{\beta}_{21} & -\bar{\beta}_{31} & \cdots & -\bar{\beta}_{n1} \\
-\bar{\beta}_{12} & \sum_{j\neq 2}\bar{\beta}_{2j} & -\bar{\beta}_{32} & \cdots & -\bar{\beta}_{n2} \\
-\bar{\beta}_{13} & -\bar{\beta}_{23} & \sum_{j\neq 3}\bar{\beta}_{3j} & \cdots & -\bar{\beta}_{n3} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-\bar{\beta}_{1n} & -\bar{\beta}_{2n} & -\bar{\beta}_{3n} & \cdots & \sum_{j\neq n}\bar{\beta}_{nj}
\end{bmatrix}. \tag{1.16}
$$

We notice that each column of $\bar{B}$ sums to zero and all the off-diagonal entries of $\bar{B}$ are negative. By Lemma 1.1.2, the linear equation $\bar{B}\mathbf{w} = 0$ has a positive solution $\mathbf{w} = [w_1, \ldots, w_n]^T$. The kth row of the equation $\bar{B}\mathbf{w} = 0$ is equivalent to

$$\sum_{j=1}^{n} \bar{\beta}_{kj} w_k = \sum_{j=1}^{n} \bar{\beta}_{jk} w_j. \tag{1.17}$$

We define $g(x) = x - 1 - \ln x$. By checking the derivative of $g(x)$ we easily get $g(x) \geq 0$ for $x > 0$ and $g(1) = \min_{x>0} g(x) = 0$. We define the Lyapunov function

$$\begin{aligned} V(t) &= \sum_{k=1}^{n} w_k \Big[S_k^* g\Big(\frac{S_k(t)}{S_k^*}\Big) + I_k^* g\Big(\frac{I_k(t)}{I_k^*}\Big)\Big] \qquad (1.18) \\ &= \sum_{k=1}^{n} w_k \Big[S_k(t) - S_k^* - S_k^* \ln\Big(\frac{S_k(t)}{S_k^*}\Big) + I_k(t) - I_k^* - I_k^* \ln\Big(\frac{I_k(t)}{I_k^*}\Big)\Big]. \end{aligned}$$

Differentiating $V(t)$, and using the equilibrium condition (1.4, 1.5), we have

$$\begin{aligned} \dot{V}(t) = &\sum_{k=1}^{n} w_k \Big[\Big(1 - \frac{S_k^*}{S_k}\Big)\Big(\sum_{j=1}^{n} \beta_{kj} S_k^* I_j^* + d_k S_k^* - \sum_{j=1}^{n} \beta_{kj} S_k I_j - d_k S_k \Big) \\ &+ \Big(1 - \frac{I_k^*}{I_k}\Big)\Big(\sum_{j=1}^{n} \beta_{kj} S_k I_j - (d_k + \gamma_k) I_k \Big)\Big] \\ = &\sum_{k=1}^{n} w_k d_k S_k^* \Big(2 - \frac{S_k}{S_k^*} - \frac{S_k^*}{S_k}\Big) + \sum_{j,k=1}^{n} w_k \beta_{kj} S_k^* I_j^* \Big(2 - \frac{S_k^*}{S_k} + \frac{I_j}{I_j^*} - \frac{S_k I_j I_k^*}{S_k^* I_j^* I_k} - \frac{I_k}{I_k^*}\Big) \\ \leq &\sum_{j,k}^{n} w_k \beta_{kj} S_k^* I_j^* \Big[-g\Big(\frac{S_k^*}{S_k}\Big) + g\Big(\frac{I_j}{I_j^*}\Big) - g\Big(\frac{S_k I_j I_k^*}{S_k^* I_j^* I_k}\Big) - g\Big(\frac{I_k}{I_k^*}\Big)\Big] \\ \leq &\sum_{j,k=1}^{n} w_k \bar{\beta}_{kj} g\Big(\frac{I_j}{I_j^*}\Big) - \sum_{j,k=1}^{n} w_k \bar{\beta}_{kj} g\Big(\frac{I_k}{I_k^*}\Big) \\ = &\sum_{j,k=1}^{n} w_k \bar{\beta}_{kj} g\Big(\frac{I_j}{I_j^*}\Big) - \sum_{k=1}^{n} g\Big(\frac{I_k}{I_k^*}\Big) \sum_{j=1}^{n} \bar{\beta}_{jk} w_j \\ = &\ 0, \end{aligned}$$

where the second to last equality is a result of (1.17). So we have that $\dot{V}(t) \leq 0$ and equality holds if $S_k(t) = S_k^*$, $I_k(t) = I_k^*$. Therefore, the endemic equilibrium E^* is globally stable.

$\square$

1.1.2 Vaccination

In this subsection we introduce the vaccination strategy. Periodical repetition of vaccinations are provided to a certain portion of the susceptible group. Those who receive vaccination will be immune to the disease. Introducing vaccination into the system (1.2), we have

$$\begin{cases} \dot{S}_k = \Lambda_k - \sum_j \beta_{kj} S_k I_j - d_k S_k, \\ \dot{I}_k = \sum_j \beta_{kj} S_k I_j - (d_k + \gamma_k) I_k, \end{cases} \qquad t \in (t_i, t_{i+1}), \tag{1.19}$$

with impulse condition

$$\begin{cases} S_k(t_i^+) = S_k(t_i^-)(1 - c_{ki}), \\ I_k(t_i^+) = I_k(t_i^-), \end{cases} \tag{1.20}$$

where c_{ki} is the portion of the susceptible from group i who receive vaccination at time t_k. We call the system (1.19,1.20) impulsive SIR model.

1.2 Deterministic Impulsive Control Problem

In this section we will discuss the main part of the first chapter. We will give a detailed statement of the impulsive optimal control problem, and study the necessary conditions that the optimal control must satisfy. We consider the system whose evolution satisfies the following ordinary differential equation:

$$\dot{x}(t) = f_k(x(t), u(t), t), \qquad t \in (t_k, t_{k+1}). \tag{1.21}$$

At time $t = t_k$, $k = 1, 2, \ldots, N-1$, the system satisfies the following jump condition

$$x(t_k^+) = g_k(x(t_k^-), c_k). \tag{1.22}$$

We call $x(t) \in \mathbb{R}^n$ the state variable, $u(t) \in \mathbb{R}^m$ the continuous control variable, and $c_k \in \mathbb{R}^M$ the impulsive control variable. The impulsive optimal control problem is to find a law for the control $u(t)$ and c_k such that the following cost functional

$$J(u(\cdot), c) = \sum_{k=1}^{N-1} \phi_k(x(t_k^-), c_k) + \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} L_k(x, u, t)\mathrm{d}t + \phi_N(x(t_N^-)), \tag{1.23}$$

is minimized. We assume that

$$\begin{aligned} f_k(x, u, t) &: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \mapsto \mathbb{R}^n, \\ g_k(x, c) &: \mathbb{R}^n \times \mathbb{R}^M \mapsto \mathbb{R}^n. \\ L_k(x, u, t) &: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \mapsto \mathbb{R}, \\ \phi_k(x, c) &: \mathbb{R}^n \times \mathbb{R}^M \mapsto \mathbb{R}. \end{aligned}$$

are smooth functions which have continuous derivatives of all orders.

1.2.1 Necessary Conditions by Methods of Variation of Calculus

In this subsection, we will study the necessary condition for the impulse optimal control problem. We introduce a small perturbation to the control and derive the variation for the cost functional. The adjoint variable is defined and the adjoint equation is obtained, which will lead us to the variational inequalities. We conclude this part with the maximum principle.

We assume that $\{\hat{u}(\cdot), \hat{c}_k\}$ is the optimal control set and $\hat{x}(t)$ is the state corresponding to $\{\hat{u}(\cdot), \hat{c}_k\}$. Let $\{\tilde{u}(\cdot, \theta), \tilde{c}_k(\theta)\}$ be another set of controls where

$$\tilde{u}(t,\theta) = \hat{u}(t) + \theta v(t), \tag{1.24}$$

$$\tilde{c}_k(\theta) = \hat{c}_k + \theta c_k, \qquad k = 1, \dots, N-1 \tag{1.25}$$

$v(\cdot)$, c_k are arbitrary perturbations, and $0 < \theta \ll 1$. Let $\tilde{x}_\theta(t)$ be the state corresponding to $\{\tilde{u}(\cdot,\theta), \tilde{c}_k(\theta)\}$. Define

$$y(t) = \frac{1}{\theta}(\tilde{x}_\theta(t) - \hat{x}(t)). \tag{1.26}$$

The function $y(t)$ in the interval (t_k, t_{k+1}) satisfies

$$\dot{y}(t) = \frac{\partial f_k}{\partial x}(\hat{x}(t), \hat{u}(t), t)y + \frac{\partial f_k}{\partial u}(\hat{x}(t), \hat{u}(t), t)v + \theta\eta(t) \tag{1.27}$$

and

$$y(t_k^+) = \frac{\partial g_k}{\partial x}(\hat{x}(t_k^-), \hat{c}_k)y(t_k^-) + \frac{\partial g_k}{\partial c}(\hat{x}(t_k^-), \hat{c}_k)c_k + \theta\zeta. \tag{1.28}$$

Then,

$$\begin{aligned} y(t) = {} & \Phi_k(t,t_k)\Big(\frac{\partial g_k}{\partial x}(\hat{x}(t_k^-), \hat{c}_k)y(t_k^-) + \frac{\partial g_k}{\partial c}(\hat{x}(t_k^-), \hat{c}_k)c_k\Big) \\ & + \int_{t_k}^{t} \Phi_k(t,s)\frac{\partial f_k}{\partial u}(\hat{x}(s), \hat{u}(s), s)v(s)\mathrm{d}s + \theta\Big(\Phi_k(t,t_k)\zeta + \int_{t_k}^{t} \Phi_k(t,s)\eta(s)\mathrm{d}s\Big), \quad t \in (t_k, t_{k+1}), \end{aligned} \tag{1.29}$$

where $\Phi_k(t,s)$ is the fundamental solution for the linear system

$$
\begin{cases} \dot{z} = \frac{\partial f_k}{\partial x}(\hat{x}(t), \hat{u}(t), t)z, \quad t < t_{k+1}, \\ z(s) = \mathrm{I}. \end{cases} \tag{1.30}
$$

We have the following fact.

Lemma 1.2.1. *There is a function $p(t) \in L(t_0, t_N; \mathbb{R}^n)$, satisfying the differential equation*

$$
-\dot{p}(s) = \Big(\frac{\partial L_k}{\partial x}(\hat{x}(s), \hat{u}(s), s)\Big)^T + \Big(\frac{\partial f_k}{\partial x}(\hat{x}(s), \hat{u}(s), s)\Big)^T p(s), \qquad s \in (t_k, t_{k+1}), \tag{1.31}
$$

and the jump condition

$$
p^T(t_k^-) = p^T(t_k^+)\frac{\partial g_k}{\partial x}(\hat{x}(t_k^-), \hat{c}_k) + \frac{\partial \phi_k}{\partial x}(\hat{x}(t_k^-), \hat{c}_k), \tag{1.32}
$$

then the variation of the cost functional has the following form

$$
\begin{aligned} & J(\hat{u} + \theta v, \hat{c} + \theta c) - J(\hat{u}, \hat{c}) \qquad (1.33) \\ = \quad & \theta \sum_{k=1}^{N-1} \alpha_k c_k + \theta \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \frac{\partial H_k}{\partial u}(\hat{x}(s), \hat{u}(s), p(s), s) v(s) \mathrm{d}s + o(\theta), \end{aligned}
$$

where

$$
\alpha_k^T = \frac{\partial \phi_k}{\partial c}(\hat{x}(t_k), \hat{c}_k) + p^T(t_k^+)\frac{\partial g_k}{\partial c}(\hat{x}(t_k), \hat{c}_k), \tag{1.34}
$$

$$
H_k(x, u, p, t) = L_k(x, u, t) + p^T f_k(x, u, t). \tag{1.35}
$$

Proof. To prove the lemma, we compute the difference between the perturbed cost and the minimal cost:

$$
\begin{aligned}
& J(\hat{u}+\theta v,\hat{c}+\theta c)-J(\hat{u},\hat{c}) \\
= \; & \theta\Big\{\sum_{k=1}^{N-1}\frac{\partial\phi_k}{\partial c}c_k+\sum_{k=0}^{N-1}\Big(\frac{\partial\phi_{k+1}}{\partial x}y(t_{k+1}^-)+\int_{t_k}^{t_{k+1}}\frac{\partial L_k}{\partial x}y(t)+\frac{\partial L_k}{\partial u}v(t)\mathrm{d}t\Big)\Big\}+o(\theta) \\
= \; & \theta\Big\{\frac{\partial\phi_{N-1}}{\partial c}c_{N-1}+\frac{\partial\phi_N}{\partial x}\Big(\Phi_{N-1}(t_N,t_{N-1})\Big(\frac{\partial g_{N-1}}{\partial x}y(t_{N-1}^-)+\frac{\partial g_{N-1}}{\partial c}c_{N-1}\Big) \\
& +\int_{t_{N-1}}^{t_N}\Phi_{N-1}(t_N,s)\frac{\partial f_{N-1}}{\partial u}v(s)\mathrm{d}s\Big) \\
& +\int_{t_{N-1}}^{t_N}\Big[\frac{\partial L_{N-1}}{\partial x}\Big(\Phi_{N-1}(t,t_{N-1})\Big(\frac{\partial g_{N-1}}{\partial x}y(t_{N-1}^-)+\frac{\partial g_{N-1}}{\partial c}c_{N-1}\Big) \\
& +\int_{t_{N-1}}^{t}\Phi_{N-1}(t,s)\frac{\partial f_{N-1}}{\partial u}v(s)\mathrm{d}s\Big)+\frac{\partial L_{N-1}}{\partial u}v(t)\mathrm{d}t\Big] \\
& +\sum_{k=1}^{N-2}\frac{\partial\phi_k}{\partial c}c_k+\sum_{k=0}^{N-2}\Big(\frac{\partial\phi_{k+1}}{\partial x}y(t_{k+1}^-)+\int_{t_k}^{t_{k+1}}\frac{\partial L_k}{\partial x}y(t)+\frac{\partial L_k}{\partial u}v(t)\mathrm{d}t\Big)\Big\}+o(\theta) \\
= \; & \theta\Big\{\Big(\frac{\partial\phi_{N-1}}{\partial c}+\frac{\partial\phi_N}{\partial x}\Phi_{N-1}(t_N,t_{N-1})\frac{\partial g_{N-1}}{\partial c}+\int_{t_{N-1}}^{t_N}\frac{\partial L_{N-1}}{\partial x}\Phi_{N-1}(t,t_{N-1})\frac{\partial g_{N-1}}{\partial c}\mathrm{d}t\Big)c_{N-1} \\
& +\int_{t_{N-1}}^{t_N}\Big[\Big(\frac{\partial\phi_N}{\partial x}\Phi_{N-1}(t_N,s)+\int_s^{t_N}\frac{\partial L_{N-1}}{\partial x}\Phi_{N-1}(t,s)\mathrm{d}t\Big)\frac{\partial f_{N-1}}{\partial u}v(s)+\frac{\partial L_{N-1}}{\partial u}v(s)\Big]\,\mathrm{d}s \\
& +\Big(\frac{\partial\phi_N}{\partial x}\Phi_{N-1}(t_N,t_{N-1})\frac{\partial g_{N-1}}{\partial x}+\int_{t_{N-1}}^{t_N}\frac{\partial L_{N-1}}{\partial x}\Phi_{N-1}(t,t_{N-1})\mathrm{d}t\frac{\partial g_{N-1}}{\partial x}\Big)y(t_{N-1}^-) \\
& +\sum_{k=1}^{N-2}\frac{\partial\phi_k}{\partial c}c_k+\sum_{k=0}^{N-2}\big(\frac{\partial\phi_{k+1}}{\partial x}y(t_{k+1}^-)+\int_{t_k}^{t_{k+1}}\frac{\partial L_k}{\partial x}y(t)+\frac{\partial L_k}{\partial u}v(t)\mathrm{d}t\Big)\Big\}+o(\theta) \\
= \; & \theta\Big\{\alpha_{N-1}^Tc_{N-1}+\int_{t_{N-1}}^{t_N}\Big(p_{N-1}^T\frac{\partial f_{N-1}}{\partial u}v(s)+\frac{\partial L_{N-1}}{\partial u}v(s)\Big)\mathrm{d}s+\beta_{N-1}^Ty(t_{N-1}^-) \\
& +\sum_{k=1}^{N-2}\frac{\partial\phi_k}{\partial c}c_k+\sum_{k=1}^{N-2}\Big(\frac{\partial\phi_{k+1}}{\partial x}y(t_{k+1}^-)+\int_{t_k}^{t_{k+1}}\frac{\partial L_k}{\partial x}y(t)+\frac{\partial L_k}{\partial u}v(t)\mathrm{d}t\Big)\Big\}+o(\theta).
\end{aligned}
$$

The last equation above is obtained by defining

$$\alpha_{N-1}^T \triangleq \frac{\partial \phi_{N-1}}{\partial c} + \frac{\partial \phi_N}{\partial x}\Phi_{N-1}(t_N, t_{N-1})\frac{\partial g_{N-1}}{\partial c} + \int_{t_{N-1}}^{t_N} \frac{\partial L_{N-1}}{\partial x}\Phi_{N-1}(t, t_{N-1})\frac{\partial g_{N-1}}{\partial c}\mathrm{d}t, \tag{1.36}$$

$$p^T(s) \triangleq \frac{\partial \phi_N}{\partial x}\Phi_{N-1}(t_N, s) + \int_s^{t_N} \frac{\partial L_{N-1}}{\partial x}\Phi_{N-1}(t, s)\mathrm{d}t, \quad s \in (t_{N-1}, t_N), \tag{1.37}$$

$$\beta_{N-1}^T \triangleq \frac{\partial \phi_N}{\partial x}\Phi_{N-1}(t_N, t_{N-1})\frac{\partial g_{N-1}}{\partial x} + \int_{t_{N-1}}^{t_N} \frac{\partial L_{N-1}}{\partial x}\Phi_{N-1}(t, t_{N-1})\mathrm{d}t\frac{\partial g_{N-1}}{\partial x}, \tag{1.38}$$

Now we use induction to prove the following claim for $l = N-1, N-2, \ldots, 1$:

$$\begin{aligned} & J(\hat{u} + \theta v, \hat{c} + \theta c) - J(\hat{u}, \hat{c}) \qquad (1.39) \\ = \; & \theta \Big\{ \sum_{k=l}^{N-1} \left(\alpha_k^T c_k + \int_{t_k}^{t_{k+1}} \Big(p^T \frac{\partial f_k}{\partial u} v(s) + \frac{\partial L_k}{\partial u} v(s) \Big) \mathrm{d}s \right) + \beta_l^T y(t_l^-) \\ & + \sum_{k=1}^{l-1} \frac{\partial \phi_k}{\partial c} c_k + \sum_{k=0}^{l-1} \Big(\frac{\partial \phi_{k+1}}{\partial x} y(t_{k+1}^-) + \int_{t_k}^{t_{k+1}} \frac{\partial L_k}{\partial x} y(t) + \frac{\partial L_k}{\partial u} v(t) \mathrm{d}t \Big) \Big\} + o(\theta). \end{aligned}$$

The $l = N-1$ case is already shown. Assuming the case for $l = j$, we compute

$$\begin{aligned} & \beta_j^T y(t_j^-) + \frac{\partial \phi_{j-1}}{\partial c} c_{j-1} + \frac{\partial \phi_j}{\partial x} y(t_j^-) + \int_{t_{j-1}}^{t_j} \frac{\partial L_{j-1}}{\partial x} y(t) + \frac{\partial L_{j-1}}{\partial u} v(t) \mathrm{d}t \\ = \; & (\beta_j^T + \frac{\partial \phi_j}{\partial x}) \Big\{ \Phi_{j-1}(t_j, t_{j-1}) \Big(\frac{\partial g_{j-1}}{\partial x} y(t_{j-1}^-) + \frac{\partial g_{j-1}}{\partial c} c_{j-1} \Big) + \int_{t_{j-1}}^{t_j} \Phi_{j-1}(t_j, s) \frac{\partial f_{j-1}}{\partial u} v(s) \mathrm{d}s \Big\} \\ & + \frac{\partial \phi_{j-1}}{\partial c} c_{j-1} + \int_{t_{j-1}}^{t_j} \Big\{ \frac{\partial L_{j-1}}{\partial x} \Big[\Phi_{j-1}(t, t_{j-1}) \Big(\frac{\partial g_{j-1}}{\partial x} y(t_{j-1}^-) + \frac{\partial g_{j-1}}{\partial c} c_{j-1} \Big) \\ & + \int_{t_{j-1}}^{t} \Phi_{j-1}(t_j, s) \frac{\partial f_{j-1}}{\partial u} v(s) \mathrm{d}s \Big] + \frac{\partial L_{j-1}}{\partial u} v(s) \Big\} \mathrm{d}t \\ = \; & \alpha_{j-1}^T c_{j-1} + \int_{t_{j-1}}^{t_j} \Big(p^T \frac{\partial f_{j-1}}{\partial u} v(s) + \frac{\partial L_{j-1}}{\partial u} v(s) \Big) \mathrm{d}s + \beta_{j-1}^T y(t_{j-1}^-). \end{aligned}$$

The last equation is obtained by defining

$$\alpha_{j-1}^T \triangleq \Big[\Big(\beta_j^T + \frac{\partial \phi_j}{\partial x}\Big)\Phi_{j-1}(t_j, t_{j-1}) + \int_{t_{j-1}}^{t_j} \frac{\partial L_{j-1}}{\partial x}\Phi_{j-1}(t, t_{j-1})\mathrm{d}t\Big]\frac{\partial g_{j-1}}{\partial c} + \frac{\partial \phi_{j-1}}{\partial c}, \tag{1.40}$$

$$p^T(s) \triangleq \Big(\beta_j^T + \frac{\partial \phi_j}{\partial x}\Big)\Phi_{j-1}(t_j, s) + \int_{s}^{t_j} \frac{\partial L_{j-1}}{\partial x}\Phi_{j-1}(t, s)\mathrm{d}t, \quad s \in (t_{j-1}, t_j), \tag{1.41}$$

$$\beta_{j-1}^T \triangleq \Big[\Big(\beta_j^T + \frac{\partial \phi_j}{\partial x}\Big)\Phi_{j-1}(t_j, t_{j-1}) + \int_{t_{j-1}}^{t_j} \frac{\partial L_{j-1}}{\partial x}\Phi_{j-1}(t, t_{j-1})\mathrm{d}t\Big]\frac{\partial g_{j-1}}{\partial x}. \tag{1.42}$$

Then (1.39) could be written as

$$\begin{aligned}
& J(\hat{u} + \theta v, \hat{c} + \theta c) - J(\hat{u}, \hat{c}) \\
= \; & \theta\Big\{ \sum_{k=j-1}^{N-1} \Big(\alpha_k^T c_k + \int_{t_k}^{t_{k+1}} \Big(p^T \frac{\partial f_k}{\partial u} v(s) + \frac{\partial L_k}{\partial u} v(s)\Big)\mathrm{d}s\Big) + \beta_{j-1}^T y(t_{j-1}^-) \\
& + \sum_{k=1}^{j-2} \frac{\partial \phi_k}{\partial c} c_k + \sum_{k=0}^{j-2} \Big(\frac{\partial \phi_{k+1}}{\partial x} y(t_{k+1}^-) + \int_{t_k}^{t_{k+1}} \frac{\partial L_k}{\partial x} y(t) + \frac{\partial L_k}{\partial u} v(t)\mathrm{d}t\Big)\Big\} + o(\theta),
\end{aligned}$$

which closes the induction. Then, we have

$$
\begin{aligned}
&J(\hat{u}+\theta v, \hat{c}+\theta c) - J(\hat{u}, \hat{c}) \\
= \ & \theta\Big\{ \sum_{k=1}^{N-1} \Big(\alpha_k^T c_k + \int_{t_k}^{t_{k+1}} \Big(p^T \frac{\partial f_k}{\partial u} v(s) + \frac{\partial L_k}{\partial u} v(s) \Big) \mathrm{d}s \Big) + \beta_1^T y(t_1^-) \\
& + \frac{\partial \phi_1}{\partial x} y(t_1^-) + \int_{t_0}^{t_1} \frac{\partial L_0}{\partial x} y(t) + \frac{\partial L_0}{\partial u} v(t) \mathrm{d}t \Big\} + o(\theta) \\
= \ & \theta\Big\{ \sum_{k=1}^{N-1} \Big(\alpha_k^T c_k + \int_{t_k}^{t_{k+1}} \Big(p^T \frac{\partial f_k}{\partial u} v(s) + \frac{\partial L_k}{\partial u} v(s) \Big) \mathrm{d}s \Big) \\
& + \Big(\beta_1^T + \frac{\partial \phi_1}{\partial x} \Big) \int_{t_0}^{t_1} \Phi_0(t_1, s) \frac{\partial f_0}{\partial u} v(s) \mathrm{d}s \\
& + \int_{t_0}^{t_1} \frac{\partial L_0}{\partial x} \int_{t_0}^{t} \Phi_0(t, s) \frac{\partial f_0}{\partial u} v(s) \mathrm{d}s \mathrm{d}t + \int_{t_0}^{t_1} \frac{\partial L_0}{\partial u} v(t) \mathrm{d}t \Big\} + o(\theta) \\
= \ & \theta\Big\{ \sum_{k=1}^{N-1} \alpha_k^T c_k + \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \Big(p^T \frac{\partial f_k}{\partial u} v(s) + \frac{\partial L_k}{\partial u} v(s) \Big) \mathrm{d}s \Big\} + o(\theta).
\end{aligned}
$$

The last equation is obtained by defining

$$
p^T(s) \triangleq \Big(\beta_1 + \frac{\partial \phi_1}{\partial x} \Big) \Phi_0(t_1, s) + \int_s^{t_1} \frac{\partial L_0}{\partial x} \Phi_0(t, s) \mathrm{d}t, \quad s \in (t_0, t_1). \tag{1.43}
$$

From (1.40-1.43), we will have that

$$
\begin{aligned}
\alpha_{j-1}^T &= \frac{\partial \phi_{j-1}}{\partial c} + p_{j-1}^T(t_{j-1}^+) \frac{\partial g_{j-1}}{\partial c} && (1.44) \\
\beta_{j-1}^T &= p^T(t_{j-1}^+) \frac{\partial g_{j-1}}{\partial x} && (1.45) \\
-\dot{p}(s) &= \Big(\frac{\partial L_{j-1}}{\partial x} \Big)^T + \Big(\frac{\partial f_{j-1}}{\partial x} \Big)^T p, \quad s \in (t_{j-1}, t_j) && (1.46) \\
p^T(t_j^-) &= \beta_j + \frac{\partial \phi_j}{\partial x} = p^T(t_j^+) \frac{\partial g_j}{\partial x} + \frac{\partial \phi_j}{\partial x} && (1.47)
\end{aligned}
$$

By defining the Hamiltonian H_k:

$$H_k(x,u,p,t) = L_k(x,u,t) + p^T f_k(x,u,t), \quad t \in (t_k, t_{k+1}), \tag{1.48}$$

we have the result

$$\begin{aligned} & J(\hat{u}+\theta v, \hat{c}+\theta c) - J(\hat{u},\hat{c}) \\ = \quad & \theta \sum_{k=1}^{N-1} \alpha_k c_k + \theta \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \frac{\partial H_k}{\partial u}(\hat{x}(s), \hat{u}(s), p(s), s) v(s) \mathrm{d}s + o(\theta), \end{aligned}$$

□

Assuming there is no constraint on the control variables, we have the following result.

Theorem 1.2.2. *If $\hat{x}(t)$ is the solution of the impulse optimal control problem stated as above, we have*

$$\frac{\partial \phi_k}{\partial c}(\hat{x}(t_k), \hat{c}_k) + p^T(t_k^+) \frac{\partial g_k}{\partial c}(\hat{x}(t_k), \hat{c}_k) = 0, \tag{1.49}$$

$$\frac{\partial H_k}{\partial u}(\hat{x}(s), \hat{u}(s), p(s), s) = 0, \qquad s \in (t_k, t_{k+1}). \tag{1.50}$$

Remark 1.2.3. Although Pontryagin's maximum principle gives a stronger set of necessary conditions than Theorem 1.2.2, it is (1.49,1.50) that we will use in the sequel. In the following we will prove the Pontryagin's maximum principle by employing the "spike variation methods".

Theorem 1.2.4. *Let $\{\hat{u}, \hat{c}\}$ be optimal control pair, $\hat{x}(\cdot)$ be the optimal state variable corresponding to $\{\hat{u}, \hat{c}\}$, $p(\cdot)$ be the adjoint variable defined by (1.31), and $H_k(x,u,p,t)$ be the Hamiltonian defined by (1.35). Then, we have*

$$H_k(\hat{x}(\tau), v, p(\tau), \tau) \geq H_k(\hat{x}(\tau), \hat{u}(\tau), p(\tau), \tau), \qquad \tau \in (t_k, t_{k+1}). \tag{1.51}$$

for any admissible control v.

Proof. Let $u_\varepsilon(\cdot)$ be defined as

$$u_\varepsilon(t) = \begin{cases} \hat{u}(t) & t \in (t_k, \tau) \\ v & t \in (\tau, \tau+\varepsilon) \\ \hat{u}(t) & t \in (\tau+\varepsilon, t_{k+1}) \\ \hat{u}(t) & t \in (t_j, t_j+1),\ j \neq k \end{cases} \tag{1.52}$$

with $0 < \varepsilon \ll 1$, and $x_\varepsilon(\cdot)$ be the state variable corresponding to $\{u_\varepsilon, \hat{c}\}$. We consider z_ε^k defined by

$$\frac{\mathrm{d}z_\varepsilon^k}{\mathrm{d}t} = \frac{\partial f_k}{\partial x}(\hat{x}, \hat{u}, t) z_\varepsilon^k, \qquad t \in (\tau, t_{k+1}), \tag{1.53}$$

$$z_\varepsilon^k(\tau) = f(\hat{x}(\tau), v, \tau) - \frac{1}{\varepsilon}\int_\tau^{\tau+\varepsilon} f(\hat{x}(s), \hat{u}(s), s)\mathrm{d}s, \tag{1.54}$$

Then, by taking $\varepsilon \to 0$ and Grönwall's Inequality, we have $z_\varepsilon^k(t) \to 0$ for $t \in (\tau, t_{k+1})$. Then we consider z_ε^j defined by

$$\frac{\mathrm{d}z_\varepsilon^j}{\mathrm{d}t} = \frac{\partial f_j}{\partial x}(\hat{x}, \hat{u}, t) z_\varepsilon^j, \qquad t \in (t_j, t_{j+1}), \tag{1.55}$$

$$z_\varepsilon^j(t_j^+) = \frac{\partial g_j}{\partial x} z_\varepsilon^{j-1}(t_j^-). \tag{1.56}$$

It could be proved that by induction that $z_\varepsilon^j(t) \to 0$, $t \in (t_j, t_{j+1})$ as $\varepsilon \to 0$ for $j = k+1, \ldots, N-1$.

We let $y_\varepsilon(t) = \frac{1}{\varepsilon}(x_\varepsilon(t) - \hat{x}(t) - \varepsilon z_\varepsilon^k(t))$, $t \in (\tau+\varepsilon, t_{k+1})$. Then, we have

$$\frac{\mathrm{d}y_\varepsilon}{\mathrm{d}t} = \int_0^1 \frac{\partial f_k}{\partial x}(\hat{x}+\lambda(x_\varepsilon-\hat{x}),\hat{u},t)\mathrm{d}\lambda y_\varepsilon + \int_0^1 \Big(\frac{\partial f_k}{\partial x}(\hat{x}+\lambda(x_\varepsilon-\hat{x}),\hat{u},t) - \frac{\partial f_k}{\partial x}(\hat{x},\hat{u},t)\Big) z_\varepsilon^k \mathrm{d}t, \tag{1.57}$$

$$y_\varepsilon(\tau+\varepsilon) = \frac{1}{\varepsilon}\int_\tau^{\tau+\varepsilon} f_k(x_\varepsilon(t),v,t) - f_k(\hat{x}(\tau),v,\tau)\mathrm{d}t - \int_\tau^{\tau+\varepsilon} \frac{\partial f_k}{\partial x}(\hat{x},\hat{u},t) z_\varepsilon^k \mathrm{d}t, \tag{1.58}$$

By Grönwall's Inequality again we have $y_\varepsilon(t) \to 0$ for $t \in (\tau+\varepsilon, t_{k+1})$ letting $\varepsilon \to 0$.

Then, let $y_\varepsilon(t) = \frac{1}{\varepsilon}(x_\varepsilon(t) - \hat{x}(t) - \varepsilon z_\varepsilon^j(t))$, $t \in (t_j, t_{j+1})$ for $j = k+1, \ldots, N-1$, and we have

$$\frac{\mathrm{d}y_\varepsilon}{\mathrm{d}t} = \int_0^1 \frac{\partial f_j}{\partial x}(\hat{x}+\lambda(x_\varepsilon-\hat{x}),\hat{u},t)\mathrm{d}\lambda y_\varepsilon + \int_0^1 \Big(\frac{\partial f_j}{\partial x}(\hat{x}+\lambda(x_\varepsilon-\hat{x}),\hat{u},t) - \frac{\partial f_j}{\partial x}(\hat{x},\hat{u},t)\Big) z_\varepsilon^j \mathrm{d}t, \tag{1.59}$$

$$\begin{aligned} y_\varepsilon(t_j^+) = {} & \int_0^1 \frac{\partial g_j}{\partial x}(\hat{x}(t_j^-)+\lambda(x_\varepsilon(t_j^-)-\hat{x}(t_j^-)),\hat{c}_j) y_\varepsilon(t_j^-)\mathrm{d}\lambda \\ & + \int_0^1 \Big(\frac{\partial g_j}{\partial x}(\hat{x}(t_j^-)+\lambda(x_\varepsilon(t_j^-)-\hat{x}(t_j^-)),\hat{c}_j) - \frac{\partial g_k}{\partial x}(\hat{x}(t_j^-),\hat{c}_j)\Big) z_\varepsilon^j(t_j^-)\mathrm{d}\lambda, \end{aligned} \tag{1.60}$$

By using Grönwall's Inequality it could be proved by induction that $y_\varepsilon(t) \to 0$, $t \in (t_j, t_{j+1})$ for $j = k+1, \ldots, N-1$, as $\varepsilon \to 0$.

Then, we have

$$
\begin{aligned}
& \frac{1}{\varepsilon}(J(u_\varepsilon)-J(u)) \qquad (1.61)\\
= \; & \frac{1}{\varepsilon}\int_\tau^{\tau+\varepsilon} L_k(x_\varepsilon,v,t)-L_k(\hat{x},\hat{u},t)\mathrm{d}t \\
& +\frac{1}{\varepsilon}\int_{\tau+\varepsilon}^{t_{k+1}} L_k(x_\varepsilon,\hat{u},t)-L_k(\hat{x},\hat{u},t)\mathrm{d}t+\frac{1}{\varepsilon}(\phi_{k+1}(x_\varepsilon(t_{k+1}^-))-\phi_{k+1}(\hat{x}(t_{k+1}^-))) \\
& +\sum_{j=k+1}^{N-1}\Big[\frac{1}{\varepsilon}\int_{t_j}^{t_{j+1}} L_j(x_\varepsilon,\hat{u},t)-L_j(\hat{x},\hat{u},t)\mathrm{d}t+\frac{1}{\varepsilon}(\phi_{j+1}(x_\varepsilon(t_{j+1}^-))-\phi_{j+1}(\hat{x}(t_{j+1}^-)))\Big] \\
= \; & L_k(\hat{x}(\tau),v,\tau)-\frac{1}{\varepsilon}\int_\tau^{\tau+\varepsilon} L_k(\hat{x},\hat{u},t)\mathrm{d}t+\frac{1}{\varepsilon}\int_\tau^{\tau+\varepsilon} L_k(x_\varepsilon,v,t)-L_k(\hat{x}(\tau),v,\tau)\mathrm{d}t \\
& +\int_\tau^{t_{k+1}}\frac{\partial L_k}{\partial x}(\hat{x},\hat{u},t)z_\varepsilon^k\mathrm{d}t+\frac{1}{\varepsilon}\int_{\tau+\varepsilon}^{t_{k+1}} L_k(x_\varepsilon,\hat{u},t)-L_k(\hat{x},\hat{u},t)-\varepsilon\frac{\partial L_k}{\partial x}(\hat{x},\hat{u},t)z_\varepsilon^k\mathrm{d}t \\
& -\int_\tau^{\tau+\varepsilon}\frac{\partial L_k}{\partial x}(\hat{x},\hat{u},t)z_\varepsilon^k\mathrm{d}t+\frac{\partial\phi_{k+1}}{\partial x}(\hat{x}(t_{k+1}^-))z_\varepsilon^k(t_{k+1}^-) \\
& +\frac{1}{\varepsilon}\Big(\phi_{k+1}(x_\varepsilon(t_{k+1}^-))-\phi_{k+1}(\hat{x}(t_{k+1}^-))-\varepsilon\frac{\partial\phi_{k+1}}{\partial x}(\hat{x}(t_{k+1}^-))z_\varepsilon^k(t_{k+1}^-)\Big) \\
& +\sum_{j=k+1}^{N-1}\Big[\int_{t_j}^{t_{j+1}}\frac{\partial L_j}{\partial x}(\hat{x},\hat{u},t)z_\varepsilon^j\mathrm{d}t+\frac{1}{\varepsilon}\int_{t_j}^{t_{j+1}}\Big(L_j(x_\varepsilon,\hat{u},t)-L_j(\hat{x},\hat{u},t)-\varepsilon\frac{\partial L_j}{\partial x}(\hat{x},\hat{u},t)z_\varepsilon^j\Big)\mathrm{d}t \\
& +\frac{\partial\phi_{j+1}}{\partial x}(\hat{x}(t_{j+1}^-))z_\varepsilon^j(t_{j+1}^-)+\frac{1}{\varepsilon}\Big(\phi_{j+1}(x_\varepsilon(t_{j+1}^-))-\phi_{j+1}(\hat{x}(t_{j+1}^-))-\varepsilon\frac{\partial\phi_{j+1}}{\partial x}(\hat{x}(t_{j+1}^-))z_\varepsilon^j(t_{j+1}^-)\Big)\Big].
\end{aligned}
$$

Since

$$
\frac{\partial p^T(t)z_\varepsilon^j}{\partial t}=-(\frac{\partial L_j}{\partial x}+p^T\frac{\partial f_j}{\partial x})z_\varepsilon^j+p^T\frac{\partial f_j}{\partial x}z_\varepsilon^j=-\frac{\partial L_j}{\partial x}z_\varepsilon^j,\quad t\in(t_j,t_{j+1}), \qquad (1.62)
$$

we have

$$\int_{\tau}^{t_{k+1}} \frac{\partial L_k}{\partial x}(\hat{x},\hat{u},t) z_\varepsilon^k \mathrm{d}t = p^T(\tau) z_\varepsilon^j(\tau) - p^T(t_{k+1}^-) z_\varepsilon^j(t_{k+1}^-) \tag{1.63}$$

$$= p^T(\tau)\Big(f(\hat{x}(\tau),v) - \frac{1}{\varepsilon}\int_{\tau}^{\tau+\varepsilon} f(\hat{x},\hat{u})\mathrm{d}t\Big) - p^T(t_{k+1}^-) z_\varepsilon^k(t_{k+1}^-).$$

$$= p^T(\tau)\Big(f(\hat{x}(\tau),v) - \frac{1}{\varepsilon}\int_{\tau}^{\tau+\varepsilon} f(\hat{x},\hat{u})\mathrm{d}t\Big) - \Big(p^T(t_{k+1}^+)\frac{\partial g_{k+1}}{\partial x} + \frac{\partial \phi_{k+1}}{\partial x}\Big) z_\varepsilon^k(t_{k+1}^-).$$

$$\int_{t_j}^{t_{j+1}} \frac{\partial L_j}{\partial x}(\hat{x},\hat{u},t) z_\varepsilon^j \mathrm{d}t = p^T(t_j^+) z_\varepsilon^j(t_j^+) - p^T(t_{j+1}^-) z_\varepsilon^j(t_{j+1}^-) \tag{1.64}$$

$$= p^T(t_j^+)\frac{\partial g_j}{\partial x} z_\varepsilon^j(t_j^-) - \Big(p^T(t_{j+1}^+)\frac{\partial g_{j+1}}{\partial x} + \frac{\partial \phi_{j+1}}{\partial x}\Big) z_\varepsilon^j(t_{j+1}^-).$$

Therefore, we have

$$
\begin{aligned}
& \frac{1}{\varepsilon}(J(u_\varepsilon) - J(\hat{u})) \qquad (1.65)\\
= \;& L_k(\hat{x}(\tau), v, \tau) - \frac{1}{\varepsilon}\int_\tau^{\tau+\varepsilon} L_k(\hat{x}, \hat{u}, t)\mathrm{d}t + \frac{1}{\varepsilon}\int_\tau^{\tau+\varepsilon} L_k(x_\varepsilon, v, t) - L_k(\hat{x}(\tau), v, \tau)\mathrm{d}t \\
& + p^T(\tau)\Big(f(\hat{x}(\tau), v) - \frac{1}{\varepsilon}\int_\tau^{\tau+\varepsilon} f(\hat{x}, \hat{u})\mathrm{d}t\Big) - p^T(t_{k+1}^-) z_\varepsilon^k(t_{k+1}^-) \\
& + \frac{1}{\varepsilon}\int_{\tau+\varepsilon}^{t_{k+1}} L_k(x_\varepsilon, \hat{u}, t) - L_k(\hat{x}, \hat{u}, t) - \varepsilon\frac{\partial L_k}{\partial x}(\hat{x}, \hat{u}, t) z_\varepsilon^k \mathrm{d}t - \int_\tau^{\tau+\varepsilon} \frac{\partial L_k}{\partial x}(\hat{x}, \hat{u}, t) z_\varepsilon^k \mathrm{d}t \\
& + \frac{\partial \phi_{k+1}}{\partial x}(x(t_{k+1}^-)) z_\varepsilon^k(t_{k+1}^-) + \frac{1}{\varepsilon}\Big(\phi_{k+1}(x_\varepsilon(t_{k+1}^-)) - \phi_{k+1}(\hat{x}(t_{k+1}^-)) - \varepsilon\frac{\partial \phi_{k+1}}{\partial x}(\hat{x}(t_{k+1}^-)) z_\varepsilon^k(t_{k+1}^-)\Big) \\
& + \sum_{j=k+1}^{N-1}\Bigg[p^T(t_j^+) z_\varepsilon^j(t_j^+) - p^T(t_{j+1}^-) z_\varepsilon^j(t_{j+1}^-) \\
& + \frac{1}{\varepsilon}\int_{t_j}^{t_{j+1}} L_j(x_\varepsilon, \hat{u}, t) - L_j(\hat{x}, \hat{u}, t) - \varepsilon\frac{\partial L_j}{\partial x}(\hat{x}, \hat{u}, t) z_\varepsilon^j \mathrm{d}t + \frac{\partial \phi_{j+1}}{\partial x}(\hat{x}(t_{j+1}^-)) z_\varepsilon^j(t_{j+1}^-) \\
& + \frac{1}{\varepsilon}\Big(\phi_{j+1}(x_\varepsilon(t_{j+1}^-)) - \phi_{j+1}(\hat{x}(t_{j+1}^-)) - \varepsilon\frac{\partial \phi_{j+1}}{\partial x}(\hat{x}(t_{j+1}^-)) z_\varepsilon^j(t_{j+1}^-)\Big)\Bigg] \\
= \;& L_k(\hat{x}(\tau), v, \tau) + p^T(\tau) f_k(\hat{x}(\tau), v, \tau) - \frac{1}{\varepsilon}\int_\tau^{\tau+\varepsilon} L_k(\hat{x}, \hat{u}, t) + p^T(t) f_k(\hat{x}, \hat{u}, t)\mathrm{d}t \\
& + \frac{1}{\varepsilon}\int_\tau^{\tau+\varepsilon} (p(t) - p(\tau))^T f_k(\hat{x}, \hat{u}, t)\mathrm{d}t + \frac{1}{\varepsilon}\int_\tau^{\tau+\varepsilon} L_k(x_\varepsilon, v, t) - L_k(\hat{x}(\tau), v, \tau)\mathrm{d}t \\
& + \frac{1}{\varepsilon}\int_{\tau+\varepsilon}^{t_{k+1}} L_k(x_\varepsilon, \hat{u}, t) - L_k(\hat{x}, \hat{u}, t) - \varepsilon\frac{\partial L_k}{\partial x} z_\varepsilon^k \mathrm{d}t - \int_\tau^{\tau+\varepsilon} \frac{\partial L_k}{\partial x} z_\varepsilon^k \mathrm{d}t \\
& + \frac{1}{\varepsilon}\Big(\phi_{k+1}(x_\varepsilon(t_{k+1}^-)) - \phi_{k+1}(\hat{x}(t_{k+1}^-)) - \varepsilon\frac{\partial \phi_{k+1}}{\partial x}(\hat{x}(t_{k+1}^-)) z_\varepsilon^k(t_{k+1}^-)\Big) \\
& + \sum_{j=k+1}^{N-1}\Bigg[\int_{t_j}^{t_{j+1}} L_j(x_\varepsilon, \hat{u}, t) - L_j(\hat{x}, \hat{u}, t) - \varepsilon\frac{\partial L_j}{\partial x} z_\varepsilon^j \mathrm{d}t \\
& + \frac{1}{\varepsilon}\Big(\phi_{j+1}(x_\varepsilon(t_{j+1}^-)) - \phi_{j+1}(\hat{x}(t_{j+1}^-)) - \varepsilon\frac{\partial \phi_{j+1}}{\partial x}(\hat{x}(t_{j+1}^-)) z_\varepsilon^j(t_{j+1}^-)\Big)\Bigg] \\
= \;& H_k(\hat{x}(\tau), p(\tau), v, \tau) - \frac{1}{\varepsilon}\int_\tau^{\tau+\varepsilon} H_k(\hat{x}, p_k, \hat{u}, t)\mathrm{d}t + X_\varepsilon \\
\to \;& H_k(\hat{x}(\tau), p(\tau), v, \tau) - H_k(\hat{x}(\tau), p_k(\tau), \hat{u}(\tau), \tau) \quad \text{ as } \varepsilon \to 0,
\end{aligned}
$$

where we use the fact that

$$\begin{aligned}
&\frac{1}{\varepsilon}\Big(L_j(x_\varepsilon,u,t)-L_j(x,u,t)-\varepsilon\frac{\partial L_j}{\partial x}z_\varepsilon^j\Big) && (1.66)\\
= &\int_0^1\frac{\partial L_j}{\partial x}(x+\lambda(x_\varepsilon-x),u,t)y_\varepsilon \mathrm{d}\lambda+\int_0^1\Big(\frac{\partial L_j}{\partial x}(x+\lambda(x_\varepsilon-x),u,t)-\frac{\partial L_j}{\partial x}(x,u,t)\Big)z_\varepsilon^j\mathrm{d}\lambda\\
\to &\ 0 \quad \text{as } \varepsilon\to 0
\end{aligned}$$

and that $y_\varepsilon(t), z_\varepsilon^k(t)\to 0$. Thus, the necessary condition for optimal control u satisfies

$$H_k(x(\tau),v,p_k(\tau),\tau)\geq H_k(x(\tau),u(\tau),p_k(\tau),\tau), \qquad \tau\in(t_k,t_{k+1}). \tag{1.67}$$

for all admissible v in control domain.

□

1.3 Numerical Solution to SIR Model

In this section, we will apply the impulse control methodology to the SIR model. Previous works [25] have provided a continuous and impulsive vaccination strategies to hold the epidemics in a stable state. Here we consider the multi-group SIR model:

$$\begin{cases}
\dot{S}_i=\Lambda_i-d_iS_i-\sum_{j=1}^n\beta_{ij}S_iI_j,\\
\dot{I}_i=\sum_{j=1}^n\beta_{ij}S_iI_j-(d_i+\gamma_i)I_i,\\
S_i(t_k^+)=S_i(t_k^-)(1-c_{ik}),\\
I_i(t_k^+)=I_i(t_k^-),
\end{cases} \tag{1.68}$$

where S_i and I_i are the population of susceptible and infected in group i, Λ_i and d_i represent the birth and death rates of the individuals in group i, and γ_i represents the recovery rate of the infected individuals in group i, β_{ij} is the infection rate in group i caused by infected population from group j, and c_{ik} is the proportion of susceptible population in group i receiving the vaccination at time t_k. We know that c_{ik} takes value in $[0,1]$. To keep the disease under control, each group will enforce a migration policy to restrict incoming populations from other groups, at the expense of retarding economic growth. We will define the infected rate $\beta_{ij} = \bar{\beta}_{ij} - u_{ij}$, where $u_{ij} \in [0, \bar{\beta}_{ij}]$ represents the control of migration.

Now, we consider the cost function

$$J = \frac{a}{2} \sum_{k=0}^{N-1} \sum_{i=1}^{n} \left(c_{ik} S_i(t_k^-) \right)^2 + \sum_{k=0}^{N-1} \sum_{i=1}^{n} \int_{t_k}^{t_{k+1}} \left(\frac{b}{2} I_i^2 + \frac{1}{2} \sum_{j=1}^{n} u_{ij}^2 \right) \mathrm{d}t + \sum_{i=1}^{n} \frac{e}{2} I_i^2(t_N). \qquad (1.69)$$

From the necessary condition (1.49, 1.50), we know that the optimal control has the form $\hat{u}_{ij} = (\eta_i - \xi_i) S_i I_j$, and the optimal impulsive control has the form $\hat{c}_{ik} = \xi_i(t_k^-)/a$, where $[\xi, \eta]^T$ is the adjoint variable corresponding to the SIR system. For the numerical experiment, we set the parameters of the model as follows:

Λ	d	β	γ	a	b	e	T	Δt
$\begin{pmatrix} .4 \\ .3 \\ .2 \end{pmatrix}$	$\begin{pmatrix} .4 \\ .3 \\ .2 \end{pmatrix}$	$\begin{pmatrix} 1 & .05 & .05 \\ .05 & 1 & .05 \\ .05 & .05 & 1 \end{pmatrix}$	$\begin{pmatrix} .5 \\ .5 \\ .5 \end{pmatrix}$	1	20	1	10	2

and initial conditions

$S(0)$ $(\times 10^5)$	$I(0)$ $(\times 10^5)$
$\begin{pmatrix} 0.9 \\ 0.8 \\ 0.7 \end{pmatrix}$	$\begin{pmatrix} 0.1 \\ 0.1 \\ 0.1 \end{pmatrix}$

Here is the table showing the cost values in case of constant controls.

Table 1.1: List of Costs Tested by Varying Controls

u	c	J
0	0	3.4658
0.1β	0	2.3479
0.15β	0	2.0632
0.2β	0	1.9561
0.25β	0	2.0133
0	0.1	2.5498
0	0.2	2.0776
0	0.3	1.8720
0	0.4	1.8240
0	0.5	1.8710
0.2β	0.4	1.8602
0.2β	0.3	1.7443
0.2β	0.2	1.6906
0.2β	0.1	1.7374
0.1β	0.1	1.8430
0.1β	0.2	1.6320
0.1β	0.3	1.5877
0.1β	0.4	1.6413
$\hat{u}$	$\hat{c}$	1.4626

Remark 1.3.1. The Table 1.1 lists the cost of the system driven by different control pairs. The first row shows that the cost is 3.4658 if no controls are applied to the system. The second row shows that the cost is 2.3479 if the migration rate is reduced by 10%. The last row shows that the cost is 1.4626 if the system is driven by optimal control pair $\{\hat{u}, \hat{c}\}$, which is derived by solving the necessary conditions (1.49,1.50). The Table 1.1 shows that the cost of the system under the optimal control pair is lower than the cost if the system is driven by other controls. The Figure 1.1 shows the susceptible population if the system is driven by optimal control. The jumps of the curves represents the effect of vaccination. Those people receiving vaccination are removed from the susceptible group as they are immune to this disease. The Figures 1.3, 1.4 and 1.5 display the optimal migration restriction of the three cities. It is reasonable that the restriction decreases as the sizes of the infected population is under control. The Figures 1.7

shows that the size of the infected population increases for a period of time if no control is applied.

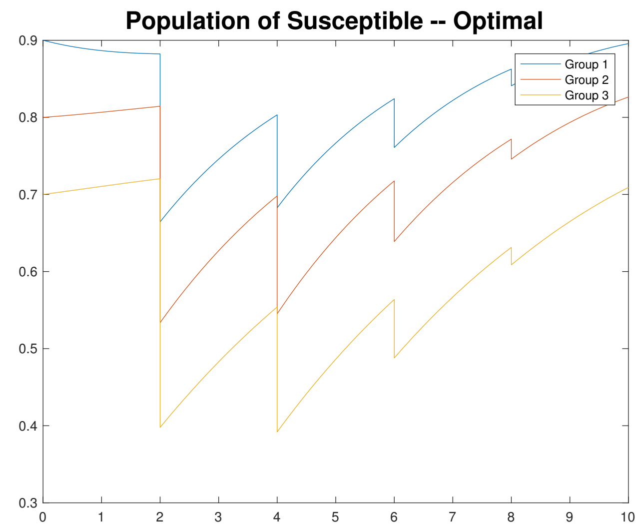

Figure 1.1: Populations of Susceptibles in All Groups under Optimal Controls

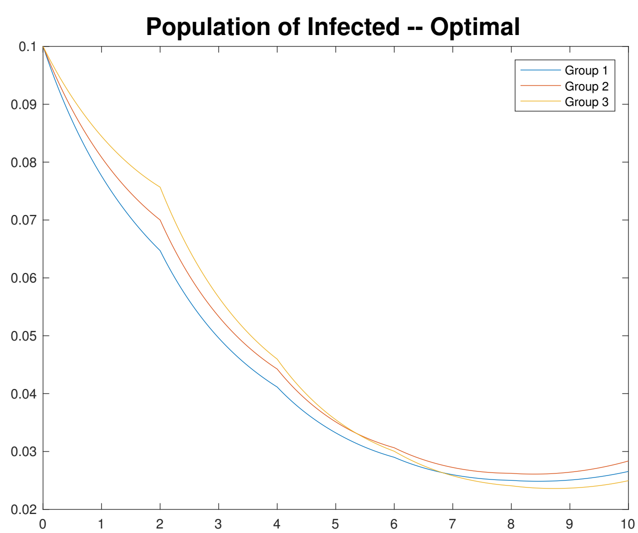

Figure 1.2: Populations of Infected in All Groups under Optimal Controls

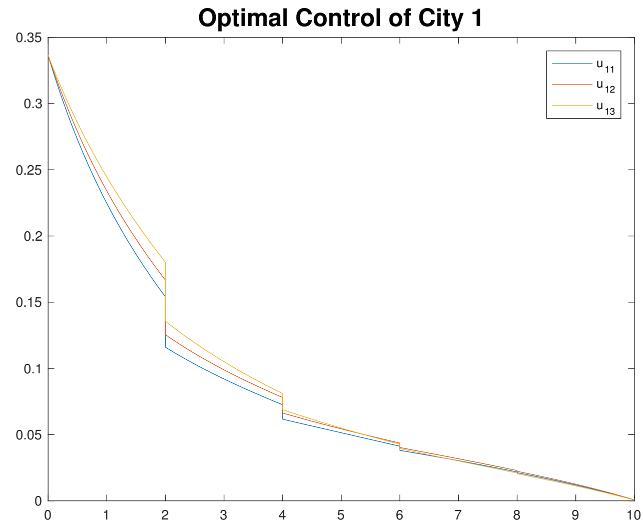

Figure 1.3: Optimal Control of City 1: $\hat{u}_{1j} = (\eta_1 - \xi_1)S_1 I_j$

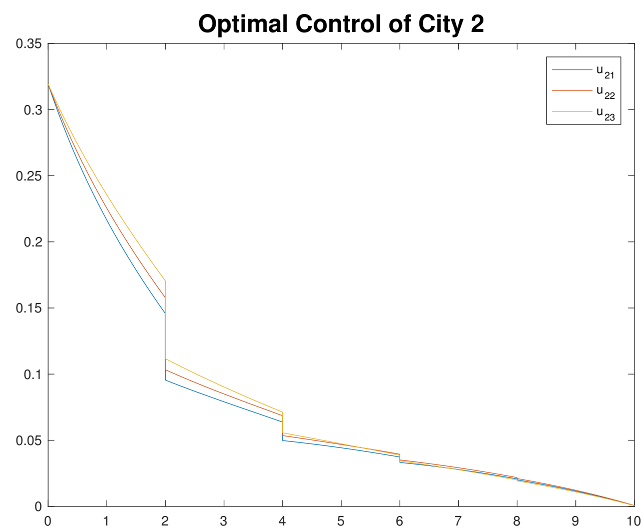

Figure 1.4: Optimal Control of City 2: $\hat{u}_{2j} = (\eta_2 - \xi_2)S_2 I_j$

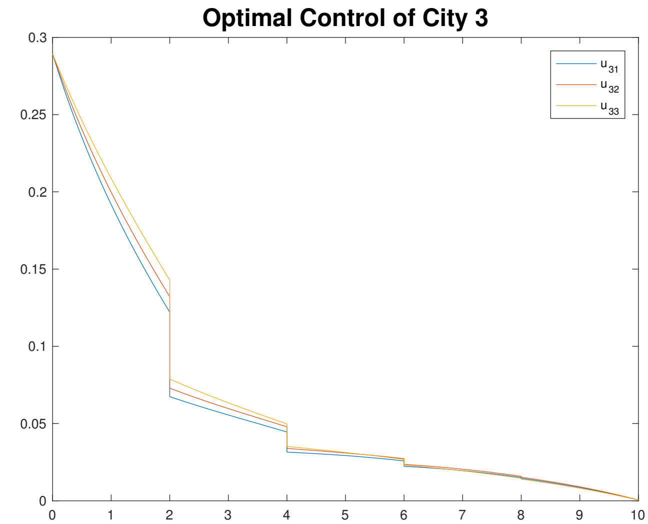

Figure 1.5: Optimal Control of City 3: $\hat{u}_{3j} = (\eta_3 - \xi_3)S_3 I_j$

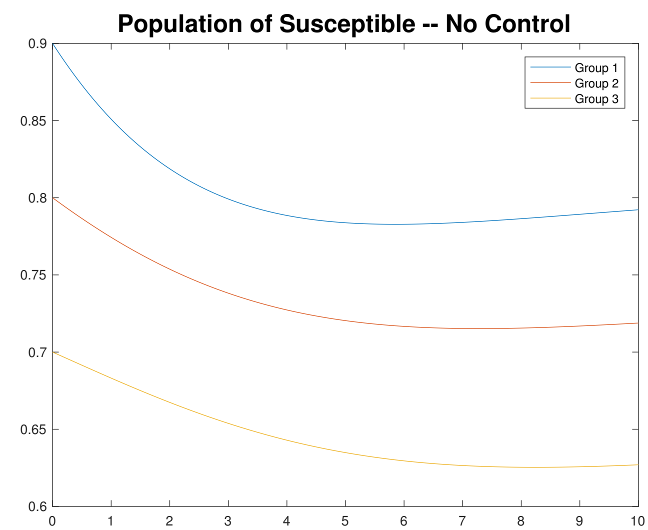

Figure 1.6: Populations of Susceptibles in All Groups under Zero Controls

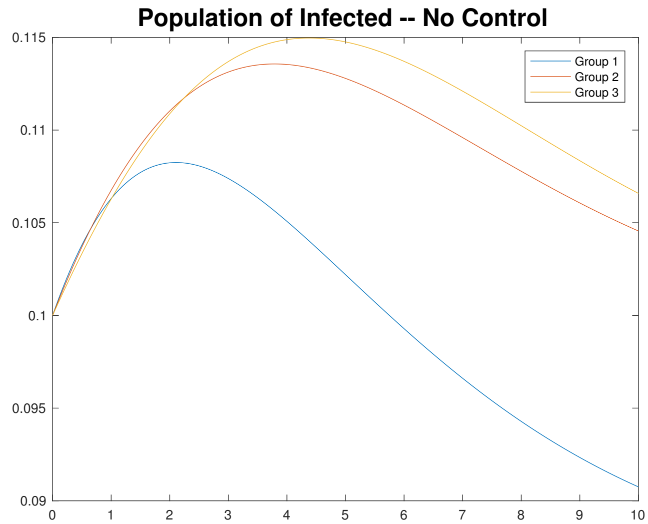

Figure 1.7: Populations of Infected in All groups under Zero Controls

Chapter 2

Stochastic Impulsive Control

In this chapter we study stochastic impulsive optimal control problems. In practice, the real world is a world of uncertainty. We receive weather forecast telling the chance of rain, we gain or lose money as the stock price goes up and down, and we hear the sounds of the radio, which receives signals contaminated by noises. In the perspective of quantum physics, every particle of the world behaves in a random manner. Stochastic model is more interesting and receives more attention than deterministic model due to its wider applicability.

There are not many papers dealing with stochastic multi-group SIR models. The stability properties depends on the reproduction number as is the case in deterministic model. In fact, the diffusion coefficients are also critical to the stability([36]). We will give details and corrections to some of the arguments in [36].

Stochastic optimal control problem is one of the hot topics in applied mathematics research([4], [5], [35], [37]). The necessary condition of the optimal control involves solving a coupled system of forward backward stochastic differential equations(FBSDEs). Solvability and explicit scheme of FBSDEs were discussed in [27], [30], [35]. In general, FBSDEs might not necessarily have a solution. There are solutions when the FBSDEs are derived as necessary conditions of an optimal control problem.

We follow the same plan of action as the first chapter. In Section 2.1, we discuss the stochastic multi-group SIR model and its stability properties. In Section 2.2, we give the statement of the stochastic impulsive optimal control and derive the necessary conditions from two directions. Solutions to the forward backward stochastic differential equations will be studied and the relation between maximum principle and dynamic programming will be discussed. In Section 2.3 we give numerical results on the stochastic impulse SIR model. At the end of the chapter, we give a proof of a lemma which is used in discussion of stability.

2.1 Stochastic Multi-group SIR Model

In this section we will include randomness into the SIR model. One option is to replace the death rate d_k with $d_k + \sigma_k \mathrm{d}B_k(t)$. It is reasonable to consider this kind of replacement since there are unpredictable natural disasters such as earthquake and tsunami that will cause unpredictable number of deaths. Let us now consider the stochastic multi-group SIR model:

$$\begin{cases} \mathrm{d}S_k = \Big(\Lambda_k - \sum_{j=1}^n \beta_{kj} S_k I_j - d_k S_k\Big)\mathrm{d}t + \rho_k S_k \mathrm{d}W_k(t), \\ \mathrm{d}I_k = \Big(\sum_{j=1}^n \beta_{kj} S_k I_j - (d_k + \gamma_k) I_k\Big)\mathrm{d}t + \theta_k I_k \mathrm{d}B_k(t), \end{cases} \tag{2.1}$$

where $W_k(t)$, $B_k(t)$, $1 \le k \le n$, are independent Brownian motions, and σ_k, ρ_k, $1 \le k \le n$, are nonnegative numbers describing the volatility.

Because of the presence of random noise, the study of the long term behavior of the SIR model becomes more complicated. In the rest of the section, we will address two questions: first, does there exist a limit for the susceptible population and infected population? Second, if there exists a limit, then in what sense does the population converge to that limit? Does it have almost sure convergence, $\mathcal{L}^2$ convergence, weak convergence or convergence in probability? As we have seen in Theorem 1.1.1, the reproduction number $\mathcal{R}_0$ is the threshold regarding the long term stability. One reasonable guess is that in case of low volatility, the stochastic process will converge to the equilibrium of the deterministic model.

The following lemma is the stochastic version of Lemma 1.1.3, which will be used in the discussion of stability of the stochastic SIR model.

Lemma 2.1.1. *The system (2.1) will almost surely have a nonnegative solution $\{S_k(t), I_k(t)\}_k$, $t \in (0, \infty)$.*

Proof of Lemma 2.1.1. Let τ_e denote the explosion time, and we know that the system (2.1) has unique solution $\{S_k(t), I_k(t)\}$ on $t \in (0, \tau_e)$. We define the stopping time

$$\tau_m = \inf\{t : \min_k \{S_k(t), I_k(t)\} \le m^{-1} \text{ or } \max_k \{S_k(t), I_k(t)\} \ge m\}. \tag{2.2}$$

For any ω in sample space, we have $\tau_m(\omega) \leq \tau_n(\omega)$ if $m \leq n$. So the limit of τ_m exists and we define $T = \lim_{m\to\infty} \tau_m$. We claim that $T = \infty$ almost sure. Otherwise assuming $P(T < \infty) > 0$, we will have

$$P(T < \infty) = \sum_{k=1}^{\infty} P(k-1 \leq T < k) > 0. \tag{2.3}$$

There exists K and $\varepsilon > 0$ such that $P(K-1 \leq T < K) = \varepsilon$. We define the set $A_K = \{\omega : T(\omega) < K\}$, then we have $\tau_m(\omega) < K$ for $\omega \in A_K$, $\forall m$. We define

$$V(t) = \sum_k (S_k - a_k - a_k \ln \frac{S_k}{a_k}) + (I_k - 1 - \ln I_k), \tag{2.4}$$

where a_k's are chosen in the same manner as in Lemma 1.1.3. By Ito's lemma, we have

$$\begin{aligned}
\mathrm{d}V = \sum_k \Big(& (1 - \frac{a_k}{S_k})(\Lambda_k - \sum_j \beta_{kj} S_k I_j - d_k S_k) + \frac{a_k \rho_k^2}{2} \\
& + (1 - \frac{1}{I_k})(\sum_j \beta_{kj} S_k I_j - (d_k + \gamma_k) I_k) + \frac{\theta_k^2}{2} \Big) \mathrm{d}t + (\dots)\mathrm{d}W + (\dots)\mathrm{d}B \\
= \sum_k \Big(& \Lambda_k - d_k S_k - \frac{a_k \Lambda_k}{S_k} + a_k \sum_j \beta_{kj} I_j + a_k d_k - (d_k + \gamma_k) I_k \\
& - \sum_j \frac{\beta_{kj} S_k I_j}{I_k} + d_k + \gamma_k + \frac{a_k \rho_k^2}{2} + \frac{\theta_k^2}{2} \Big) \mathrm{d}t + (\dots)\mathrm{d}W + (\dots)\mathrm{d}B
\end{aligned} \tag{2.5}$$

We integrate the above equation from 0 to $K \wedge \tau_m$, and we will have

$$
\begin{aligned}
V(K\wedge\tau_m) &= V(0)+\int_0^{K\wedge\tau_m}\sum_k\Big((1-\frac{a_k}{S_k})(\Lambda_k-\sum_j\beta_{kj}S_kI_j-d_kS_k)+\frac{a_k\rho_k^2}{2}\\
&\quad +(1-\frac{1}{I_k})(\sum_j\beta_{kj}S_kI_j-(d_k+\gamma_k)I_k)+\frac{\theta_k^2}{2}\Big)\mathrm{d}t+M_{K\wedge\tau_m}\\
&\le V(0)+\int_0^{K\wedge\tau_m}\sum_k(\Lambda_k+a_kd_k+d_k+\gamma_k+\frac{a_k\rho_k^2}{2}+\frac{\theta_k^2}{2})\mathrm{d}t+M_{K\wedge\tau_m}\\
&\le V(0)+K\sum_k(\Lambda_k+a_kd_k+d_k+\gamma_k+\frac{a_k\rho_k^2}{2}+\frac{\theta_k^2}{2})+M_{K\wedge\tau_m}. \qquad (2.6)
\end{aligned}
$$

Taking expectation of the above inequality (2.6), we will have

$$
EV(K\wedge\tau_m)\le EV(0)+K\sum_k(\Lambda_k+a_kd_k+d_k+\gamma_k+\frac{a_k\rho_k^2}{2}+\frac{\theta_k^2}{2}). \qquad (2.7)
$$

Since the function $V(t)$ is always nonnegative, we have

$$
\begin{aligned}
EV(K\wedge\tau_m) &\ge P(\tau_m<K)E[V(\tau_m)|\tau_m<K]\ge P(T<K)E[V(\tau_m)|\tau_m<K] \qquad (2.8)\\
&\ge \varepsilon\max\{m-a_k-\ln\frac{m}{a_k},m^{-1}-a_k-\ln ma_k,m-1-\ln m,m^{-1}-1+\ln m\}.
\end{aligned}
$$

Combining (2.7) and (2.8) we have

$$
\begin{aligned}
&\varepsilon\max\{m-a_k-\ln\frac{m}{a_k},m^{-1}-a_k-\ln ma_k,m-1-\ln m,m^{-1}-1+\ln m\}\\
&\quad\le EV(0)+K\sum_k(\Lambda_k+a_kd_k+d_k+\gamma_k+\frac{a_k\rho_k^2}{2}+\frac{\theta_k^2}{2}), \qquad (2.9)
\end{aligned}
$$

which leads to a contradiction since the left hand side of (2.9) could be arbitrarily large if we take $m\to\infty$. Therefore we have $\lim_{m\to\infty}\tau_m=\infty$ and the the solution $\{S_k(t),I_k(t)\}$ to system (2.1) is nonnegative on $t\in(0,\infty)$.

$\square$

2.1.1 Stability of Disease Free Equilibrium

In this part we consider the case when $\mathcal{R}_0 < 1$.

Consider the system:

$$\begin{cases} \mathrm{d}X_k = \Big(\Lambda_k - d_k X_k\Big)\mathrm{d}t + \rho_k X_k \mathrm{d}W_k, \\ X_k(0) = S_k(0) \end{cases} \tag{2.10}$$

By the comparison principle, we have that $S_k(t) \le X_k(t)$ almost surely.

Let $\mathbf{v} = [v_1, \ldots, v_n]^T$ be the same vector defined in Theorem 1.1.1, which satisfies

$$\sum_{k=1}^{n} v_k \frac{\beta_{kj}\Lambda_k}{d_k(d_k+\gamma_k)} = \mathcal{R}_0 v_j. \tag{2.11}$$

We consider the Lyapunov function $V(t) = \sum_{k=1}^{n} e_k I_k$, where $e_k = \dfrac{v_k}{d_k+\gamma_k}$. By Ito's formula, we have

$$
\begin{aligned}
\mathrm{d}\log V &= \frac{1}{V}\Big(\sum_{k,j} e_k\beta_{kj}S_kI_j - \sum_k w_kI_k\Big)\mathrm{d}t - \frac{1}{2V^2}\sum_k e_k^2\theta_k^2I_k^2\mathrm{d}t + \frac{1}{V}\sum_k e_k\theta_kI_k\mathrm{d}B_k \\
&\le \frac{1}{V}\Big(\sum_{k,j} e_k\beta_{kj}X_kI_j - \sum_k w_kI_k\Big)\mathrm{d}t - \frac{1}{2V^2}\sum_k e_k^2\theta_k^2I_k^2\mathrm{d}t + \frac{1}{V}\sum_k e_k\theta_kI_k\mathrm{d}B_k \\
&= \frac{1}{V}\Big(\sum_{k,j} e_k\beta_{kj}\frac{\Lambda_k}{d_k}I_j - \sum_k w_kI_k\Big)\mathrm{d}t - \frac{1}{2V^2}\sum_k e_k^2\theta_k^2I_k^2\mathrm{d}t + \frac{1}{V}\sum_k e_k\theta_kI_k\mathrm{d}B_k \\
&\quad + \frac{1}{V}\Big[\sum_{k,j} e_k\beta_{kj}\Big(X_k - \frac{\Lambda_k}{d_k}\Big)I_j\Big]\mathrm{d}t \\
&= \frac{1}{V}\Big((\mathcal{R}_0 - 1)\sum_k w_kI_k\Big)\mathrm{d}t - \frac{1}{2V^2}\sum_k e_k^2\theta_k^2I_k^2\mathrm{d}t + \frac{1}{V}\sum_k e_k\theta_kI_k\mathrm{d}B_k \\
&\quad + \frac{1}{V}\Big[\sum_{k,j} e_k\beta_{kj}\Big(X_k - \frac{\Lambda_k}{d_k}\Big)I_j\Big]\mathrm{d}t
\end{aligned}
$$

Integrating the above equation, then computing time average, we have

$$
\begin{aligned}
\frac{\log V(T) - \log V(0)}{T} \le\ & \frac{1}{T}\int_0^T \frac{1}{V}(\mathcal{R}_0 - 1)\sum_k w_kI_k\mathrm{d}t - \frac{1}{T}\int_0^T \frac{1}{2V^2}\sum_k e_k^2\theta_k^2I_k^2\mathrm{d}t \\
& + \sum_k \frac{1}{T}\int_0^T \frac{1}{V}e_k\theta_kI_k\mathrm{d}B_k + \sum_k \frac{1}{T}\int_0^T \frac{1}{V}\sum_j e_k\beta_{kj}\Big(X_k - \frac{\Lambda_k}{d_k}\Big)I_j\mathrm{d}t.
\end{aligned} \tag{2.12}
$$

Now we look at the first integral on the right hand side. Notice that

$$
\frac{\sum_k w_kI_k}{V} = \frac{\sum_k w_kI_k}{\sum_j e_jI_j} = \frac{\sum_k w_kI_k}{\sum_j \frac{w_j}{d_j+\gamma_j}I_j} \le \frac{\sum_k w_kI_k}{\frac{1}{\max_j\{d_j+\gamma_j\}}\sum_k w_kI_k} = \max_j\{d_j + \gamma_j\}, \tag{2.13}
$$

So the first integral is bounded by

$$\frac{1}{T}\int_0^T \frac{1}{V}(\mathcal{R}_0-1)\sum_k w_k I_k \mathrm{d}t \le \begin{cases} (\mathcal{R}_0-1)\max_j\{d_j+\gamma_j\}, & \mathcal{R}_0 \ge 1, \\ 0, & \mathcal{R}_0 < 1, \end{cases}. \tag{2.14}$$

Looking at the second integral, we compute

$$\frac{\sum_k e_k^2\theta_k^2 I_k^2}{V^2} = \frac{\sum_k e_k^2\theta_k^2 I_k^2}{\left(\sum_j e_j I_j\right)^2} = \frac{\sum_k e_k^2\theta_k^2 I_k^2}{\left(\sum_j e_j \theta_j I_j \theta_j^{-1}\right)^2} \ge \frac{\sum_k e_k^2\theta_k^2 I_k^2}{\sum_j e_j^2\theta_j^2 I_j^2 \sum_j \theta_j^{-2}} = \frac{1}{\sum_k \theta_k^{-2}}. \tag{2.15}$$

The second integral is bounded by

$$-\frac{1}{T}\int_0^T \frac{1}{2V^2}\sum_k e_k^2\theta_k^2 I_k^2 \mathrm{d}t \le -\frac{1}{2\sum_k \theta_k^{-2}}. \tag{2.16}$$

Now we look at the third integral. Define a martingale $\mathcal{M}(t) = \displaystyle\int_0^t \frac{1}{V} e_k\theta_k I_k \mathrm{d}B_k$. We will show that

$$\lim_{T\to\infty} \frac{1}{T}\mathcal{M}(T) = 0 \qquad \text{a.s.} \tag{2.17}$$

We define

$$A^{(m)} = \left\{\omega :\ \overline{\lim}_{T\to\infty}\frac{1}{T}|\mathcal{M}(T)| > \frac{1}{m}\right\}, \tag{2.18}$$

$$A = \cup_{m=1}^{\infty} A^{(m)}, \tag{2.19}$$

$$B = \left\{\omega :\ \lim_{T\to\infty}\frac{1}{T}\mathcal{M}(T) = 0\right\}. \tag{2.20}$$

Let $\omega \notin A$, then it is true for all $m \in \mathbb{N}$ that

$$\overline{\lim}_{T\to\infty}\frac{1}{T}|M(T)| \le \frac{1}{m}, \tag{2.21}$$

which implies $\lim\limits_{T\to\infty}\frac{1}{T}\mathcal{M}(T) = 0$. Then, we have $A^c = B$. To show the almost sure convergence, we need to show

$$\mathbf{P}(B) = 1 \iff \mathbf{P}(A) = 0 \iff \mathbf{P}(A^{(m)}) = 0,\ \forall m \tag{2.22}$$

For a fixed m, we define

$$A_{m,l} = \Big\{\omega :\ \sup_{T>2^l}\frac{1}{T}|\mathcal{M}(T)| > \frac{1}{m}\Big\}, \tag{2.23}$$

then we will have $A^{(m)} = \cap_{l=1}^{\infty}A_{m,l}$. Notice that

$$A_{m,l} \supset A_{m,l+1}, \tag{2.24}$$

we have

$$\mathbf{P}(A^{(m)}) = \lim_{l\to\infty}\mathbf{P}(A_{m,l}). \tag{2.25}$$

By Doob's Martingale Inequality, we have

$$\begin{aligned}\mathbf{P}\Big(\sup_{2^{r-1}<T\le 2^r}|\mathcal{M}(T)| > \varepsilon\Big) &\le \mathbf{P}\Big(\sup_{T\le 2^r}|\mathcal{M}(T)| > \varepsilon\Big) \le \frac{1}{\varepsilon}E|\mathcal{M}(2^r)| \\ &\le \frac{1}{\varepsilon}\Big[E(|\mathcal{M}(2^r)|^2)\Big]^{\frac{1}{2}} = \frac{1}{\varepsilon}\Big\{E\Big[\int_0^{2^r}\frac{1}{V^2}e_k^2\theta_k^2 I_k^2\mathrm{d}t\Big]\Big\}^{\frac{1}{2}}.\end{aligned} \tag{2.26}$$

Notice that

$$\frac{1}{V^2}e_k^2\theta_k^2 I_k^2 = \frac{e_k^2\theta_k^2 I_k^2}{\left(\sum_j e_j I_j\right)^2} \leq \frac{e_k^2\theta_k^2 I_k^2}{e_k^2 I_k^2} = \theta_k^2. \tag{2.27}$$

We have that

$$\mathbf{P}\Big(\sup_{2^{r-1}<T\leq 2^r} |\mathcal{M}(T)| > \varepsilon\Big) \leq \frac{\theta_k 2^{\frac{r}{2}}}{\varepsilon}. \tag{2.28}$$

We choose $\varepsilon = \frac{1}{m}2^{r-1}$, then we have that

$$\begin{aligned} \mathbf{P}\Big(\sup_{2^{r-1}<T\leq 2^r} \frac{1}{T}|\mathcal{M}(T)| > \frac{1}{m}\Big) &= \mathbf{P}\Big(\sup_{2^{r-1}<T\leq 2^r} \frac{1}{T}|\mathcal{M}(T)| > \frac{\varepsilon}{2^{r-1}}\Big) \qquad (2.29) \\ &\leq \mathbf{P}\Big(\sup_{2^{r-1}<T\leq 2^r} |\mathcal{M}(T)| > \varepsilon\Big) \\ &\leq \frac{2m\theta_k}{2^{\frac{r}{2}}}. \end{aligned}$$

Then, we have that

$$\mathbf{P}(A_{m,l}) \leq \sum_{r=l+1}^{\infty} \mathbf{P}\Big(\sup_{2^{r-1}<T\leq 2^r} \frac{1}{T}|\mathcal{M}(T)| > \frac{1}{m}\Big) \leq \sum_{r=l+1}^{\infty} \frac{2m\theta_k}{2^{\frac{r}{2}}} \to 0 \tag{2.30}$$

as $l \to \infty$. Thus, the almost sure convergence is proved.

Now we look at the fourth integral under the limit $T \to \infty$.

$$
\begin{aligned}
& \lim_{T\to\infty}\sum_{k,j}\frac{1}{T}\int_0^T \frac{1}{V}e_k\beta_{kj}I_j(X_k-\frac{\Lambda_k}{d_k})\mathrm{d}t \\
\leq\quad & \lim_{T\to\infty}\sum_{k,j}\frac{1}{T}\int_0^T \frac{e_k\beta_{kj}I_j}{V}\Big|X_k-\frac{\Lambda_k}{d_k}\Big|\mathrm{d}t \\
\leq\quad & \sum_{k,j}\frac{e_k\beta_{kj}}{e_j}\lim_{T\to\infty}\frac{1}{T}\int_0^T \Big|X_k-\frac{\Lambda_k}{d_k}\Big|\mathrm{d}t \\
=\quad & \sum_{k,j}\frac{e_k\beta_{kj}}{e_j}\int_0^\infty \Big|x-\frac{\Lambda_k}{d_k}\Big|\nu(x)\mathrm{d}x \\
\leq\quad & \sum_{k,j}\frac{e_k\beta_{kj}}{e_j}\Big[\int_0^\infty \Big(x-\frac{\Lambda_k}{d_k}\Big)^2\nu(x)\mathrm{d}x\Big]^{\frac{1}{2}}
\end{aligned}
\tag{2.31}
$$

In the fourth line of the we used the ergodic property of the process $X_k(t)$, i.e. for any measurable function $f(x)$, we have

$$
\lim_{T\to\infty}\frac{1}{T}\int_0^T f(X_k(t))\mathrm{d}t=\int_{\mathbb{R}} f(x)\nu(x)\mathrm{d}x,\ \text{a.s.,} \tag{2.32}
$$

where $\nu(x)$ is the stationary distribution of $X_k(t)$. We will provide the proof of the ergodic property in the end of this chapter. Using this property of X_k with function $f(x)=(x-\frac{\Lambda_k}{d_k})\wedge m$, we have

$$
\begin{aligned}
\int_0^\infty \Big[\Big(x-\frac{\Lambda_k}{d_k}\Big)^2\wedge m\Big]\nu(x)\mathrm{d}x &= \lim_{T\to\infty}\frac{1}{T}\int_0^T \Big[\Big(X_k-\frac{\Lambda_k}{d_k}\Big)^2\wedge m\Big]\mathrm{d}t \\
&= \lim_{T\to\infty}\frac{1}{T}\int_0^T E\Big[\Big(X_k-\frac{\Lambda_k}{d_k}\Big)^2\wedge m\Big]\mathrm{d}t \\
&\leq \lim_{T\to\infty}\frac{1}{T}\int_0^T E\Big(X_k-\frac{\Lambda_k}{d_k}\Big)^2\mathrm{d}t
\end{aligned}
\tag{2.33}
$$

We can solve $X_k(t)$ in explicit form as

$$X_k(t) = e^{-(d_k+\frac{\rho_k^2}{2})t+\rho B_t} S_k(0) + \int_0^t e^{(d_k+\frac{\rho_k^2}{2})(s-t)+\rho(B_t-B_s)} \Lambda_k \mathrm{d}s. \tag{2.34}$$

By cumbersome calculation, which will be provided at the end of this chapter, we have

$$\lim_{T\to\infty} E\Big(X_k(T) - \frac{\Lambda_k}{d_k}\Big)^2 = \frac{\rho_k^2}{d_k^2(2d_k - \rho_k^2)} \tag{2.35}$$

Then, letting $m \to \infty$ in (2.33) we have

$$\int_0^\infty \Big(x - \frac{\Lambda_k}{d_k}\Big)^2 \nu(x)\mathrm{d}x \le \frac{\rho_k^2}{d_k^2(2d_k - \rho_k^2)} \tag{2.36}$$

Then, the inequality (2.31) could be rewritten as

$$\begin{aligned}
\lim_{T\to\infty} \sum_{k,j} \frac{1}{T}\int_0^T \frac{1}{V} e_k\beta_{kj} I_j (X_k - \frac{\Lambda_k}{d_k})\mathrm{d}t &\le \sum_{k,j} \frac{e_k\beta_{kj}}{e_j} \frac{\rho_k}{d_k\sqrt{2d_k - \rho_k^2}} \\
&= \max_l \Big\{ \frac{\rho_l}{\sqrt{2d_l - \rho_l^2}} \Big\} \sum_{k,j} \frac{w_k\beta_{kj}}{d_k(d_k+\gamma_k)e_j} \\
&= \max_l \Big\{ \frac{\rho_l}{\sqrt{2d_l - \rho_l^2}} \Big\} \sum_j \frac{\mathcal{R}_0 w_j}{e_j} \\
&= \max_l \Big\{ \frac{\rho_l}{\sqrt{2d_l - \rho_l^2}} \Big\} \sum_j \mathcal{R}_0 (d_j + \gamma_j)
\end{aligned} \tag{2.37}$$

Therefore, we have the estimate

$$\lim_{T\to\infty}\frac{1}{T}\log V(T) \le (\mathcal{R}_0-1)\max_k\{d_j+\gamma_k\} - \frac{1}{2\sum_k \theta_k^{-2}} + \max_l\left\{\frac{\rho_l}{\sqrt{2d_l-\rho_l^2}}\right\}\sum_j \mathcal{R}_0(d_j+\gamma_j). \tag{2.38}$$

Remark 2.1.2. From the above proof we could see the limit in the estimate (2.38) is taken in the almost sure sense. The first two terms on the right hand side of (2.38) are negative, the third term is a small positive number with the assumption that the volatility ρ is small. Then, the estimation (2.38) states that in case of $\mathcal{R}_0 < 1$, the Lyapunov function $V(t)$, which is equivalent to the infected population, decrease to zero exponentially almost surely.

2.1.2 Stability of Endemic Equilibrium

Now we consider the case when $\mathcal{R}_0 > 1$. By Theorem 1.1.1 we know that the deterministic multi-group SIR system (1.2) has an endemic equilibrium $E^* = [S_1^*, S_2^*, \ldots, S_n^*, I_1^*, I_2^*, \ldots, I_n^*]$, and E^* is globally stable. We have the equilibrium condition

$$\Lambda_k - \sum_{j=1}^{n} \beta_{kj} S_k^* I_j^* - d_k S_k^* = 0, \tag{2.39}$$

$$\sum_{j=1}^{n} \beta_{kj} S_k^* I_j^* - (d_k+\gamma_k) I_k^* = 0. \tag{2.40}$$

Let $\bar{B}$ and $\mathbf{w} = [w_1, \ldots, w_n]$ be defined as in Theorem 1.1.1, where

$$\bar{B} = \begin{bmatrix} -\sum_{j\neq 1}\bar{\beta}_{1j} & \bar{\beta}_{21} & \bar{\beta}_{31} & \cdots & \bar{\beta}_{n1} \\ \bar{\beta}_{12} & -\sum_{j\neq 2}\bar{\beta}_{2j} & \bar{\beta}_{32} & \cdots & \bar{\beta}_{n2} \\ \bar{\beta}_{13} & \bar{\beta}_{23} & -\sum_{j\neq 3}\bar{\beta}_{3j} & \cdots & \bar{\beta}_{n3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \bar{\beta}_{1n} & \bar{\beta}_{2n} & \bar{\beta}_{3n} & \cdots & -\sum_{j\neq n}\bar{\beta}_{nj} \end{bmatrix}, \tag{2.41}$$

and $\mathbf{w}$ satisfies

$$\sum_{j=1}^{n} \bar{\beta}_{kj} w_k = \sum_{j=1}^{n} \bar{\beta}_{jk} w_j. \tag{2.42}$$

We define $g(x) = 1 + \ln x$, and the fact that $x \geq g(x)$ will be used repeatedly.

We consider the function

$$V_1 = \sum_{k=1}^{N} w_k \Big[S_k^* \Big(\frac{S_k(t)}{S_k^*} - g\Big(\frac{S_k(t)}{S_k^*}\Big)\Big) + I_k^* \Big(\frac{I_k(t)}{I_k^*} - g\Big(\frac{I_k(t)}{I_k^*}\Big)\Big)\Big] \tag{2.43}$$

Differentiating (2.43), we have

$$
\begin{aligned}
\mathrm{d}V_1 = &\sum_k w_k\Big[\Lambda_k - d_kS_k - (d_k+\gamma_k)I_k - \frac{S_k^*}{S_k}(\Lambda_k - \sum_j \beta_{kj}S_kI_j - d_kS_k) \\
&-\frac{I_k^*}{I_k}(\sum_j \beta_{kj}S_kI_j - (d_k+\gamma_k)I_k) + \frac{1}{2}\rho_k^2S_k^* + \frac{1}{2}\theta_k^2I_k^*\Big] + (...)\mathrm{d}B_k + (...)\mathrm{d}W_k \\
= &\sum_k w_k\Big[\sum_j \beta_{kj}S_k^*I_j^* + d_kS_k^* - d_kS_k - \frac{S_k^*}{S_k}(\sum_j \beta_{kj}S_k^*I_j^* + d_kS_k^*) + \sum_j \beta_{kj}S_k^*I_j \\
&+d_kS_k^* - \frac{I_k^*}{I_k}\sum_j \beta_{kj}S_kI_j + \sum_j S_k^*I_j^* + +\frac{1}{2}\rho_k^2S_k^* + \frac{1}{2}\theta_k^2I_k^*\Big] + (...)\mathrm{d}B_k + (...)\mathrm{d}W_k \\
= &\sum_k w_k\Big[d_kS_k^*(2 - \frac{S_k}{S_k^*} - \frac{S_k^*}{S_k}) + \sum_j \beta_{kj}S_k^*I_j^*(2 - \frac{I_k}{I_k^*} - \frac{S_k^*}{S_k} + \frac{I_j}{I_j^*} - \frac{I_k^*S_kI_j}{I_kS_k^*I_j^*}) \\
&+\frac{1}{2}\rho_k^2S_k^* + \frac{1}{2}\theta_k^2I_k^*\Big] + (...)\mathrm{d}B_k + (...)\mathrm{d}W_k \\
= &\sum_k w_kd_kS_k^*\Big(2 - \frac{S_k}{S_k^*} - \frac{S_k^*}{S_k}\Big) + \sum_{k,j} w_k\beta_{kj}S_k^*I_j^*\Big(g(\frac{I_k}{I_k^*}) - \frac{I_k}{I_k^*} - g(\frac{I_j}{I_j^*}) + \frac{I_j}{I_j^*}\Big) \\
&+\sum_{k,j} w_k\beta_{kj}S_k^*I_j^*\Big(g(\frac{S_k^*}{S_k}) - \frac{S_k^*}{S_k} + g(\frac{I_k^*S_kI_j}{I_kS_k^*I_j^*}) - \frac{I_k^*S_kI_j}{I_kS_k^*I_j^*}\Big) + \sum_k w_k(\frac{1}{2}\rho_k^2S_k^* + \frac{1}{2}\theta_k^2I_k^*) \\
&+(...)\mathrm{d}B_k + (...)\mathrm{d}W_k. \qquad (2.44)
\end{aligned}
$$

From (2.42) we have

$$
\sum_{k,j} w_k\beta_{kj}S_k^*I_j^*\Big(g(\frac{I_k}{I_k^*}) - \frac{I_k}{I_k^*}\Big) = \sum_{k,j} w_j\beta_{jk}S_j^*I_k^*\Big(g(\frac{I_k}{I_k^*}) - \frac{I_k}{I_k^*}\Big), \qquad (2.45)
$$

Therefore, we have

$$
\mathrm{d}V_1 \le \sum_k w_kd_kS_k^*\Big(2 - \frac{S_k}{S_k^*} - \frac{S_k^*}{S_k}\Big) + \sum_k w_k(\frac{1}{2}\rho_k^2S_k^* + \frac{1}{2}\theta_k^2I_k^*) + (\dots)\mathrm{d}B + (\dots)\mathrm{d}W. \quad (2.46)
$$

Consider the function

$$V_2 = \sum_{k=1}^{N} w_k I_k^* \Big[\frac{I_k}{I_k^*} - g(\frac{I_k}{I_k^*})\Big]. \tag{2.47}$$

Then, we have

$$\begin{aligned}
\mathrm{d}V_2 &= \sum_k w_k \Big[(1-\frac{I_k^*}{I_k})\Big(\sum_j \beta_{kj} S_k I_j - (d_k+\gamma_k) I_k\Big) + \frac{1}{2}\theta_k^2 I_k^*\Big] + (...)\mathrm{d}B_k + (...)\mathrm{d}W_k \\
&= \sum_k w_k \Big[\sum_j \beta_{kj} S_k I_j - \frac{I_k}{I_k^*}\sum_j \beta_{kj} S_k^* I_j^* - \frac{I_k^*}{I_k}\sum_j \beta_{kj} S_k I_j + \sum_j \beta_{kj} S_k^* I_j^* + \frac{1}{2}\theta_k^2 I_k^*\Big] \\
&\quad + (...)\mathrm{d}B_k + (...)\mathrm{d}W_k \\
&= \sum_k w_k \Big[\sum_j \beta_{kj}(S_k - S_k^*)(I_j - I_j^*) + \sum_j \beta_{kj} S_k^* I_j^* (\frac{S_k}{S_k^*} + \frac{I_j}{I_j^*} - \frac{I_k}{I_k^*} - \frac{I_k^* S_k I_j}{I_k S_k^* I_j^*}) + \frac{1}{2}\theta_k^2 I_k^*\Big] \\
&\quad + (...)\mathrm{d}B_k + (...)\mathrm{d}W_k \\
&\le \sum_k w_k \Big[\sum_j \beta_{kj}(S_k - S_k^*)(I_j - I_j^*) + \sum_j \beta_{kj} S_k^* I_j^* (\frac{S_k}{S_k^*} + \frac{I_j}{I_j^*} - \frac{I_k}{I_k^*}) + \frac{1}{2}\theta_k^2 I_k^* \\
&\quad + \sum_{kj} S_k^* I_j^* (-1 - \ln \frac{I_k^* S_k I_j}{I_k S_k^* I_j^*})\Big] + (...)\mathrm{d}B_k + (...)\mathrm{d}W_k \\
&= \sum_k w_k \Big[\sum_j \beta_{kj}(S_k - S_k^*)(I_j - I_j^*) + \sum_j \beta_{kj} S_k^* I_j^* (\frac{I_j}{I_j^*} - \ln\frac{I_j}{I_j^*} - \frac{I_k}{I_k^*} + \ln\frac{I_k}{I_k^*}) \\
&\quad + \sum_j \beta_{kj} S_k^* I_j^* (\frac{S_k}{S_k^*} - 2 + g(\frac{S_k^*}{S_k})) + \frac{1}{2}\theta_k^2 I_k^*\Big] + (...)\mathrm{d}B_k + (...)\mathrm{d}W_k \\
&\le \sum_k w_k \Big[\sum_j \beta_{kj}(S_k - S_k^*)(I_j - I_j^*) + \sum_j \beta_{kj} S_k^* I_j^* (\frac{S_k}{S_k^*} + \frac{S_k^*}{S_k} - 2) + \frac{1}{2}\theta_k^2 I_k^*\Big] \\
&\quad + (...)\mathrm{d}B_k + (...)\mathrm{d}W_k.
\end{aligned} \tag{2.48}$$

Consider the function

$$V_3 = \sum_{k=1}^{N} w_k \frac{(S_k - S_k^*)^2}{2S_k^*}. \tag{2.49}$$

Then, we have

$$\mathrm{d}V_3 = \sum_k w_k \Big[\frac{S_k - S_k^*}{S_k^*} (\Lambda_k - \sum_j \beta_{kj} S_k I_j - d_k S_k) + \frac{\rho_k^2 S_k^2}{2S_k^*} \Big] + (...)\mathrm{d}B_k + (...)\mathrm{d}W_k \tag{2.50}$$

$$\begin{aligned}
&= \sum_k w_k \Big[\frac{S_k - S_k^*}{S_k^*} (\sum_j \beta_{kj} S_k^* I_j^* + d_k S_k - \sum_j \beta_{kj} S_k I_j - d_k S_k) + \frac{\rho_k^2 (S_k - S_k^* + S_k^*)^2}{2S_k^*} \Big] \\
&\quad + (...)\mathrm{d}B_k + (...)\mathrm{d}W_k \\
&= w_k \Big[- \frac{d_k - \rho_k^2}{S_k^*} (S_k - S_k^*)^2 - \frac{1}{S_k^*} \sum_j \beta_{kj} (S_k - S_k^*)^2 I_j - \sum_j \beta_{kj} (S_k - S_k^*)(I_j - I_j^*) \\
&\quad + \rho_k^2 S_k^* \Big] + (...)\mathrm{d}B_k + (...)\mathrm{d}W_k \\
&\leq w_k \Big[- \frac{d_k - \rho_k^2}{S_k^*} (S_k - S_k^*)^2 - \sum_j \beta_{kj} (S_k - S_k^*)(I_j - I_j^*) + \rho_k^2 S_k^* \Big] \\
&\quad + (...)\mathrm{d}B_k + (...)\mathrm{d}W_k.
\end{aligned} \tag{2.51}$$

Consider the function

$$V_4 = \sum_k (S_k - S_k^* + I_k - I_k^*)^2. \tag{2.52}$$

Then, we have

$$
\begin{aligned}
\mathrm{d}V_4 = & \sum_k 2(S_k - S_k^* + I_k - I_k^*)(\Lambda_k - d_k S_k - (d_k+\gamma_k)I_k) + (\rho_k^2 S_k^2 + \theta_k^2 I_k^2) \\
& +(...)\mathrm{d}B_k + (...)\mathrm{d}W_k \\
= & \sum_k 2(S_k - S_k^* + I_k - I_k^*)((d_k+\gamma_k)I^* + d_k S_k^* - d_k S_k - (d_k+\gamma_k)I_k) \\
& +(\rho_k^2 S_k^2 + \theta_k^2 I_k^2) + (...)\mathrm{d}B_k + (...)\mathrm{d}W_k \\
= & \sum_k -2d_k(S_k - S_k^*)^2 - 2(d_k+\gamma_k)(I_k - I_k^*)^2 - 2(2d_k+\gamma_k)(S_k - S_k^*)(I_k - I_k^*) \\
& +(\rho_k^2 S_k^2 + \theta_k^2 I_k^2) + (...)\mathrm{d}B_k + (...)\mathrm{d}W_k.
\end{aligned} \tag{2.53}
$$

Note that

$$
-2(2d_k+\gamma_k)(S_k - S_k^*)(I_k - I_k^*) \leq (d_k+\gamma_k)(I_k - I_k^*)^2 + \frac{(2d_k+\gamma_k)^2}{d_k+\gamma_k}(S_k - S_k^*)^2, \tag{2.54}
$$

Then, we have

$$
\begin{aligned}
\mathrm{d}V_4 \leq & \sum_k (\frac{(2d_k+\gamma_k)^2}{d_k+\gamma_k} - 2d_k + 2\rho_k^2)(S_k - S_k^*)^2 - (d_k+\gamma_k - 2\theta_k^2)(I_k - I_k^*)^2 \\
& +2\rho_k^2 S_k^{*2} + 2\theta_k^2 I_k^{*2} + (...)\mathrm{d}B_k + (...)\mathrm{d}W_k.
\end{aligned} \tag{2.55}
$$

Choose $\lambda = \max_k\{\sum_j \beta_{kj} I_j^*/d_k\}$ and $\varepsilon \leq \min_k\{\frac{d_k - \rho_k^2}{S_k^*}(\frac{(2d_k+\gamma_k)^2}{d_k+\gamma_k} - 2d_k + 2\rho_k^2)^{-1}\}$. Now we compute

$$
\begin{aligned}
& \mathrm{d}(\lambda V_1 + V_2 + V_3 + \varepsilon V_4) \\
\leq & \sum_k -A_k(S_k - S_k^*)^2 - B_k(I_k - I_k^*)^2 + C_k\rho_k^2 + D_k\theta_k^2 + (...)\mathrm{d}B_k + (...)\mathrm{d}W_k,
\end{aligned} \tag{2.56}
$$

where

$$A_k = \frac{d_k - \rho_k^2}{S_k^*} - \varepsilon(\frac{(2d_k + \gamma_k)^2}{d_k + \gamma_k} - 2d_k + 2\rho_k^2), \tag{2.57}$$

$$B_k = \varepsilon(d_k + \gamma_k - 2\theta_k^2), \tag{2.58}$$

$$C_k = \frac{1}{2}\lambda w_k S_k^* + w_k S_k^* + 2\varepsilon S_k^{*2}, \tag{2.59}$$

$$D_k = \frac{1}{2}\lambda w_k I_k^* + w_k I_k^* + 2\varepsilon I_k^{*2}. \tag{2.60}$$

Integrate (2.56) from 0 to T and then take expectation, we have

$$\begin{aligned} & E(\lambda V_1 + V_2 + V_3 + \varepsilon V_4)(T) - E(\lambda V_1 + V_2 + V_3 + \varepsilon V_4)(0) \\ \leq & \sum_k E\int_0^T -A_k(S_k - S_k^*)^2 - B_k(I_k - I_k^*)^2 \mathrm{d}t + (C_k\rho_k^2 + D_k\theta_k^2)T. \end{aligned} \tag{2.61}$$

Divide (2.61) by T, we will have

$$\begin{aligned} & \frac{1}{T}E(\lambda V_1 + V_2 + V_3 + \varepsilon V_4)(T) - \frac{1}{T}E(\lambda V_1 + V_2 + V_3 + \varepsilon V_4)(0) \\ \leq & \sum_k E\frac{1}{T}\int_0^T -A_k(S_k - S_k^*)^2 - B_k(I_k - I_k^*)^2 \mathrm{d}t + C_k\rho_k^2 + D_k\theta_k^2. \end{aligned} \tag{2.62}$$

Then, we have

$$\begin{aligned} \sum_k E\frac{1}{T}\int_0^T A_k(S_k - S_k^*)^2 + B_k(I_k - I_k^*)^2 \mathrm{d}t \leq & \sum_k (C_k\rho_k^2 + D_k\theta_k^2) \\ & + \frac{1}{T}E(\lambda V_1 + V_2 + V_3 + \varepsilon V_4)(0) - \frac{1}{T}E(\lambda V_1 + V_2 + V_3 + \varepsilon V_4)(T) \\ \leq & \sum_k (C_k\rho_k^2 + D_k\theta_k^2) + \frac{1}{T}E(\lambda V_1 + V_2 + V_3 + \varepsilon V_4)(0). \end{aligned} \tag{2.63}$$

Assuming $T \geq 1$, we know that the right hand side of (2.63) is bounded by constant. Then we take $T \to \infty$ on both side of (2.63). By dominating convergence theorem, it is valid to exchange limit operator and expectation. So we will have

$$\sum_k E \lim_{T\to\infty} \frac{1}{T} \int_0^T A_k(S_k - S_k^*)^2 + B_k(I_k - I_k^*)^2 \mathrm{d}t \leq \sum_k (C_k \rho_k^2 + D_k \theta_k^2). \tag{2.64}$$

Remark 2.1.3. From the estimation (2.64) we could see that the limit is taken in the $\mathcal{L}^2$ sense. With the assumption that the volatility $\{\rho_k, \theta_k\}$ is small, the susceptible population and the infected population $[S_1, I_1, \ldots, S_n, I_n]$ will be close to the endemic equilibrium $[S_1^*, I_1^*, \ldots, S_n^*, I_n^*]$.

2.1.3 Vaccination

We will apply pulse vaccination strategy to the stochastic SIR model. We have the following system:

$$\begin{cases} \dot{S}_k = \Lambda_k - \sum_{j=1}^n \beta_{kj} S_k I_j - d_k S_k + \sigma_k S_k \mathrm{d}W_k(t), \\ \dot{I}_k = \sum_{j=1}^n \beta_{kj} S_k I_j - (d_k + \gamma_k) I_k + \rho_k I_k \mathrm{d}B_k(t), \end{cases} \qquad t \in (t_i, t_{i+1}), \tag{2.65}$$

with vaccination condition

$$\begin{cases} S_k(t_i^+) = S_k(t_i^-)(1 - c_{ki}), \\ I_k(t_i^+) = I_k(t_i^-). \end{cases} \tag{2.66}$$

We will study the optimal strategy for vaccination and give numerical results in the later sections.

2.2 Stochastic Impulsive Control Problems

In this section we will give detailed description of the impulsive optimal control problem, and study the necessary condition that the optimal controls must satisfy. We have two approaches: methods of variation of calculus and dynamic programming. In addition we will discuss the solutions to coupled forward backward stochastic differential equations.

We look at the system whose evolution satisfies the following stochastic differential equation

$$\mathrm{d}x = f_k(x,u,t)\mathrm{d}t + \sum_{j=1}^{d} \sigma_k^j(x,u,t)\mathrm{d}W_j(t), \qquad t \in (t_k, t_{k+1}), \tag{2.67}$$

where $W(\cdot) = (W_1(\cdot), ..., W_d(\cdot))^T$ is a standard d-dimensional Wiener process defined on a complete probability space $(\Omega, \mathscr{F}, P)$ with filtration $\mathscr{F}^t = \sigma\{W(s); 0 \leq s \leq t\}$. Impulsive control c_k's are applied to the system at time t_k, $k = 1, \ldots, N-1$, and the state variable satisfies the following jump conditions

$$x(t_k^+) = g_k(x(t_k^-), c_k). \tag{2.68}$$

The stochastic impulsive optimal control problem is to find a continuous control $u(t)$ adapted to $\mathscr{F}^t$ and impulses c_k's , such that the cost functional

$$J(u(\cdot), c) = E\Big\{ \sum_{k=1}^{N-1} \phi_k(x(t_k^-), c_k) + \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} L_k(x,u,t)\mathrm{d}t + \phi_N(x(t_N^-)) \Big\} \tag{2.69}$$

is minimized. We assume that

$$f_k(x,u,t) : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \mapsto \mathbb{R}^n,$$

$$g_k(x,c) : \mathbb{R}^n \times \mathbb{R}^M \mapsto \mathbb{R}^n.$$

$$\sigma_k^j(x,u,t) : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \mapsto \mathbb{R}^n.$$

$$L_k(x,u,t) : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \mapsto \mathbb{R},$$

$$\phi_k(x,c) : \mathbb{R}^n \times \mathbb{R}^M \mapsto \mathbb{R}.$$

are smooth functions which have continuous derivatives of all orders.

2.2.1 Necessary Conditions by Methods of Variation of Calculus

As in the deterministic version, we derive the variational equation by adding a small perturbation to the optimal control. An adjoint variable is defined by a backward stochastic differential equation, and the optimal continuous control is found by minimizing the Hamiltonian, and the optimal impulse is determined in the process.

Assume $\{\hat{u}(\cdot), \hat{c}_k\}$ is the optimal control pair and $\hat{x}(\cdot)$ is the state variable of the system corresponding the control $\{\hat{u}(\cdot), \hat{c}_k\}$. Let us define another set of control $\{u^\theta(\cdot), c_k^\theta\}$ by

$$\begin{aligned} u^\theta(t) &\triangleq \hat{u}(t) + \theta v(t), \\ c_k^\theta &\triangleq \hat{c}_k + \theta c_k, \qquad k = 1, \dots, N-1, \end{aligned}$$

where $v(\cdot)$, c_k are arbitrary perturbations, and $0 < \theta \ll 1$. Let $x^\theta(\cdot)$ be the state variable corresponding to $\{u^\theta(t), c_k^\theta\}$.

Let us consider $z_k(t), k = 0, \dots N-1$, which solves the system,

$$\mathrm{d}z_k = \Big(\frac{\partial f_k}{\partial x}(\hat{x}(t), \hat{u}(t), t) z_k + \frac{\partial f_k}{\partial u} v\Big)\mathrm{d}t + \sum_{j=1}^{d} \Big(\frac{\partial \sigma_k^j}{\partial x} z_k + \frac{\partial \sigma_k^j}{\partial u} v\Big)\mathrm{d}W_j, \quad t \in (t_k, t_{k+1}), \tag{2.70}$$

$$z_k(t_k^+) = \frac{\partial g_k}{\partial x}(\hat{x}(t_k^-), \hat{c}_k) z_{k-1}(t_k^-) + \frac{\partial g_k}{\partial c} c_k, \tag{2.71}$$

for $k = 0, \dots, N-1$. We define $z_{-1}(t_0^-) = 0$ and $c_0 = 0$ just to ease the notation. Then, we will have the following estimation.

Lemma 2.2.1. *Let $\hat{x}(\cdot)$, $x^\theta(\cdot)$ and $z_k(\cdot)$ be defined as above, then we have*

1. $E\{x^\theta(t) - \hat{x}(t) - \theta z_k(t)\} = O(\theta^2)$, $t \in (t_k, t_{k+1})$, *and that*

2.

$$\frac{\mathrm{d}J(u^\theta,c^\theta)}{\mathrm{d}\theta}\Big|_{\theta=0} = E\Big\{\sum_{k=0}^{N-1}\Big[\frac{\partial\phi_k}{\partial c}(\hat{x}(t_k^-),\hat{c}_k)c_k + \frac{\partial\phi_{k+1}}{\partial x}(\hat{x}(t_k^-),\hat{c}_k)z_k(t_{k+1}^-) \tag{2.72}$$

$$+\int_{t_k}^{t_{k+1}}\Big(\frac{\partial L_k}{\partial x}(\hat{x}(s),\hat{u}(s),s)z_k(s) + \frac{\partial L_k}{\partial u}(\hat{x}(s),\hat{u}(s),s)z_k(s)v(s)\Big)\mathrm{d}t\Big]\Big\}.$$

Proof. 1. let

$$y^\theta(t) = \frac{1}{\theta}(x^\theta(t) - \hat{x}(t) - \theta z_k(t)), \qquad t\in(t_k,t_{k+1}), \tag{2.73}$$

then we have

$$\begin{aligned}
\mathrm{d}y^\theta \quad = \quad & \Big(\int_0^1 \frac{\partial f_k}{\partial x}(\hat{x}+\lambda(x^\theta-\hat{x}),\hat{u}+\lambda\theta v,t)y^\theta\mathrm{d}\lambda\Big)\mathrm{d}t \\
&+\Big(\int_0^1\Big(\frac{\partial f_k}{\partial x}(\hat{x}+\lambda(x^\theta-\hat{x}),\hat{u}+\lambda\theta v,t) - \frac{\partial f_k}{\partial x}(\hat{x},\hat{u},t)\Big)z_k\mathrm{d}\lambda\Big)\mathrm{d}t \\
&+\Big(\int_0^1\Big(\frac{\partial f_k}{\partial u}(\hat{x}+\lambda(x^\theta-\hat{x}),\hat{u}+\lambda\theta v,t) - \frac{\partial f_k}{\partial u}(\hat{x},\hat{u},t)\Big)v\mathrm{d}\lambda\Big)\mathrm{d}t \\
&+\sum_j\Big(\int_0^1 \frac{\partial\sigma_k^j}{\partial x}(\hat{x}+\lambda(x^\theta-\hat{x}),\hat{u}+\lambda\theta v,t)y^\theta\mathrm{d}\lambda\Big)\mathrm{d}W_j \\
&+\sum_j\Big(\int_0^1\Big(\frac{\partial\sigma_k^j}{\partial x}(\hat{x}+\lambda(x^\theta-\hat{x}),\hat{u}+\lambda\theta v,t) - \frac{\partial\sigma_k^j}{\partial x}(\hat{x},\hat{u},t)\Big)z_k\mathrm{d}\lambda\Big)\mathrm{d}W_j \\
&+\sum_j\Big(\int_0^1\Big(\frac{\partial\sigma_k^j}{\partial u}(\hat{x}+\lambda(x^\theta-\hat{x}),\hat{u}+\lambda\theta v,t) - \frac{\partial\sigma_k^j}{\partial u}(\hat{x},\hat{u},t)\Big)v\mathrm{d}\lambda\Big)\mathrm{d}W_j.
\end{aligned} \tag{2.74}$$

Now we refer to the useful fact (see [37] lemma 3.4.2) that if $Y(t)$ is the solution of the following

$$
\begin{cases} \mathrm{d}Y(t) = (A(t)Y(t) + a(t))\mathrm{d}t + \sum_j (B_j(t)Y(t) + b_j(t))\mathrm{d}W_j, \\ Y(0) = Y_0, \end{cases} \tag{2.75}
$$

and $|A(t)|, |B_j(t)| \leq L$, a.e. $t \in [0, T]$. Then,

$$
\sup_{t\in(0,T)} E|Y(t)|^2 \leq K\Big\{E|Y(0)|^2 + E\int_0^T (|a(s)|^2 + \sum_j |b_j(s)|^2)\mathrm{d}s\Big\}. \tag{2.76}
$$

Using the above result, we have

$$
\sup_{t\in(t_0,t_1)} E|y^\theta(t)|^2 \leq K\Big\{E|y^\theta(t_0)|^2 + E\int_0^T (|a_0(s)|^2 + \sum_j |b_0^j(s)|^2)\mathrm{d}s\Big\}, \tag{2.77}
$$

where

$$
\begin{aligned}
a_k(t) &= \int_0^1 \Big(\frac{\partial f_k}{\partial x}(\hat{x} + \lambda(x^\theta - \hat{x}), \hat{u} + \lambda\theta v, t) - \frac{\partial f_k}{\partial x}(\hat{x}, \hat{u}, t)\Big) z_k \mathrm{d}\lambda \qquad (2.78) \\
&\quad + \int_0^1 \Big(\frac{\partial f_k}{\partial u}(\hat{x} + \lambda(x^\theta - \hat{x}), \hat{u} + \lambda\theta v, t) - \frac{\partial f_k}{\partial u}(\hat{x}, \hat{u}, t)\Big) v \mathrm{d}\lambda = O(\theta), \\
b_k^j(t) &= \int_0^1 \Big(\frac{\partial \sigma_k^j}{\partial x}(\hat{x} + \lambda(x^\theta - \hat{x}), \hat{u} + \lambda\theta v, t) - \frac{\partial \sigma_k^j}{\partial x}(\hat{x}, \hat{u}, t)\Big) z_k \mathrm{d}\lambda\Big)\mathrm{d}W_j \qquad (2.79) \\
&\quad + \int_0^1 \Big(\frac{\partial \sigma_k^j}{\partial u}(\hat{x} + \lambda(x^\theta - \hat{x}), \hat{u} + \lambda\theta v, t) - \frac{\partial \sigma_k^j}{\partial u}(\hat{x}, \hat{u}, t)\Big) v \mathrm{d}\lambda\Big)\mathrm{d}W_j = O(\theta).
\end{aligned}
$$

Since $y^\theta(t_0) = 0$, we have

$$\sup_{t\in(t_0,t_1)} E|y^\theta(t)|^2 = O(\theta^2) \implies \sup_{t\in(t_0,t_1)} E|y^\theta(t)| = O(\theta), \tag{2.80}$$

by Hölder's Inequality.

Moving to the next interval (t_1, t_2), we have that

$$\begin{aligned} y^\theta(t_1^+) &= \frac{1}{\theta}\Big(g_1(\hat{x}^\theta(t_1^-), \hat{c}_1 + \theta c_1) - g_0(\hat{x}(t_1^-), \hat{c}_1) \\ &\quad -\theta\frac{\partial g_1}{\partial x}(\hat{x}(t_1^-), \hat{c}_1) z_0(t_1^-) - \theta\frac{\partial g_1}{\partial c}(\hat{x}(t_1^-), \hat{c}_1) c_1\Big) \\ &= \frac{\partial g_1}{\partial x}(\hat{x}(t_1^-), \hat{c}_1) y^\theta(t_1^-) + O(\theta). \end{aligned} \tag{2.81}$$

Then, we compute that

$$E|y^\theta(t_1^+)|^2 \leq K_1 E|y^\theta(t_1^-)|^2 + K_2\theta E|y^\theta(t_1^-)| + K_3\theta^2. \tag{2.82}$$

Using the estimation

$$\sup_{t\in(t_1,t_2)} E|y^\theta(t)|^2 \leq K\Big\{E|y^\theta(t_1^+)|^2 + E\int_0^T (|a_1(s)|^2 + \sum_j |b_1^j(s)|^2)\mathrm{d}s\Big\}, \tag{2.83}$$

we will have that

$$\sup_{t\in(t_1,t_2)} E|y^\theta(t)|^2 = O(\theta^2), \tag{2.84}$$

$$\sup_{t\in(t_1,t_2)} E|y^\theta(t)| = O(\theta), \tag{2.85}$$

By induction, we have

$$\sup_{t\in(t_k,t_{k+1})} E|y^\theta(t)|^2 = O(\theta^2), \tag{2.86}$$

$$\sup_{t\in(t_k,t_{k+1})} E|y^\theta(t)| = O(\theta), \tag{2.87}$$

for $k = 0, \ldots, N-1$, which finishes the first part of the lemma.

2. The second part of the lemma is proved by computing

$$\begin{aligned}
&J(u^\theta, c^\theta) - J(\hat{u}, \hat{c}) - \theta E\Big\{ \sum_{k=0}^{N-1} \Big[\frac{\partial \phi_k}{\partial c} c_k + \frac{\partial \phi_{k+1}}{\partial x} z_k(t_{k+1}^-) + \int_{t_k}^{t_{k+1}} \Big(\frac{\partial L_k}{\partial x} z_k + \frac{\partial L_k}{\partial u} v\Big)\mathrm{d}t\Big]\Big\} \\
&= \sum_{k=0}^{N-1} E\Big\{ \int_{t_k}^{t_{k+1}} \Big[\int_0^1 \frac{\partial L_k}{\partial x}(\hat{x} + \lambda(x^\theta - \hat{x}), \hat{u} + \lambda\theta v, t)\theta y^\theta \mathrm{d}\lambda \\
&\qquad + \int_0^1 \Big(\frac{\partial L_k}{\partial x}(\hat{x} + \lambda(x^\theta - \hat{x}), \hat{u} + \lambda\theta v, t) - \frac{\partial L_k}{\partial x}(\hat{x}, \hat{u}, t)\Big)\theta z_k \mathrm{d}\lambda \\
&\qquad + \int_0^1 \Big(\frac{\partial L_k}{\partial u}(\hat{x} + \lambda(x^\theta - \hat{x}), \hat{u} + \lambda\theta v, t) - \frac{\partial L_k}{\partial u}(\hat{x}, \hat{u}, t)\Big)\theta v \mathrm{d}\lambda\Big]\mathrm{d}t \\
&\qquad + \frac{\partial \phi_{k+1}}{\partial x}(\hat{x}(t_{k+1}^-), \hat{c}_{k+1})\theta \hat{y}_\theta(t_{k+1}^-) + O(\theta^2)\Big\} \\
&\implies \frac{\mathrm{d}J(u^\theta, c^\theta)}{\mathrm{d}\theta}\Big|_{\theta=0} = E\Big\{ \sum_{k=0}^{N-1} \Big[\frac{\partial \phi_k}{\partial c} c_k + \frac{\partial \phi_{k+1}}{\partial x} z_k(t_{k+1}^-) + \int_{t_k}^{t_{k+1}} \Big(\frac{\partial L_k}{\partial x} z_k + \frac{\partial L_k}{\partial u} v\Big)\mathrm{d}t\Big]\Big\}
\end{aligned}$$

□

Let us define

$$J_k = \frac{\partial \phi_k}{\partial c} c_k + \frac{\partial \phi_{k+1}}{\partial x} z_k(t_{k+1}^-) + \int_{t_k}^{t_{k+1}} \Big(\frac{\partial L_k}{\partial x} z_k + \frac{\partial L_k}{\partial u} v \Big) \mathrm{d}t, \tag{2.88}$$

which simplifies the lemma as

$$\frac{\mathrm{d}J(u^\theta, c^\theta)}{\mathrm{d}\theta}\Big|_{\theta=0} = E\Big\{ \sum_{k=0} J_k \Big\}. \tag{2.89}$$

Now we consider the following backward stochastic differential equation:

$$\begin{aligned} \mathrm{d}p_{N-1}(t) = \Big(& - \Big(\frac{\partial L_{N-1}}{\partial x}(\hat{x}(t), \hat{u}(t), t) \Big)^T - \Big(\frac{\partial f_{N-1}}{\partial x}(\hat{x}(t), \hat{u}(t), t) \Big)^T p_{N-1}(t) \\ & - \sum_{j=1}^{d} \Big(\frac{\partial \sigma_{N-1}^j}{\partial x}(\hat{x}(t), \hat{u}(t), t) \Big)^T r_{N-1}^j(t) \Big) \mathrm{d}t + \sum_j r_{N-1}^j(t) \mathrm{d}W_j, \quad t \in (t_{N-1}, t_N), \end{aligned} \tag{2.90}$$

$$p_{N-1}(t_N^-) = \frac{\partial \phi_N}{\partial x}^T (\hat{x}(t_N^-)). \tag{2.91}$$

By theory of BSDE, there exists a unique pair of processes $\{p_{N-1}(t), r_{N-1}^j(t)\}$ which are adapted to the filtration $\mathscr{F}^t$.

By Ito's formula we will have

$$\begin{aligned} d(p_{N-1}^T z_{N-1}) = & -\frac{\partial L_{N-1}}{\partial x}(\hat{x}(t), \hat{u}(t), t) z_{N-1}(t) + p_{N-1}^T(t) \frac{\partial f_{N-1}}{\partial u}((\hat{x}(t), \hat{u}(t), t)) v(t) \\ & + \sum_j (r_{N-1}^j(t))^T \frac{\partial \sigma_{N-1}^j}{\partial u}(\hat{x}(t), \hat{u}(t), t) v(t) \mathrm{d}t + (\dots) \mathrm{d}W. \end{aligned} \tag{2.92}$$

Integrating and taking expectation, we have that

$$E\Big\{\frac{\partial \phi_N}{\partial x}(\hat{x}(t_N^-))z_{N-1}(t_N^-) + \int_{t_{N-1}}^{t_N} \frac{\partial L_{N-1}}{\partial x}(\hat{x}(t), \hat{u}(t), t) z_{N-1}(t)\mathrm{d}t\Big\} \tag{2.93}$$

$$= E\Big\{p_{N-1}^T(t_{N-1}^+)\Big(\frac{\partial g_{N-1}}{\partial x}(\hat{x}(t_{N-1}^-), \hat{c}_{N-1}))z_{N-2}(t_{N-1}^-) + \frac{\partial g_{N-1}}{\partial c}(\hat{x}(t_{N-1}^-), \hat{c}_{N-1}))c_{N-1}\Big)$$

$$+ \int_{t_{N-1}}^{t_N} p_{N-1}^T(t)\frac{\partial f_{N-1}}{\partial u}(\hat{x}(t), \hat{u}(t), t)v(t) + \sum_j (r_{N-1}^j(t))^T \frac{\partial \sigma_{N-1}^j}{\partial u}(\hat{x}(t), \hat{u}(t), t)v(t)\mathrm{d}t\Big\}.$$

Then, we could get

$$E\{J_{N-1}\} = E\Big\{\alpha_{N-1}^T c_{N-1} + \int_{t_{N-1}}^{t_N} \Big(p_{N-1}^T(t)\frac{\partial f_{N-1}}{\partial u}(\hat{x}(t), \hat{u}(t), t) \tag{2.94}$$

$$+ \sum_j (r_{N-1}^j)^T(t)\frac{\partial \sigma_{N-1}^j}{\partial u}(\hat{x}(t), \hat{u}(t), t) + \frac{\partial L_{N-1}}{\partial u}(\hat{x}(t), \hat{u}(t), t)\Big)v(t)\mathrm{d}t + \beta_{N-1}^T z_{N-2}(t_{N-1}^-)\Big\},$$

where

$$\alpha_{N-1}^T = \frac{\partial \phi_{N-1}}{\partial c}(\hat{x}(t_{N-1}^-), \hat{c}_{N-1}) + p_{N-1}^T(t_{N-1}^+)\frac{\partial g_{N-1}}{\partial c}(\hat{x}(t_{N-1}^-), \hat{c}_{N-1}) \tag{2.95}$$

$$\beta_{N-1}^T = p_{N-1}^T(t_{N-1}^+)\frac{\partial g_{N-1}}{\partial x}(\hat{x}(t_{N-1}^-), \hat{c}_{N-1}). \tag{2.96}$$

By induction we will conclude the result as follows.

Theorem 2.2.2. *For $k = N-1, N, \ldots, 0$, let $\{p_k(t), r_k^j(t)\}$ be the unique processes solving the following BSDE*

$$\begin{aligned}
\mathrm{d}p_k(t) = \Big[-\Big(\frac{\partial L_k}{\partial x}(\hat{x}(t), \hat{u}(t), t)\Big)^T - \frac{\partial f_k}{\partial x}(\hat{x}(t), \hat{u}(t), t) p_k(t) \\
- \sum_j \Big(\frac{\partial \sigma_k^j}{\partial x}(\hat{x}(t), \hat{u}(t), t)\Big)^T r_k^j(t)\Big]\mathrm{d}t + \sum_j r_k^j(t)\mathrm{d}W_j, \quad t \in (t_k, t_{k+1})
\end{aligned} \tag{2.97}$$

$$p_k(t_{k+1}^-) = p_{k+1}^T(t_{k+1}^+)\frac{\partial g_{k+1}}{\partial x}(\hat{x}(t_{k+1}^-), \hat{c}_{k+1}) + \frac{\partial \phi_{k+1}}{\partial x}^T(\hat{x}(t_{k+1}^-), \hat{c}_{k+1}), \tag{2.98}$$

then the variation of the cost function has the form:

$$\begin{aligned}
\frac{\mathrm{d}J(u^\theta, c^\theta)}{\mathrm{d}\theta}\Big|_{\theta=0} &= E\Big\{ \textstyle\sum_{k=0}^{N-1} \Big[\Big(\frac{\partial \phi_k}{\partial c}(\hat{x}(t_k^-), \hat{c}_k) + p_k^T(t_k^+)\frac{\partial g_k}{\partial c}(\hat{x}(t_k^-), \hat{c}_k)\Big)c_k \\
&\quad + \int_{t_k}^{t_{k+1}} p_k^T \frac{\partial f_k}{\partial u} v + \sum_j (r_k^j)^T \frac{\partial \sigma_k^j}{\partial u} v + \frac{\partial L_k}{\partial u} v \mathrm{d}t\Big]\Big\}.
\end{aligned} \tag{2.99}$$

The optimal control $\{\hat{u}(t), \hat{c}\}$ satisfy

$$\begin{aligned}
&\frac{\partial \phi_k}{\partial c}(\hat{x}(t_k^-), \hat{c}_k) + p_k^T(t_k^+)\frac{\partial g_k}{\partial c}(\hat{x}(t_k^-), \hat{c}_k) = 0, \\
&p_k^T \frac{\partial f_k}{\partial u}(\hat{x}(t), \hat{u}(t), t) + \sum_j (r_k^j)^T \frac{\partial \sigma_k^j}{\partial u}(\hat{x}) + \frac{\partial L_k}{\partial u}(\hat{x}, \hat{u}, t) = 0.
\end{aligned} \tag{2.100}$$

2.2.2 Forward Backward Stochastic Differential Equations

By Theorem 2.2.2 the optimal control $\hat{u}$ could be solved in a feedback form $\hat{u} = \hat{u}(\hat{x}, p, r, t)$, and we set

$$\begin{aligned}
&b_k(t, x, p_k, r_k) = f_k(x, u(t, x, p_k, r_k), t), \\
&\hat{b}_k(t, x, p_k, r_k) = -\frac{\partial L_k}{\partial x}^T(x, \hat{u}(t, x, p_k, r_k), t) - \frac{\partial f_k}{\partial x}(x, \hat{u}(..), t)p_k(t) - \sum_j \frac{\partial \sigma_k^j}{\partial x}^T(x, \hat{u}(..), t) r_k^j(t), \\
&\xi_k^j(t, x, p_k, r_k) = \sigma_k^j(x, \hat{u}(t, x, p_k, r_k), t).
\end{aligned}$$

Then, the problem is to solve the following FBSDE

$$
\begin{aligned}
\mathrm{d}x(t) &= b_k(t,x,p_k,r_k)\mathrm{d}t + \sum_{j=1}^{d} \xi_k(t,x,p_k,r_k)\mathrm{d}W_j, &(2.101)\\
\mathrm{d}p_k(t) &= -\hat{b}_k(t,x,p_k,r_k)\mathrm{d}t + \sum_{j=1}^{d} r_k^j \mathrm{d}W_j, \quad t\in(t_k,t_{k+1}), &(2.102)
\end{aligned}
$$

with jump conditions

$$
\begin{aligned}
x(t_k^+) &= g_k(x(t_k^-),c_k), &(2.103)\\
p_k(t_{k+1}^-) &= p_{k+1}^T(t_{k+1}^+)\frac{\partial g_{k+1}}{\partial x}(x(t_{k+1}^-),c_{k+1}) + \frac{\partial \phi_{k+1}}{\partial x}^T(x(t_{k+1}^-),c_{k+1}), &(2.104)
\end{aligned}
$$

where c_k satisfies

$$
\frac{\partial \phi_k}{\partial c}(x(t_k^-),c_k) + p_k^T(t_k^+)\frac{\partial g_k}{\partial c}(x(t_k^-),c_k) = 0. \tag{2.105}
$$

Previous works are done by Ma *et al.* [27] and the solvability of the adapted solution to the FBSDE has been studied. In particular, a direct scheme, called the four step scheme is provided to solve the FBSDE explicitly.

Then, our approach to solve the impulsive optimal control has the following steps.

1. Solve for $r_{N-1} = r_{N-1}(t,x,y,z)$

$$
z\xi_{N-1}(t,x,y,r_{N-1}) + r_{N-1} = 0. \tag{2.106}
$$

2. Solve the system

$$\frac{\partial \theta^j_{N-1}}{\partial t} + \frac{1}{2}\mathrm{tr}\Big(\frac{\partial^2 \theta^j_{N-1}}{\partial x^2}\xi_{N-1}(t,x,\theta_{N-1},r_{N-1}(t,x,\theta,\frac{\partial}{\partial x}\theta_{N-1}))\xi^T_{N-1}\Big) \tag{2.107}$$
$$+\frac{\partial \theta^j_{N-1}}{\partial x} b_{N-1}(t,x,\theta_{N-1},r_{N-1}(..)) + \hat{b}^j_{N-1}(t,x,\theta_{N-1},r_{N-1}(..)) = 0,$$

for $t \in [t_{N-1}, t_N]$, with the terminal condition $\theta_{N-1}(t_N, x) = \frac{\partial \phi_N}{\partial x}^T(x)$.

3. Then for $k = N-2, ..., 1, 0$, do the following steps:

   (a) Solve for $c_k(x)$ satisfying

$$\frac{\partial \phi_k}{\partial c}(x, c_k) + \theta^T_k(t_k, g_k(x, c_k))\frac{\partial g_k}{\partial c}(x, c_k) = 0. \tag{2.108}$$

   (b) Solve for $r_k = r_k(t, x, y, z)$

$$z\xi_k(t,x,y,r_k) + r_k = 0. \tag{2.109}$$

   (c) Solve the system

$$\frac{\partial \theta^j_k}{\partial t} + \frac{1}{2}\mathrm{tr}\Big(\frac{\partial^2 \theta^j_k}{\partial x^2}\xi_k(t,x,\theta_k,r_k(t,x,\theta,\frac{\partial}{\partial x}\theta_k))\xi^T_k\Big) \tag{2.110}$$
$$+\frac{\partial \theta^j_k}{\partial x} b_k(t,x,\theta_k,r_k(..)) + \hat{b}^j_k(t,x,\theta_k,r_k(..)) = 0$$

   for $t \in [t_k, t_{k+1}]$ with terminal condition

$$\theta_k(t_{k+1}, x) = \frac{\partial g_{k+1}}{\partial x}^T (x, c_{k+1}(x))\theta_{k+1}(t_{k+1}, g_{k+1}(x, c_{k+1}(x))) + \frac{\partial \phi_k}{\partial x}(x, c_k(x)) \tag{2.111}$$

4. Solve the FSDE

$$\begin{aligned}
\mathrm{d}X_k(t) &= b_k(t, X_k, \theta_k(t, X_k), r_k(t, X_k, \theta_k(t, X_k), \partial_x\theta_k(t, X_k)))\mathrm{d}t & (2.112)\\
&\quad +\xi_k(t, X_k, \theta_k(t, X_k), r_k(..))\mathrm{d}W(t) \\
X_k(t_k) &= g_k(X_{k-1}(t_k), c_k(X_{k-1}(t_k))) & (2.113)
\end{aligned}$$

   for $k = 0, 1, ..., N-1$, and $x_{-1}(t_0) = x_0$.

5. Compute the adjoint processes by

$$p_k(t) = \theta_k(t, X_k(t)), \tag{2.114}$$

$$r_k(t) = r_k(t, X_k, \theta_k(t, X_k), \partial_x\theta_k(t, X_k)). \tag{2.115}$$

6. The optimal control is determined by

$$c_k = c_k(X_k(t_k)), \tag{2.116}$$

$$u_k(t) = u_k(t, X_k(t), p_k, r_k). \tag{2.117}$$

Remark 2.2.3. The realization of the scheme depends on the solvability of the pde in step (2) and (3c), and the existence of the smooth function $c_k(x)$ in step (3a).

2.2.3 Necessary Conditions by Dynamic Programming Approach

In the study of optimal control dynamic programming is another frequently used approach. However, we cannot find any study of dynamic programming applied to impulsive optimal control problem. In this subsection, we set up the value function which involves the impulse cost and we derive the Hamilton-Jacobi-Bellman equation which characterize the optimal control and value function. The relation between the value function and the maximum principle will be discussed.

Consider the time interval $t \in (t_m, t_{m+1})$, and define the value function as follows

$$V_m(x,t) = \inf_{\substack{u(s),\ t<s<t_{N-1} \\ \{c_k\}_{k=m+1}^{N-1}}} E\Big\{ \int_t^{t_{m+1}} L_m(Y(s),u(s),s)\mathrm{d}s \tag{2.118}$$
$$+ \sum_{k=m+1}^{N-1} \Big[\phi_k(Y(t_k^-),c_k) + \int_{t_k}^{t_{k+1}} L_k(Y(s),u(s),s)\mathrm{d}s\Big] + \phi_N(Y(t_N^-))\Big\},$$

where $Y(s)$ is the process defined by

$$\begin{cases} \mathrm{d}Y(s) = f_k(Y(s),u(s),s)\mathrm{d}s + \sum_{j=1}^d \sigma_k^j(Y(s),u(s),t)\mathrm{d}W_j(s), \\ \qquad\qquad\qquad\qquad\qquad s \in (t,t_{m+1}) \bigcup \Big(\bigcup_{k=m+1}^{N-1}(t_k,t_{k+1})\Big), \\ Y(t_k^+) = g_k(Y(t_k^-),c_k), \qquad k = m+1,...,N-1 \\ Y(t) = x \end{cases} \tag{2.119}$$

Assuming we use an arbitrary constant control u in a short time interval $(t,t+h)$, and then guide the system by optimal control, we will have

$$\begin{aligned} V_m(x,t) &\le E\Big\{ \int_t^{t+h} L_m(Y(s),u,t)\mathrm{d}s + V_m(Y(t+h),t+h)\Big\} \quad \Longrightarrow \\ 0 &\le E\Big\{ \int_t^{t+h} L_m(Y(s),u,t)\mathrm{d}s + V_m(Y(t+h),t+h) - V_m(Y(t),t)\Big\}. \end{aligned} \tag{2.120}$$

Applying Ito's formula to $V_m(X(t),t)$, we have

$$\mathrm{d}V_m = \frac{\partial V_m}{\partial t}\mathrm{d}t + \frac{\partial V_m}{\partial x} f_m \mathrm{d}t + \frac{\partial V_m}{\partial x}\sigma_m \mathrm{d}W + \frac{1}{2}\sum_{i,j,k}\frac{\partial^2 V_m}{\partial x_i x_k}\sigma_m^{ij}\sigma_m^{kj}\mathrm{d}t. \tag{2.121}$$

Then we have

$$\begin{aligned} & E\Big\{V_m(Y(t+h), t+h) - V_m(Y(t), t)\Big\} \qquad (2.122) \\ = \quad & E\Big\{\int_t^{t+h}\frac{\partial V_m}{\partial t} + \frac{\partial V_m}{\partial x} f_m + \frac{1}{2}\sum_{i,j,k}\frac{\partial^2 V_m}{\partial x_i x_k}\sigma_m^{ij}\sigma_m^{kj}\mathrm{d}s\Big\}. \end{aligned}$$

We combine (2.122) and inequality (2.120), then

$$\begin{aligned} E\Big\{\int_t^{t+h} & L_m(Y(s), u, s) + \frac{\partial V_m}{\partial t}(Y(s), s) + \frac{\partial V_m}{\partial x}(Y(s), s) f_m(Y(s), u, s) \qquad (2.123) \\ & + \frac{1}{2}\sum_{i,j,k}\frac{\partial^2 V_m}{\partial x_i x_k}(Y(s), s)\sigma_m^{ij}(Y(s), s)\sigma_m^{kj}(Y(s), s)\mathrm{d}s\Big\} \geq 0. \end{aligned}$$

For $s \to t$, we have $Y(s) \to Y(t) = x$ almost surely, thus we derive that

$$\frac{\partial V_m}{\partial t}(x,t) + \frac{\partial V_m}{\partial x}(x,t) f_m(x,u,t) + \frac{1}{2}\sum_{i,j,k}\frac{\partial^2 V_m}{\partial x_i x_k}(x,t)\sigma_m^{ij}(x,t)\sigma_m^{kj}(x,t) + L_m(x,u,t) \geq 0$$

by taking $h \to 0$. Equality will be reached when u is chosen to be optimal, i.e.

$$\begin{aligned} \min_u \Big\{ & \frac{\partial V_m}{\partial t}(x,t) + \frac{\partial V_m}{\partial x}(x,t) f_m(x,u,t) \qquad (2.124) \\ & + \frac{1}{2}\sum_{i,j,k}\frac{\partial^2 V_m}{\partial x_i x_k}(x,t)\sigma_m^{ij}(x,t)\sigma_m^{kj}(x,t) + L_m(x,u,t)\Big\} = 0. \end{aligned}$$

The above equation is the Hamilton-Jacobi-Bellman Equation.

Now we consider the jump condition of the value function at time t_m, and by definition (2.118) we have

$$
\begin{aligned}
V_{m-1}(x, t_m^-) = \inf_{\substack{u(s),\ t_m < s < t_{N-1} \\ \{c_k\}_{k=m}^{N-1}}} & E\Big\{ \sum_{k=m}^{N-1} \phi_k(Y(t_k^-), c_k) \\
& + \sum_{k=m}^{N-1} \int_{t_k}^{t_{k+1}} L_k(Y(s), u(s), s)\mathrm{d}s + \phi_N(Y(t_N^-)) \Big\}.
\end{aligned}
$$

We refer to the fact that

$$
\inf_{x,y} f(x, y) = \inf_x \{\inf_y f(x, y)\} \tag{2.125}
$$

where $f(x, y)$ is smooth and minimum of f exists. Then, we have that

$$
\begin{aligned}
V_{m-1}(x, t_m^-) &= \inf_{c_m} \Bigg\{ \inf_{\substack{u(s),\ t_m < s < t_{N-1} \\ \{c_k\}_{k=m+1}^{N-1}}} E\Big\{ \sum_{k=m}^{N-1} \phi_k(Y(t_k^-), c_k) \\
&\quad + \sum_{k=m}^{N-1} \int_{t_k}^{t_{k+1}} L_k(Y(s), u(s), s)\mathrm{d}s + \phi_N(Y(t_N^-)) \Big\} \Bigg\} \\
&= \inf_{c_m} \Bigg\{ \phi_m(x, c_m) + \inf_{\substack{u(s),\ t_m < s < t_{N-1} \\ \{c_k\}_{k=m+1}^{N-1}}} E\Big\{ \sum_{k=m+1}^{N-1} \phi_k(Y(t_k^-), c_k) \\
&\quad + \sum_{k=m}^{N-1} \int_{t_k}^{t_{k+1}} L_k(Y(s), u(s), s)\mathrm{d}s + \phi_N(Y(t_N^-)) \Big\} \Bigg\} \\
&= \inf_{c_m} \Big\{ \phi_m(x, c_m) + V_m(g_k(x, c_m), t_k^+) \Big\}.
\end{aligned} \tag{2.126}
$$

For a regular optimal control problem, the necessary condition derived from Pontryagin's Maximum Principle is equivalent to HJB equation. In fact for the impulsive optimal control problem, the two approaches are also equivalent.

Proposition 2.2.4. *If the value function $V_m(x,t)$, $m = 0, \ldots, N-1$, is sufficiently smooth, then we have that*

$$p_m(t) = \Big(\frac{\partial V_m}{\partial x}(\hat{x}(t), t)\Big)^T, \tag{2.127}$$

$$r_m^{lk} = \sum_i \frac{\partial^2 V_m}{\partial x_i \partial x_l}(\hat{x}(t), t)\sigma_m^{ik}(\hat{x}(t), t), \tag{2.128}$$

where $\{p_m(t), r_m(t)\}$ are the processes adapted to $\mathscr{F}^t$, which solve the adjoint equation defined in (2.97).

Proof. In this proof, we would use comma to indicate spatial derivatives. Ito's formula gives

$$\mathrm{d}p_m^j = (V_{m,j})_t \mathrm{d}t + \sum_i V_{m,ji} f_m^i \mathrm{d}t + \sum_{i,l} V_{m,ji}\sigma_m^{il}\mathrm{d}W_l + \frac{1}{2}\sum_{i,l,k} V_{m,jik}\sigma_m^{il}\sigma_m^{kl}\mathrm{d}t. \tag{2.129}$$

Taking derivative w.r.t. x_j in the equation (2.124), we have

$$\begin{aligned}(V_{m,j})_t + \sum_i V_{m,ji} f_m^i + \frac{\partial V_m}{\partial x}\Big(f_{m,j} + \frac{\partial f_m}{\partial u}\hat{u}_{,j}\Big) \\ + \frac{1}{2}\sum_{ilk}\Big(V_{m,ijk}\sigma_m^{il}\sigma_m^{kl} + V_{m,ik}(\sigma_m^{il}\sigma_m^{kl})_{,j} + V_{m,ik}\frac{\partial(\sigma_m^{il}\sigma_m^{kl})}{\partial u}\hat{u}_{,j}\Big) + L_{m,j} + \frac{\partial L_m}{\partial u}\hat{u}_{,j} = 0\end{aligned} \tag{2.130}$$

Recalling (2.124) we know that $\hat{u}$ satisfies

$$\frac{\partial V_m}{\partial x}\frac{\partial f_m}{\partial u} + \frac{1}{2}\sum_{i,j,k}\frac{\partial^2 V_m}{\partial x_i x_k}\frac{\sigma_m^{ij}\sigma_m^{kj}}{\partial u} + \frac{\partial L_m}{\partial u} = 0, \tag{2.131}$$

which reduces (2.130) to

$$(V_{m,j})_t + \sum_i V_{m,ji} f_m^i + \frac{\partial V_m}{\partial x} f_{m,j} + \frac{1}{2} \sum_{ilk} \Big(V_{m,ijk} \sigma_m^{il} \sigma_m^{kl} + V_{m,ik} (\sigma_m^{il} \sigma_m^{kl})_{,j} \Big) + L_{m,j} = 0 \quad (2.132)$$

Then, substitute (2.132) in (2.129) and we have

$$-\mathrm{d}p_m^j = \Big(p^T f_{m,j} + \frac{1}{2} \sum_{ilk} V_{m,ik} (\sigma_m^{il} \sigma_m^{kl})_{,j} + L_{m,j} \Big) \mathrm{d}t - \sum_{i,l} V_{m,ji} \sigma_m^{il} \mathrm{d}W_l. \quad (2.133)$$

Define $r_m^{jl} = \sum_i V_{m,ji} \sigma_m^{il}$, and the equation (3.114) could be rewritten as

$$-\mathrm{d}p_m = \Big(\frac{\partial L_m}{\partial x}^T + \frac{\partial f_m}{\partial x} p_m + \sum_k \frac{\sigma_m^k}{\partial x} r_m^k \Big) \mathrm{d}t - r_m \mathrm{d}W, \quad (2.134)$$

which is exactly the same equation as (2.97). Now we are going to check the variable p_m defined by $\Big(\frac{\partial V_m}{\partial x} (\hat{x}(t), t) \Big)^T$ satisfies the jump condition (2.98).

Look at the equation (2.126), the optimal impulsive control $\hat{c}_m$ satisfies

$$\begin{aligned}
&\frac{\partial \phi_m}{\partial c}(x, \hat{c}_m(x)) + \frac{\partial}{\partial x} V_m(g_m(x, \hat{c}_m), t_m^+) \frac{\partial g_m}{\partial c}(x, \hat{c}_m) = 0 \qquad \Longrightarrow \\
&\frac{\partial \phi_m}{\partial c}(x, \hat{c}_m(x)) + p_m^T(t_m^+) \frac{\partial g_m}{\partial c}(x, \hat{c}_m) = 0 \qquad (2.135)
\end{aligned}$$

Taking partial derivative w.r.t. x_j in equation (2.126), we get

$$p_{m-1}^j(t_m^-) = \frac{\partial \phi_m}{\partial x_j}(x, \hat{c}_m) + \frac{\partial \phi_m}{\partial c} \frac{\partial \hat{c}}{\partial x_j} + \frac{\partial V_m}{\partial x} \Big(\frac{\partial g_m}{\partial x_j} + \frac{\partial g_m}{\partial c} \frac{\partial \hat{c}}{\partial x_j} \Big), \quad (2.136)$$

which shows the variable defined in (2.127) satisfies the condition (2.98).

□

2.3 Numerical Solution to Stochastic SIR Model

Here we consider the following SIR model, where the system is affected by stochastic perturbation:

$$\begin{cases} \mathrm{d}S_i = (\Lambda_i - d_i S_i - \sum\limits_{i,j} (\bar{\beta}_{ij} - u_{ij}) S_i I_j)\mathrm{d}t + \mu_i S_i \mathrm{d}B_i, \\ \mathrm{d}I_i = \sum\limits_{i,j} ((\bar{\beta}_{ij} - u_{ij}) I_j S_i - (d_i + \gamma_i) I_i)\mathrm{d}t + \rho_i I_i \mathrm{d}W, \\ S_i(t_k^+) = S_i(t_k^-)(1 - c_{ik}), \\ I_i(t_k^+) = I_i(t_k^-), \end{cases} \tag{2.137}$$

with the cost function

$$J = E\Big\{ \frac{a}{2} \sum_{k=0}^{N-1} \sum_{i=1}^{n} \Big(c_{ik} S_i(t_k^-) \Big)^2 + \sum_{k=0}^{N-1} \sum_{i=1}^{n} \int_{t_k}^{t_{k+1}} \Big(\frac{b}{2} I_i^2 + \frac{1}{2} \sum_{j=1}^{n} u_{ij}^2 \Big) \mathrm{d}t + \frac{\alpha}{2} \sum_{i=1}^{n} I_i^2(t_N^-) \Big\}. \tag{2.138}$$

For the numerical experiment, we set the parameters of the model as follows:

d	β	γ	ρ	μ	a	b	α	N	t_0	t_1	t_2	t_3
$\begin{pmatrix} 0.4 \\ 0.4 \end{pmatrix}$	$\begin{pmatrix} 1.6 & 0.2 \\ 0.1 & 1.4 \end{pmatrix}$	$\begin{pmatrix} 0.6 \\ 0.7 \end{pmatrix}$	$\begin{pmatrix} 0.04 \\ 0.04 \end{pmatrix}$	$\begin{pmatrix} 0.04 \\ 0.04 \end{pmatrix}$	1	1	.5	3	0	0.5	1	1.5

and initial conditions:

$S(0)$ $(\times 10^5)$	$I(0)$ $(\times 10^5)$
$\begin{pmatrix} 0.7 \\ 0.6 \end{pmatrix}$	$\begin{pmatrix} 0.4 \\ 0.3 \end{pmatrix}$

Here is the table showing the cost values in case of constant controls.

Table 2.1: List of Cost Tested by Varying Controls

u	c	J
0	0	0.3210
0	0.1	0.3087
0	0.2	0.3050
0	0.3	0.3092
0.1β	0	0.3297
0.02β	0	0.3169
0.03β	0	0.3158
0.04β	0	0.3159
0.05β	0	0.3165
0.06β	0	0.3178
0.03β	0.2	0.3024
0.03β	0.22	0.3024
0.03β	0.21	0.3024
0.03β	0.19	0.3023
0.03β	0.18	0.3022
0.03β	0.17	0.3022
$\hat{u}$	$\hat{c}$	0.2937

Figure 2.1: populations of susceptible in different groups under optimal controls

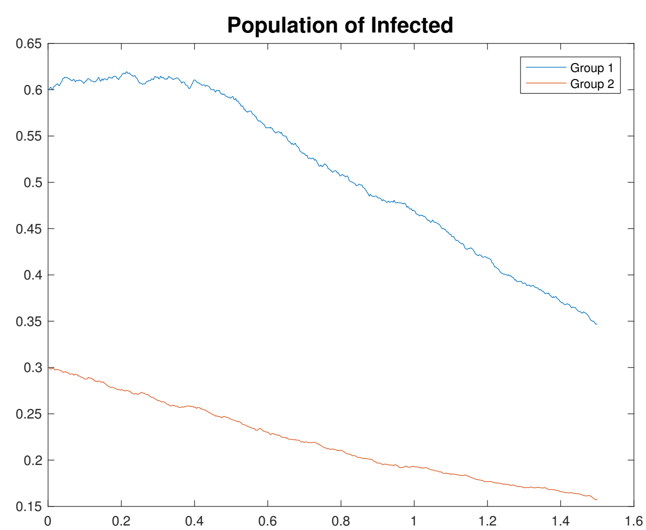

Figure 2.2: populations of infected in different groups under optimal controls

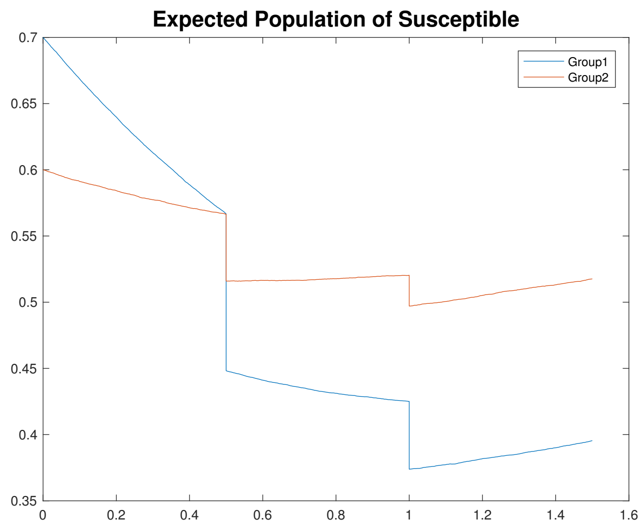

Figure 2.3: expected populations of susceptible in different groups under zero controls

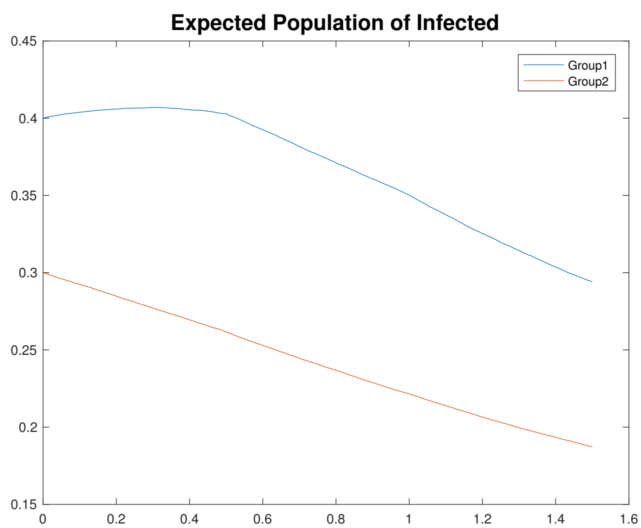

Figure 2.4: expected populations of infected in different groups under zero controls

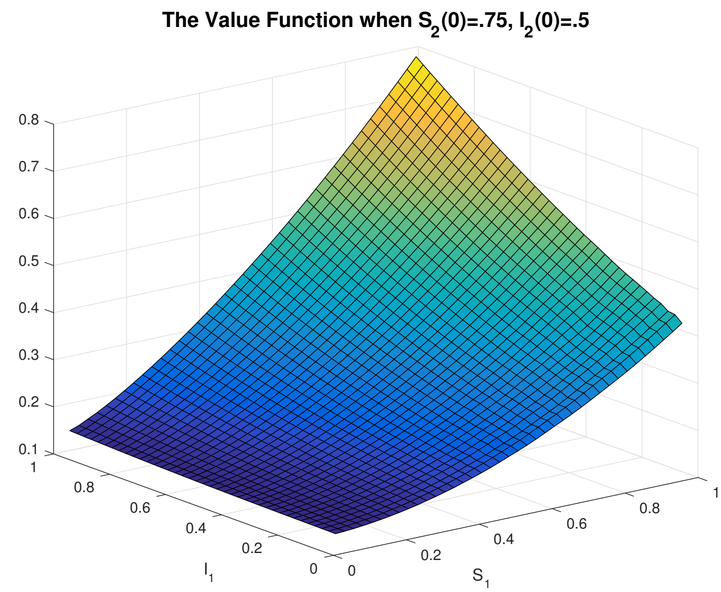

Figure 2.5: Value Function $V(S_1, S_2 = .75, I_1, I_2 = .5)$

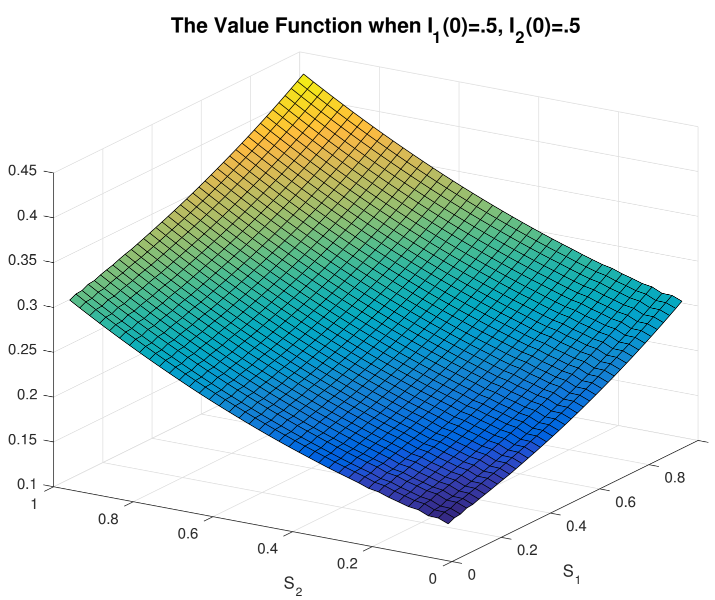

Figure 2.6: Value Function $V(S_1, S_2, I_1 = .5, I_2 = .5)$

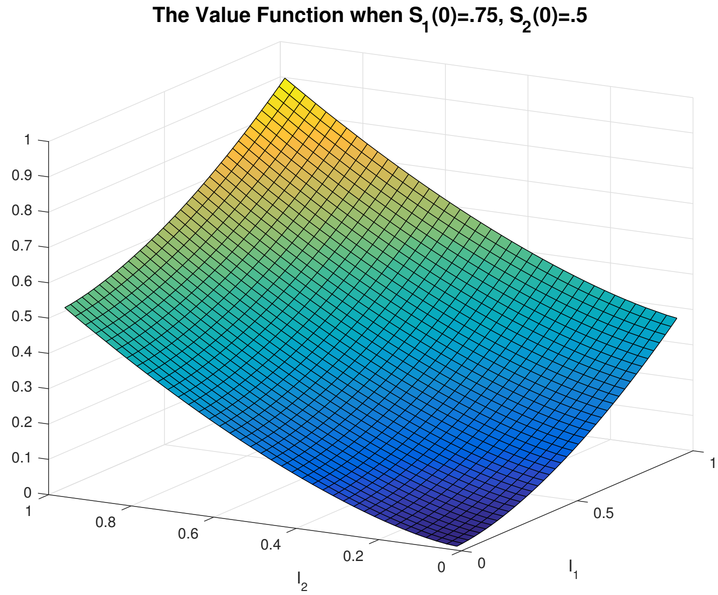

Figure 2.7: Value Function $V(S_1 = .75, S_2 = .5, I_1, I_2)$

2.4 Proof of A Lemma

This section serves as a supplement to the discussion of stability. We will give a proof to some properties of the process $X(t)$, which is defined as the solution to the following system:

$$\begin{cases} \mathrm{d}X = \Big(\Lambda - aX\Big)\mathrm{d}t + \rho X \mathrm{d}W, \\ X(0) = X_0 \end{cases} \tag{2.139}$$

Lemma 2.4.1.

1.

$$\lim_{T\to\infty} E\Big(X(T) - \frac{\Lambda}{a}\Big)^2 = \frac{\rho^2}{a^2(2a-\rho^2)} \tag{2.140}$$

2. $X(t)$ has a stationary distribution $\nu(x)$.

3. The distribution of $X(t)$ converges to $\nu(x)$.

4. The ergodic property holds for $X(t)$, i.e. for any measurable function $f(x)$, we have

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f(X(t))\mathrm{d}t = \int_{\mathbb{R}} f(x)\nu(x)\mathrm{d}x, \ \ a.s.. \tag{2.141}$$

Proof.

1. Recall the fact that

$$X(t) = e^{-(a+\frac{\rho^2}{2})t+\rho B_t}X_0 + \int_0^t e^{(a+\frac{\rho^2}{2})(s-t)+\rho(B_t-B_s)}\Lambda\mathrm{d}s, \tag{2.142}$$

and that the moment generating function for a normal random variable $X \sim \mathcal{N}(\mu,\sigma)$ is give as $M_X(u) = E[e^{Xu}] = e^{\mu u+\frac{\sigma^2 u^2}{2}}$. Then we compute

$$\begin{aligned}
E\Big(X(t)-\frac{\Lambda}{a}\Big)^2 &= E\Big[X_0^2 e^{(-2a-\rho)t}e^{2\rho B_t}\Big] + E\Big[2X_0\Lambda\int_0^t e^{(a+\frac{\rho^2}{2})(s-2t)}e^{2\rho(B_t-B_s)+\rho B_s}\mathrm{d}s\Big]\\
&\quad -E\Big[2\frac{\Lambda}{a}X_0 e^{(-a-\frac{\rho^2}{2})t}e^{\rho B_t}\Big] + E\Big[\Lambda\int_0^t e^{(a+\frac{\rho^2}{2})(s-t)+\rho(B_t-B_s)}\mathrm{d}s\Big]^2\\
&\quad -E\Big[2\frac{\Lambda^2}{a}\int_0^t e^{(a+\frac{\rho^2}{2})(s-t)+\rho(B_t-B_s)}\mathrm{d}s\Big] + \frac{\Lambda^2}{a^2}\\
&= X_0^2 e^{(-2a+\rho^2)t} + 2X_0\Lambda\int_0^t e^{a(s-2t)}e^{\rho^2(-\frac{3}{2}s+t)}\mathrm{d}s - \frac{2\Lambda}{a}X_0 e^{-at}\\
&\quad +E\Big[\Lambda\int_0^t e^{(a+\frac{\rho^2}{2})(s-t)+\rho(B_t-B_s)}\mathrm{d}s\Big]^2 - \frac{2\Lambda^2}{a}\int_0^t e^{a(s-t)}\mathrm{d}s + \frac{\Lambda^2}{a^2}
\end{aligned} \tag{2.143}$$

With the assumption that $-2a+\rho^2<0$, the first and the third term converge to zero as $t\to\infty$. Now we compute

$$2X_0\Lambda\int_0^t e^{a(s-2t)}e^{\rho^2(-\frac{3}{2}s+t)}\mathrm{d}s = 2X_0\Lambda\frac{1}{a-\frac{3}{2}\rho^2}\Big(e^{(-a-\frac{\rho^2}{2})t} - e^{(-2a+\rho^2)t}\Big) \to 0 \tag{2.144}$$

as $t\to\infty$, and

$$\frac{2\Lambda^2}{a}\int_0^t e^{a(s-t)}\mathrm{d}s = \frac{2\Lambda^2}{a^2}(1-e^{-at}) \to \frac{2\Lambda^2}{a^2} \tag{2.145}$$

as $t \to \infty$. Then we compute

$$E\Big[\Lambda\int_0^t e^{(a+\frac{\rho^2}{2})(s-t)+\rho(B_t-B_s)}\mathrm{d}s\Big]^2$$
$$= \Lambda^2 E\Big[\int_0^t\int_0^t e^{(a+\frac{\rho^2}{2})(s-t)+\rho(B_t-B_s)}e^{(a+\frac{\rho^2}{2})(u-t)+\rho(B_t-B_u)}\mathrm{d}u\mathrm{d}s\Big]. \tag{2.146}$$

We split the above integral in two parts. We have the first part as

$$\begin{aligned}
&E\int_0^t\int_0^s e^{(a+\frac{\rho^2}{2})(s-t)+\rho(B_t-B_s)}e^{(a+\frac{\rho^2}{2})(u-t)+\rho(B_t-B_u)}\mathrm{d}u\mathrm{d}s\\
&= E\int_0^t\int_0^s e^{(a+\frac{\rho^2}{2})(s+u-2t)}e^{2\rho(B_t-B_s)}e^{\rho(B_s-B_u)}\mathrm{d}u\mathrm{d}s\\
&= \int_0^t\int_0^s e^{(a+\frac{\rho^2}{2})(s+u-2t)}e^{2\rho^2(t-s)}e^{\frac{\rho^2}{2}(s-u)}\mathrm{d}u\mathrm{d}s\\
&= \frac{1}{a}\Big(\frac{1}{2a-\rho^2}(1-e^{(-2a+\rho^2)t}) - \frac{1}{a-\rho^2}(e^{-at}-e^{(-2a+\rho^2)t})\Big)\\
&\to \frac{1}{a}\frac{1}{2a-\rho^2},
\end{aligned} \tag{2.147}$$

and the second part as

$$\begin{aligned}
&E\int_0^t\int_s^t e^{(a+\frac{\rho^2}{2})(s-t)+\rho(B_t-B_s)}e^{(a+\frac{\rho^2}{2})(u-t)+\rho(B_t-B_u)}\mathrm{d}u\mathrm{d}s\\
&= E\int_0^t\int_s^t e^{(a+\frac{\rho^2}{2})(s+u-2t)}e^{2\rho(B_t-B_u)}e^{\rho(B_u-B_s)}\mathrm{d}u\mathrm{d}s\\
&= \int_0^t\int_s^t e^{(a+\frac{\rho^2}{2})(s+u-2t)}e^{2\rho^2(t-u)}e^{\frac{\rho^2}{2}(u-s)}\mathrm{d}u\mathrm{d}s\\
&= \frac{1}{a-\rho^2}\Big(\frac{1}{a}(1-e^{-at}) - \frac{1}{2a-\rho^2}(1-e^{(-2a-\rho^2)t})\Big)\\
&\to \frac{1}{a-\rho^2}\frac{1}{a} - \frac{1}{a-\rho^2}\frac{1}{2a-\rho^2}.
\end{aligned} \tag{2.148}$$

We substitute (2.144, 2.145, 2.146, 2.147, 2.148) to (2.143), then we have

$$\begin{aligned}
\lim_{t\to\infty} E\Big(X(t)-\frac{\Lambda}{a}\Big)^2 &= \frac{\Lambda^2}{a^2} - \frac{2\Lambda^2}{a^2} + \frac{\Lambda^2}{a(2a-\rho^2)} + \frac{\Lambda^2}{(a-\rho^2)a} - \frac{\Lambda^2}{(a-\rho^2)(2a-\rho^2)}\\
&= \frac{\Lambda^2\rho^2}{a^2(2a-\rho^2)}
\end{aligned} \tag{2.149}$$

2. To prove the existence of stationary distribution, we will introduce the idea of Fokker Planck Equation, which is a main tool of the next chapter. Let us consider X_t^ξ which denotes the solution to the system

$$\begin{cases} \mathrm{d}X = b(X,t)\mathrm{d}t + \sigma(X,t)\mathrm{d}B_t, \\ X(0) = \xi, \end{cases} \tag{2.150}$$

and we denote by $m(x,t)$ the distribution of X_t^ξ. By Ito's formula, we have

$$E[f(X_t^\xi) - f(X_0^\xi)] = E\Big[\int_0^t \frac{\partial f}{\partial x} b(X_s^\xi, s) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma^2(X_s^\xi, s)\mathrm{d}s\Big] \tag{2.151}$$

for any measurable function $f(\cdot)$. By definition of $m(x,t)$, we could write the equation (2.151) into

$$\int_{\mathbb{R}} f(x)m(x,t)\mathrm{d}x - \int_{\mathbb{R}} f(x)m(x,0)\mathrm{d}x = \int_0^t \int_{\mathbb{R}} \Big(\frac{\partial f}{\partial x} b(x,s) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma^2(x,s)\Big)m(x,s)\mathrm{d}x\mathrm{d}s. \tag{2.152}$$

Taking the derivative of equation (2.152) with respect to t, we have

$$\int_{\mathbb{R}} f(x)\frac{\partial m(x,t)}{\partial t}\mathrm{d}x = \int_{\mathbb{R}} \Big(\frac{\partial f}{\partial x} b(x,t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma^2(x,t)\Big)m(x,t)\mathrm{d}x \tag{2.153}$$

Integration by part on the right hand side of (2.153) gives

$$\int_{\mathbb{R}} f(x)\frac{\partial m(x,t)}{\partial t}\mathrm{d}x = \int_{\mathbb{R}} -f(x)\partial_x(b(x,t)m(x,t)) + \frac{1}{2}f(x)\partial_{xx}(\sigma^2(x,t)m(x,t))\mathrm{d}x \tag{2.154}$$

Since $f(\cdot)$ could be arbitrary measurable function, then we have

$$\frac{\partial m(x,t)}{\partial t} = -\partial_x(b(x,t)m(x,t)) + \frac{1}{2}\partial_{xx}(\sigma^2(x,t)m(x,t)) \tag{2.155}$$

The above equation (2.155) is the Fokker Planck Equation for the system (2.150) and the initial condition of $m(x,t)$ is determined by the distribution of ξ. We assume the system (2.150) is homogeneous, i.e. $\mathrm{d}X = b(X)\mathrm{d}t + \sigma(X)\mathrm{d}B$, and we assume that there exists a function $\nu(x)$ satisfying $\int_{\mathbb{R}} \nu(x)\mathrm{d}x = 1$, and

$$-\partial_x(b(x)\nu(x)) + \frac{1}{2}\partial_{xx}(\sigma^2(x)\nu(x)) = 0. \tag{2.156}$$

We call $\nu(x)$ the stationary distribution of the system (2.150). Assuming $\sigma(x) \neq 0$ for $x \in G \subset \mathbb{R}$, we can integrate the above differential equation and get

$$\nu(x) = C\sigma^{-2}(x) \exp\Big(\int_\theta^x \frac{2b(z)}{\sigma^2(z)} \mathrm{d}z\Big), \tag{2.157}$$

for $[\theta, x] \subset G$. Assuming $\sigma(x) \neq 0$ for $x \in \mathbb{R}$, and

$$\int_{-\infty}^{\infty} \sigma^{-2}(x) \exp\Big(\int_\theta^x \frac{2b(z)}{\sigma^2(z)} \mathrm{d}z\Big) \mathrm{d}x < \infty, \tag{2.158}$$

then $\nu(x)$ defined in (2.157) is the stationary distribution of the system (2.150), where $C = \Big(\int_{-\infty}^{\infty} \sigma^{-2}(x) \exp\Big(\int_\theta^x \frac{2b(z)}{\sigma^2(z)} \mathrm{d}z\Big) \mathrm{d}x\Big)^{-1}$.

Since it is proved in Section 2.1 that the process defined in (2.139) is positive almost surely, it will be valid to define $Y(t) = \ln X(t)$. The process $Y(t)$ will satisfy

$$\mathrm{d}Y = (\Lambda e^{-Y} - a - \frac{1}{2}\sigma^2)\mathrm{d}t + \sigma \mathrm{d}B. \tag{2.159}$$

Let $b(x) = (\Lambda e^{-x} - a - \frac{1}{2}\sigma^2)$ and $\sigma(x) = \sigma$, it is not difficult to check that the integral in (2.158) is finite. Therefore the process $Y(t)$ has a stationary distribution, so does $X(t)$.

3. If the process $X(t)$ starts with the stationary distribution, i.e. $m(x, 0) = \nu(x)$, then we have $m(x, t) \equiv \nu(x)$ for any $t > 0$, and the equation (2.151) will lead to

$$E[f(X_t^\xi)] = E[f(X_0^\xi)]. \tag{2.160}$$

Taking $f(x) = \mathbb{1}_A(x)$ to be the indicator function of any measurable set $A \subset \mathbb{R}$, then we have

$$\int_{\mathbb{R}} P(X_t^x \in A)\nu(x)\mathrm{d}x = P(X_t^\xi \in A) = P(\xi \in A) = P^0(A), \quad \forall t > 0, \tag{2.161}$$

where $P^0(\mathrm{d}x) = \nu(x)\mathrm{d}x$ is the probability measure induced by the stationary distribution $\nu(x)$. Now we consider X_t^x and X_t^y be two copies of the process defined in (2.139) which starts from x and y. We define $Z(t) = X_t^x - X_t^y$, then $Z(t)$ satisfies

$$\mathrm{d}Z = -aZ\mathrm{d}t + \sigma Z \mathrm{d}B, \tag{2.162}$$

and the solution has the form

$$Z(t) = Ce^{-(a+\frac{1}{2}\sigma^2)t+\sigma B_t}. \tag{2.163}$$

Recall the fact that

$$\limsup_{t\to\infty} \frac{B_t}{\sqrt{2t\ln\ln t}} = 1 \qquad \text{a.s.} \tag{2.164}$$

then we will have

$$\lim_t (X_t^x - X_t^y) = 0 \qquad \text{a.s.} \tag{2.165}$$

Combine (2.165) with the equation (2.161) we could have

$$\begin{aligned}
P^0(A) &= \int_{\mathbb{R}} P(X_t^x \in A)\nu(x)\mathrm{d}x \\
&= \int_{\mathbb{R}} \int_{\Omega} \mathbb{1}_{X_t^x \in A}(\omega)\mathrm{d}P(\omega)\nu(x)\mathrm{d}x \\
&= \lim_{t\to\infty} \int_{\Omega} \int_{\mathbb{R}} \mathbb{1}_{X_t^x \in A}(\omega)\nu(x)\mathrm{d}x\mathrm{d}P(\omega) \\
&= \lim_{t\to\infty} \int_{\Omega} \int_{\mathbb{R}} \mathbb{1}_{X_t^y \in A}(\omega)\nu(x)\mathrm{d}x\mathrm{d}P(\omega) \\
&= \lim_{t\to\infty} \int_{\Omega} \mathbb{1}(X_t^y \in A)(\omega)\mathrm{d}P(\omega) \\
&= \lim_{t\to\infty} P(X_t^y \in A)
\end{aligned}$$

for any $y \in \mathbb{R}$. So the process X_t^y has a limiting distribution $\nu(x)$ regardless of its initial distribution. The stationary distribution should be unique. Otherwise let $\tilde{P}(\mathrm{d}x)$ be a stationary distribution, and let X_t^ξ have initial distribution $P(X_0^\xi \in A) = \tilde{P}(A)$. Since the distribution of X_t^ξ is stationary, we will have $\tilde{P}(A) = P(X_t^\xi \in A)$ for all t. Then $\tilde{P}(A) = \lim_{t\to\infty} P(X_t^\xi \in A) = P^0(A)$.

Now we give another characterization for the stationary distribution. With the assumption that the process is positive recurrent, i.e. $T_{yy} \triangleq \inf\{t > 0 : X_t = y, X_0 = y\} < \infty$ a.s., we will prove that the stationary distribution satisfies $P^0(A) = \frac{m(A)}{E^y T_{yy}}$, where $m(A)$ is the mean time X_t spend in set A before X_t returns y, i.e.

$$m(A) = E^y \int_0^{T_{yy}} \mathbb{1}(X_t \in A)\mathrm{d}t. \tag{2.166}$$

Let $f(\cdot)$ be arbitrary measurable function. Integration of $f(\cdot)$ with respect to measure m gives

$$\begin{aligned}
\int_{z\in\mathbb{R}} f(z)m(\mathrm{d}z) &= \int_{z\in\mathbb{R}} f(z)E^y \int_0^{T_{yy}} \mathbb{1}(X_t \in \mathrm{d}z)\mathrm{d}t \\
&= \int_{z\in\mathbb{R}} f(z)E^y \int_0^{\infty} \mathbb{1}(X_t \in \mathrm{d}z, t < T_{yy})\mathrm{d}t \\
&= E^y \int_0^{\infty} f(X_t)\mathbb{1}(t < T_{yy})\mathrm{d}t \\
&= E^y \int_0^{T_{yy}} f(X_t)\mathrm{d}t. \qquad (2.167)
\end{aligned}$$

Now we check that m is a stationary measure. Define $g(z) \triangleq E^z f(X_s)$, we have

$$\begin{aligned}
&\int_{z\in\mathbb{R}} E^z f(X_s)m(\mathrm{d}z) = \int_{z\in\mathbb{R}} g(z)m(\mathrm{d}z) = E^y \int_0^{T_{yy}} g(X_t)\mathrm{d}t \\
&= E^y \int_0^{T_{yy}} E^{X_t} f(X_s)\mathrm{d}t = \int_0^{\infty} E^y\big[\mathbb{1}(t < T_{yy})E^{X_t} f(X_s)\big]\mathrm{d}t \\
&= \int_0^{\infty} E^y\Big[\mathbb{1}(t < T_{yy})E^y\big[f(X_{t+s})|\mathscr{F}_t\big]\Big]\mathrm{d}t = \int_0^{\infty} E^y E^y\big[\mathbb{1}(t < T_{yy})f(X_{t+s})|\mathscr{F}_t\big]\mathrm{d}t \\
&= \int_0^{\infty} E^y\big[f(X_{t+s})\mathbb{1}(t < T_{yy})\big]\mathrm{d}t = E^y \int_0^{T_{yy}} f(X_{t+s})\mathrm{d}t = E^y \int_s^{T_{yy}+s} f(X_u)\mathrm{d}u \\
&= E^y \int_0^{T_{yy}} f(X_u)\mathrm{d}u + E^y \int_{T_{yy}}^{T_{yy}+s} f(X_u)\mathrm{d}u - E^y \int_0^{s} f(X_u)\mathrm{d}u \\
&= E^y \int_0^{T_{yy}} f(X_u)\mathrm{d}u = \int_{z\in\mathbb{R}} f(z)m(\mathrm{d}z) \qquad (2.168)
\end{aligned}$$

Substituting $f(z) = \mathbb{1}(z \in A)$ in (2.168) we have

$$\int \mathbb{1}(z \in A)m(\mathrm{d}z) = m(A). \qquad (2.169)$$

On the other hand we have

$$\int_{z\in\mathbb{R}} E^z \mathbb{1}(X_s \in A)m(\mathrm{d}z) = \int_{z\in\mathbb{R}} P(X_s^z \in A)m(\mathrm{d}z). \qquad (2.170)$$

Equating (2.169) with (2.170) we conclude that m is a stationary measure. Notice that

$$\int_{z\in\mathbb{R}} m(\mathrm{d}z) = \int_{z\in\mathbb{R}} E^y \int_0^{T_{yy}} \mathbb{1}(X_t \in \mathrm{d}z)\mathrm{d}t = \int_0^{\infty} E^y \mathbb{1}(t < T_{yy})\mathrm{d}t = E^y T_{yy}, \quad (2.171)$$

we define

$$\tilde{P}(\mathrm{d}z) \triangleq \frac{m(\mathrm{d}z)}{E^y T_{yy}}, \quad (2.172)$$

and we will have $\tilde{P}$ is stationary measure of X_t and $\tilde{P}(\mathbb{R}) = 1$. Then we know that $\tilde{P}$ is equivalent to the stationary distribution P^0.

4. To prove the ergodic property of the process $X(t)$ defined in (2.150), it is sufficient to prove the ergodic property of $Y(t) \triangleq \ln X(t)$. We know that $Y(t)$ satisfies

$$\mathrm{d}Y = b(Y)\mathrm{d}t + \sigma \mathrm{d}B, \quad (2.173)$$

where $b(x) = (\Lambda e^{-x} - a - \frac{1}{2}\sigma^2)$. Let us define

$$\varphi(x) = \int_0^x \exp\Big(\int_0^y -\frac{2b(z)}{\sigma^2}\mathrm{d}z\Big)\mathrm{d}y \quad (2.174)$$

It is not difficult to check that $\lim\limits_{x\to\infty} \varphi(x) = \infty$ and $\lim\limits_{x\to-\infty} \varphi(x) = -\infty$. Then let us define $Z(t) = \varphi(Y(t))$, and we have

$$\mathrm{d}Z(t) = \exp\Big(\int_0^{Y(t)} -\frac{2b(z)}{\sigma^2}\mathrm{d}z\Big)\sigma \mathrm{d}B_t,$$

which implies $Z(t)$ is a martingale. Assume $Y(0) = y$ and $a < y < b$. Denote by T_a the stopping time when $Y(t)$ first hits a, and $\tau = T_a \wedge T_b$, we have

$$\varphi(y) = E\varphi(Y_\tau) = \varphi(a)P^y(T_a < T_b) + \varphi(b)(1 - P^y(T_a < T_b)), \quad (2.175)$$

and solving we have

$$P^y(T_a < T_b) = \frac{\varphi(b) - \varphi(y)}{\varphi(b) - \varphi(a)}. \quad (2.176)$$

Recall that $\lim\limits_{x\to\infty} \varphi(x) = \infty$, we have $P^y(T_a < \infty) = 1$. Similarly we have $P^y(T_b < \infty) = 1$. Since y, a, b are arbitrary, we know that the process $Y(t)$ is recurrent and $P(T_{yy} < \infty) = 1$, where T_{yy} is the next time of $Y(t)$ hitting y after leaving y. Let $\tau_0 = 0$, define τ_k recursively that $\tau_k = \inf\{t : t > \tau_{k-1}, Y(t) = y\}$. We denote $\eta_k = \int_{\tau_k}^{\tau_{k+1}} f(Y(t))\mathrm{d}t$ and $\Delta_k = \tau_{k+1} - \tau_k$.

By strong Markov property of $Y(t)$, $\{\eta_k\}_{k>0}$ are i.i.d. and $\{\Delta_k\}_{k>0}$ are i.i.d., and by Strong Law of Large Numbers, we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\eta_k = E\eta_0, \qquad \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\Delta_k = ET_{yy}. \tag{2.177}$$

Recall (2.167, 2.172) we have

$$E\eta_0 = E^y\int_0^{T_{yy}} f(Y(t))\mathrm{d}t = \int_{z\in\mathbb{R}} f(z)\tilde{m}(\mathrm{d}z) = E^yT_{yy}\int_{z\in\mathbb{R}} f(z)\tilde{P}(\mathrm{d}z), \tag{2.178}$$

where $\tilde{P}$ is the unique stationary distribution for the process Y_t. Let $\kappa(T)=\max\{k:\tau_k< T\}$, we claim that

$$\lim_{T\to\infty}\frac{\tau_{\kappa(T)}}{T}=1, \tag{2.179}$$

since

$$1\geq\lim_{T\to\infty}\frac{\tau_{\kappa(T)}}{T}\geq\lim_{T\to\infty}\frac{\tau_{\kappa(T)}}{\tau_{\kappa(T)+1}}=\lim_{T\to\infty}\frac{\sum_{k=0}^{\kappa(T)}\Delta_k}{\kappa(T)}\frac{\kappa(T)+1}{\sum_{k=1}^{\kappa(T)+1}\Delta_k}=1. \tag{2.180}$$

Then we have

$$\frac{1}{T}\int_0^T f(Y(t))\mathrm{d}t=\frac{1}{T}\Big(\sum_{k=0}^{\kappa(T)-1}\int_{\tau_k}^{\tau_{k+1}} f(Y(t))\mathrm{d}t+\int_{\tau_{\kappa(T)}}^{T} f(Y(t))\mathrm{d}t\Big), \tag{2.181}$$

which leads to

$$\begin{aligned}\lim_{T\to\infty}\frac{1}{T}\int_0^T f(Y(t))\mathrm{d}t &= \lim_{T\to\infty}\frac{1}{\tau_{\kappa(T)}}\sum_{k=0}^{\kappa(T)-1}\eta_k\cdot\frac{\tau_{\kappa(T)}}{T}\\ &= \lim_{T\to\infty}\frac{1}{\kappa(T)}\sum_{k=0}^{\kappa(T)-1}\eta_k\cdot\frac{1}{\frac{1}{\kappa(T)}\sum_0^{\kappa(T)-1}\Delta_k}\\ &= \int_{\mathbb{R}} f(z)\tilde{P}(\mathrm{d}z).\end{aligned}$$

$\square$

Chapter 3

Mean Field Control

The concept of mean field comes from particle physics, which deals with a large number of interacting particles. The enormous number of interaction makes it unreasonable to describe the dynamics of each individual particle. So the "mean field" idea is brought up to reduce the large system to a simple model by only considering an averaged influence from a representative agent instead of interactions from every individual.

Mean field game studies the control problem of the representative agent, where the dynamic system can be described as McKean-Vlasov equation([1], [6], [9], [10]). In particular, the linear quadratic mean field control problem was studied by Yong [38], where a feedback form control was explicitly solved and uniqueness of the solution was discussed.

In Section 3.1, we will introduce the Fokker-Planck equation by studying the evolution of the probability distribution of stochastic SIR model. In Sections 3.2 and 3.3, we will study the control problem of a large interacting system. When the number of players $N \to \infty$, the problem turns to mean field game. We will study the multi-banks model given by Carmona [9], and will give justification for his approach. In Section 3.4, an interesting problem of mean field game with a dominating player is discussed. In Section 3.5 we will consider the impulse mean field control problem.

3.1 Master Equation for the SIR Model

In this section we will look at the distribution of the SIR model. In general, the master equation demonstrates the evolution of $P(\mathbf{n}, t)$, which represents the probability of finding the system in state $\mathbf{n}$ at time t:

$$\frac{\partial P(\mathbf{n},t)}{\partial t} = \sum_{\mathbf{n}'\neq\mathbf{n}} T(\mathbf{n}|\mathbf{n}')P(\mathbf{n}',t) - \sum_{\mathbf{n}'\neq\mathbf{n}} T(\mathbf{n}'|\mathbf{n})P(\mathbf{n},t), \tag{3.1}$$

where $T(\mathbf{n}'|\mathbf{n})$ represents the transition rate from state $\mathbf{n}$ to state $\mathbf{n}'$.

Let us consider the single group SIR model. Assuming that the birth rate equals the death rate, so that the population remains as a constant N. Let S, I, R denote the number of people who are susceptible, infected and recovered, then we have $R = N - S - I$. So the state of the population is determined by a two dimensional vector (S, I), which takes value in $\{(n,m) \in \mathbb{N}^2 | n + m \leq N\}$.

Consider the transition probabilities of the following type:

(1) Infection: a susceptible individual getting the disease by contact with infectious person, resulting in an increase of infectious population and a decrease of susceptible population.

$$T(S = n-1, I = m+1 | S = n, I = m) = \beta \frac{n}{N} m.$$

(2) Recovery: an infectious individual is cured from the disease, results in an decrease of infectious population.

$$T(S = n, I = m-1 | S = n, I = m) = \gamma m.$$

(3) Death of an infectious individual: with the assumption that birth rate and death rate are equal, the total population is constant. With another assumption that all newborns are susceptible, then a death of an infectious individual results in a decrease of infectious population and an increase of susceptible population .

$$T(S = n+1, I = m-1 | S = n, I = m) = \mu m.$$

(4) Death of a recovered individual: with the same reasoning as in the last scenario, a death of a recovered individual results in a decrease of recovered population and an increase of susceptible population.

$$T(S = n+1, I = m | S = n, I = m) = \mu R = \mu(N - n - m).$$

Substituting the transition probabilities into the master equation, we have

$$
\begin{aligned}
\frac{\partial P(n,m,t)}{\partial t} &= P(n+1,m-1,t)\beta\frac{m-1}{N}(n+1) + P(n,m+1,t)\gamma(m+1) \\
&+P(n-1,m+1,t)\mu(m+1) + P(n-1,m,t)\mu(N-n-m+1) \\
&-P(n,m,t)(\beta\frac{m}{N}n + \gamma m + \mu m + \mu(N-n-m))
\end{aligned} \tag{3.2}
$$

Now, we let $x = \frac{n}{N}$, $y = \frac{m}{N}$, and define

$$p(x,y,t) = P(n,m,t) = P(Nx,Ny,t),$$

then we can rewrite the master equation of SIR model (3.2) as

$$
\begin{aligned}
\frac{\partial p(x,y,t)}{\partial t} &= p(x+\frac{1}{N}, y-\frac{1}{N}, t)N\beta(y-\frac{1}{N})(x+\frac{1}{N}) + p(x, y+\frac{1}{N}, t)N\gamma(y+\frac{1}{N}) \\
&+ p(x-\frac{1}{N}, y+\frac{1}{N}, t)N\mu(y+\frac{1}{N}) + p(x-\frac{1}{N}, y, t)N\mu(1-x-y+\frac{1}{N}) \\
&- p(x,y,t)N(\beta xy + \gamma y + \mu y + \mu(1-x-y)) \\
&= [p + \frac{1}{N}(p_x - p_y) + \frac{1}{2N^2}(p_{xx} - 2p_{xy} + p_{yy})]N\beta(xy + \frac{1}{N}(y-x) - \frac{1}{N^2}) \\
&+ (p + \frac{1}{N}p_y + \frac{1}{2N^2}p_{yy})N\gamma(y+\frac{1}{N}) + [p + \frac{1}{N} + \frac{1}{2N^2}(p_{xx} - 2p_{xy} + p_{yy})]N\mu(y+\frac{1}{N}) \\
&+ [p - \frac{1}{N} + \frac{1}{2N^2}p_{xx}]N\mu(1-x-y-\frac{1}{N}) - pN(\beta xy + \gamma y + \mu y + \mu(1-x-y)) \\
&= p\beta(y-x) + p\gamma + 2p\mu - p\frac{\beta}{N} + p_x(\beta xy - \mu(1-x) + \frac{\beta}{N}(y-x) - \frac{2\mu}{N}) \\
&+ p_y(-\beta xy + \gamma y + \mu y - \frac{\beta}{N}(y-x) + \frac{\gamma}{N} + \frac{\mu}{N}) + \frac{1}{2N}p_{xx}(\beta xy + \mu y + \mu(1-x-y)) \\
&- \frac{1}{N}p_{xy}(\beta xy + \mu y) + \frac{1}{2N}p_{yy}(\beta xy + \gamma y + \mu y) \\
&= [p(\beta xy - \mu(1-x))]_x + [p(-\beta xy + \gamma y + \mu y)]_y + \frac{1}{2N}[p(\beta xy + \mu(1-x))]_{xx} \\
&- \frac{1}{N}[p(\beta xy + \mu y)]_{xy} + \frac{1}{2N}[p(\beta xy + \gamma y + \mu y)]_{yy},
\end{aligned}
$$

where we have neglected the higher order term of $\frac{1}{N}$. With the notation $\mathbf{x} = [x, y]$, the above equation could be written as

$$\frac{\partial}{\partial t} p(\mathbf{x}, t) = -\mathrm{div}[f(\mathbf{x})p(\mathbf{x}, t)] + \frac{1}{2}\frac{\partial}{\partial \mathbf{x}}\frac{\partial}{\partial \mathbf{x}}[A(\mathbf{x})p(\mathbf{x}, t)], \tag{3.3}$$

where

$$f(\mathbf{x}) = \begin{bmatrix} -\beta xy + \mu(1-x) \\ \\ \beta xy - \gamma y - \mu y \end{bmatrix}, \tag{3.4}$$

$$A(\mathbf{x}) = \tfrac{1}{N}\begin{bmatrix} \beta xy + \mu(1-x) & -\beta xy - \mu y \\ \\ -\beta xy - \mu y & \beta xy + \gamma y + \mu y \end{bmatrix}. \tag{3.5}$$

The equation (3.3) is in the form of the Fokker Planck equation, where $p(\mathbf{x}, t)$ characterizes the density of the stochastic process

$$\mathrm{d}X = f(X)\mathrm{d}t + \sigma(X)\mathrm{d}W. \tag{3.6}$$

where $\sigma(\mathbf{x})$ satisfies $A(\mathbf{x}) = \sigma(\mathbf{x})\sigma(\mathbf{x})^T$.

Remark 3.1.1. The Fokker Planck equation gives information on the evolution of the probability distribution of a stochastic process based on the stochastic differential equation which describes the dynamics of the stochastic process. The Fokker Planck equation is widely used in the mean field problem.

3.2 Stochastic Interacting System

In this section we study the control problem for an interacting system, where the evolution of system satisfies a measure valued stochastic differential equation.

3.2.1 Terms and Notations

Let $\mathcal{L}(\mathbb{R})$ denote the set of all integrable functions defined in $\mathbb{R}$. We suppose that

$$f(t,x,m,u) : [0,T] \times \mathbb{R} \times \mathcal{L}(\mathbb{R}) \times \mathbb{R}^m \mapsto \mathbb{R}$$
$$\sigma(t,x,m,u) : [0,T] \times \mathbb{R} \times \mathcal{L}(\mathbb{R}) \times \mathbb{R}^m \mapsto \mathbb{R}$$
$$L(t,x,m,u) : [0,T] \times \mathbb{R} \times \mathcal{L}(\mathbb{R}) \times \mathbb{R}^m \mapsto \mathbb{R}$$
$$\phi(x,m) : \mathbb{R} \times \mathcal{L}(\mathbb{R}) \times \mapsto \mathbb{R}$$

are differentiable with respect to all arguments, where the argument m represents the measure term. In case of differentiability with respect to m, We use the notation $\dfrac{\partial f}{\partial m}(t,x,m,u,\xi)$ and it is the unique function $[0,T] \times \mathbb{R} \times \mathcal{L}(\mathbb{R}) \times \mathbb{R}^m \times \mathbb{R} \mapsto \mathbb{R}$, such that

$$\frac{\mathrm{d}}{\mathrm{d}\theta} f(t,x,m+\theta\tilde{m},u)|_{\theta=0} = \int_{\mathbb{R}} \frac{\partial f}{\partial m}(t,x,m,u,\xi)\tilde{m}(\xi)\mathrm{d}\xi \tag{3.7}$$

for all $\tilde{m} \in \mathcal{L}(\mathbb{R})$.

In this section we will consider the case when m is the empirical distribution μ_t^N, where

$$\mu_t^N(\cdot) = \frac{1}{N}\sum_{k=1}^{N} \delta_{x_t^k}(\cdot). \tag{3.8}$$

The dynamics of agent i satisfies

$$\mathrm{d}x_t^i = f(t,x_t^i,\mu_t^N,u_t^i)\mathrm{d}t + \sigma(t,x_t^i,\mu_t^N,u_t^i)\mathrm{d}W_t^i, \qquad 1 \le i \le N, \tag{3.9}$$

where u_t^i is the strategy he takes to minimize his cost functional:

$$J^i(u^1,\ldots,u^N) = E\Big\{\int_0^T L(t,x_t^i,\mu_t^N,u_t^i)\mathrm{d}t + \phi(x_T^i,\mu_T^N)\Big\}. \tag{3.10}$$

To ease the discussion, the equation (3.9) depends on the empirical measure only through the mean. Thus we rewrite the dynamics in the form:

$$\mathrm{d}x_t^i = f(t,x_t^i,\bar{x}_t,u_t^i)\mathrm{d}t + \sigma(t,x_t^i,\bar{x}_t,u_t^i)\mathrm{d}W_t^i, \qquad 1 \le i \le N. \tag{3.11}$$

where $\bar{x}_t = \dfrac{1}{N}\displaystyle\sum_{j=1}^N x_t^j$. The game is for each agent to work out a strategy u_t^i to minimize his cost functional

$$J^i(u^1,\ldots,u^N) = E\Big\{\int_0^T L(t,x_t^i,\bar{x}_t,u_t^i)\mathrm{d}t + \phi(x_T^i,\bar{x}_T)\Big\}. \tag{3.12}$$

Keep in mind the fact that the cost J^i depends on all the controls $\{u_t^j\}_{1\le j\le N}$ through the mean $\bar{x}_t$. To find the equilibrium, we assume all players except player i are already taking the optimal strategy, and player i is still in search for his optimal strategy. When $N \to \infty$, the dynamics of all the other players will not be affected by a single player x_t^i. That is to say, $\bar{x}_t$ will remain the same regardless of the behavior of player x_i. So the equilibrium could be worked out by solving the optimal control problem of one single player x_i, while all the other players x_j's keep staying at their optimal paths.

3.2.2 Inter-Banks Lending and Borrowing

We consider the following inter-bank lending and borrowing model. Assume x^i, the reserve assets of bank i, has the following dynamics:

$$\mathrm{d}x_t^i = (a(\bar{x}_t - x_t^i) + u_t^i)\mathrm{d}t + \sigma\mathrm{d}W_t^i, \tag{3.13}$$

where u_t^i represents the rate of bank i lending to (if $u_t^i < 0$) and borrowing from (if $u_t^i > 0$) the central bank. Bank i controls the rate u_t^i in order to minimize

$$J^i(u^1,\dots,u^N) = E\Big(\int_0^T \frac{1}{2}(u_t^i)^2 - qu_t^i(\bar{x}_t - x_t^i) + \frac{\varepsilon}{2}(\bar{x}_t - x_t^i)^2 \mathrm{d}t + \frac{c}{2}(\bar{x}_T - x_T^i)^2\Big). \tag{3.14}$$

Assuming all banks except bank i are taking the optimal strategy. Solving the optimization for bank i via dynamic programming approach, the value function $V^i(t,x)$ will satisfy the HJB equation:

$$\begin{aligned} \frac{\partial V^i(t,x)}{\partial t} &+ \inf_{u\in\mathbb{R}}\Big\{(a(\bar{x}-x^i)+u)\frac{\partial V^i}{\partial x^i} + \frac{1}{2}u^2 - qu(\bar{x}_t - x_t^i)\Big\} + \frac{\varepsilon}{2}(\bar{x}_t - x_t^i)^2 \\ &+ \sum_{j\neq i}(a(\bar{x}-x^j)+\hat{u}^j)\frac{\partial V^i}{\partial x^j} + \frac{\sigma^2}{2}\sum_{j=1}^N \frac{\partial^2 V^i}{\partial x^j \partial x^j} = 0, \end{aligned} \tag{3.15}$$

where we use the notation $\bar{x} = (x^1 + \cdots + x^N)/N$ and with the terminal condition $V^i(T,x) = (c/2)(\bar{x} - x^i)^2$. The infimum in the HJB equation (3.15) can be solved

$$\begin{aligned} &\inf_{u\in\mathbb{R}}\Big\{(a(\bar{x}-x^i)+u)\frac{\partial V^i}{\partial x^i} + \frac{1}{2}u^2 - qu(\bar{x}_t - x_t^i)\Big\} \\ &= a(\bar{x}-x^i)\frac{\partial V^i}{\partial x^i} - \frac{1}{2}\Big(q(\bar{x}-x^i) - \frac{\partial V^i}{\partial x^i}\Big)^2, \end{aligned} \tag{3.16}$$

and the optimal strategy of bank i is

$$\hat{u}^i = q(\bar{x}-x^i) - \frac{\partial V^i}{\partial x^i}. \tag{3.17}$$

Then, the HJB equation could be rewritten as:

$$\frac{\partial V^i}{\partial t} + \sum_k \frac{\partial V^i}{\partial x^k}\Big[(a+q)(\bar{x}-x^k) - \frac{\partial V^k}{\partial x^k}\Big] + \frac{\sigma^2}{2}\sum_k \frac{\partial V^i}{\partial x^k \partial x^k} \tag{3.18}$$
$$+\frac{1}{2}(\varepsilon - q^2)(\bar{x}-x^i) + \frac{1}{2}\Big(\frac{\partial V^i}{\partial x^i}\Big)^2 = 0,$$

Assume the value function $V^i(x,t)$ has the form

$$V^i(x,t) = \frac{1}{2}(\bar{x}-x^i)^2\eta(t) + \mu(t), \tag{3.19}$$

with undetermined functions $\eta(t)$, $\mu(t)$. Then, we have

$$\frac{\partial V^i}{\partial x^k} = \eta(t)(\bar{x}-x^i)(\frac{1}{N} - \delta_{ik}), \tag{3.20}$$
$$\frac{\partial^2 V^i}{\partial x^j \partial x^k} = \eta(t)(\frac{1}{N} - \delta_{ij})(\frac{1}{N} - \delta_{ik}). \tag{3.21}$$

Then, the HJB equation would be simplified as

$$(\bar{x}-x^i)^2\Big(\frac{\dot{\eta}}{2} - \eta(a+q+\frac{N-1}{N}\eta) + \frac{1}{2}(\varepsilon - q^2) + \frac{1}{2}\eta^2(\frac{N-1}{N})^2\Big) + \dot{\mu} + \frac{\sigma^2}{2}\frac{N-1}{N}\eta = 0. \tag{3.22}$$

Notice that the equation (3.22) holds for all $x = (x^1, \dots, x^N) \in \mathbb{R}^N$. Therefore, we have

$$\dot{\eta} = 2\eta(a+q+\frac{N-1}{N}\eta) - (\varepsilon - q^2) - \eta^2(\frac{N-1}{N})^2, \tag{3.23}$$
$$\dot{\mu} = -\frac{\sigma^2}{2}\frac{N-1}{N}\eta, \tag{3.24}$$

with terminal conditions $\eta(T) = c$ and $\mu(T) = 0$. Substituting (3.20) to (3.17), the optimal strategy could be written as

$$u_t^i = \Big(q + \frac{N-1}{N}\eta(t)\Big)(\bar{x} - x^i). \tag{3.25}$$

3.2.3 Multi-Objective Problem

In the previous inter-banks model, Carmona solves the optimal control by minimizing each bank's cost function as if no one pays attention to the cost of the other banks. However, multi-objective problem might not generally have a solution that achieves minimum for all the cost functions. In this subsection, we will give a justification for Carmona's approach. Here we assign weight w_k to the cost functional of player k, therefore the multi-objective optimal control problem could be solved by minimizing one weighted averaged cost functional:

$$J(u^1, \dots, u^N) = \sum_{i=1}^{N} w_i E\Big(\int_0^T \frac{1}{2}(u_t^i)^2 - q u_t^i(\bar{x}_t - x_t^i) + \frac{\varepsilon}{2}(\bar{x}_t - x_t^i)^2 \mathrm{d}t + \frac{c}{2}(\bar{x}_T - x_T^i)^2\Big). \tag{3.26}$$

In this case let $\bar{x}$ denote $\sum_{k=1}^{N} w_k x^k$, where $\sum_{k=1}^{N} w_k = 1$. Then, by the usual dynamic programming approach we will have the HJB equation:

$$\begin{aligned} \frac{\partial V}{\partial t} + \sum_{k=1}^{N} \inf_{u_k} \Big[\frac{\partial V}{\partial x^k}\Big(a(\bar{x} - x^k) + u^k\Big) + \frac{w_k}{2}(u^k)^2 - w_k q u^k(\bar{x} - x^k)\Big] \qquad (3.27) \\ + \frac{\sigma^2}{2}\frac{\partial V}{\partial x^k \partial x^k} + \frac{\varepsilon}{2}(\bar{x} - x^k) = 0, \end{aligned}$$

Solving the infimum of the equation (3.27), the optimal control u^k should have the feedback form:

$$u^k = q(\bar{x} - x^k) - \frac{1}{w_k}\frac{\partial V}{\partial x_k}. \tag{3.28}$$

Then, (3.27) could be rewritten as

$$\frac{\partial V}{\partial t} + \sum_{k=1}^{N} \frac{\partial V}{\partial x^k}\Big((a+q)(\bar{x} - x^k) - \frac{1}{w_k}\frac{\partial V}{\partial x^k}\Big) + \sum_{k=1}^{N} \frac{\sigma^2}{2}\frac{\partial V}{\partial x^k \partial x^k} \tag{3.29}$$
$$+ \sum_{k=1}^{N} \frac{w_k}{2}(\varepsilon - q^2)(\bar{x} - x^k) + \sum_{k=1}^{N} \frac{1}{2w_k}\Big(\frac{\partial V}{\partial x^k}\Big)^2 = 0,$$

In this problem, we assume that the value function has the form $V(x,t) = \frac{\eta(t)}{2}\sum_{k=1}^{N} w_k(\bar{x} - x^k)^2 + \mu(t)$. We compute

$$\frac{\partial V}{\partial x^k} = -\eta(t) w_k(\bar{x} - x^k); \tag{3.30}$$
$$\frac{\partial^V}{\partial x^k \partial x^k} = -\eta w_k(w_k - 1). \tag{3.31}$$

After substituting the expression for $V(x,t)$ to (3.29), we will end up with

$$\sum_{k=1}^{N} w_k(\bar{x} - x^k)\Big[\frac{\dot{\eta}}{2} - (a+q)\eta - \frac{\eta^2}{2} + \frac{1}{2}(\varepsilon - q^2)\Big] + \dot{\mu} + \frac{\sigma^2}{2}\eta\Big(1 - \sum_{k=1}^{N} w_k^2\Big) = 0. \tag{3.32}$$

It implies that the solution to the HJB equation (3.29) is

$$V(x,t) = \frac{\eta(t)}{2}\sum_{k=1}^{N} w_k(\bar{x} - x^k)^2 + \mu(t), \tag{3.33}$$

where $\eta(t)$ and $\mu(t)$ satisfies

$$\dot{\eta} = 2\eta(a+q) - (\varepsilon - q^2) + \eta^2, \tag{3.34}$$

$$\dot{\mu} = -\frac{\sigma^2}{2}\eta\Big(1 - \sum_{k=1}^{N} w_k^2\Big). \tag{3.35}$$

We assume that $w_k = O(\frac{1}{N})$. Then we could see that taking $N \to \infty$ in equations (3.23, 3.24) and (3.34, 3.35), the two systems of Riccati equations has the same limit. In the situation when each bank minimize its own cost functional, the optimal strategy of bank k is given in (3.25) as

$$u^k = \Big(q + \eta(1 - \frac{1}{N})\Big)(\bar{x} - x^k), \tag{3.36}$$

and in the situation when all the banks cooperate to minimize a single total cost functional, the strategy for bank k can be derived from (3.28) and (3.30):

$$u^k = (q + \eta)(\bar{x} - x^k). \tag{3.37}$$

Remark 3.2.1. It shows that when the number N is large, then the strategy for minimizing the total cost functional is the same as that of each player minimizing its cost functional individually. Thus it gives the justification for Carmona's approach. Every individual can just deal with his cost functional with the size of the group being large, and it will turn out a minimized total cost functional as if everyone is cooperating with each other.

3.3 Mean Field Game

Mean field games deal with a system consisting of a large number of interacting players who make their decisions under the mean behavior of the group of agents rather than the behavior of an individual. The idea of mean field was first brought up by physicists, and later, works done by French mathematicians Lasry and Lions laid a basis for this field and was followed by a lot of studies in economics, finance, social dynamics and etc.

3.3.1 Limit of Empirical Distribution

Consider that the dynamics of agent i satisfies

$$\mathrm{d}x_t^i = f(t, x_t^i, \mu_t^N, u_t^i)\mathrm{d}t + \sigma(t, x_t^i, \mu_t^N, u_t^i)\mathrm{d}W_t^i, \qquad 1 \le i \le N. \tag{3.38}$$

Assume that for each player i, the control u_t^i has the feedback form $u_t^i = u(x_t^i, \mu_t^N, t)$. For any test function $f(x) \in \mathcal{C}^\infty(\mathbb{R})$, we have

$$\begin{aligned}
\mathrm{d}\langle \phi(x), \mu_t^N(x) \rangle &= \frac{1}{N}\sum_{k=1}^N \mathrm{d}\phi(x_t^k) \\
&= \frac{1}{N}\sum_{k=1}^N \phi'(x_t^k) f(t, x_t^k, \mu_t^N, u(x_t^k, \mu_t^N, t))\mathrm{d}t \\
&\quad + \frac{1}{2N}\sum_{k=1}^N \phi''(x_t^k)\sigma^2(t, x_t^k, \mu_t^N, u(x_t^k, \mu_t^N, t))\mathrm{d}t \\
&\quad + \frac{1}{N}\sum_{k=1}^N \phi'(x_t^k)\sigma(t, x_t^k, \mu_t^N, u(x_t^k, \mu_t^N, t))\mathrm{d}W_t^k \\
&= \langle \phi'(x) f(t, x, \mu_t^N, u(x, \mu_t^N, t)) + \frac{1}{2}\phi''(x)\sigma(t, x, \mu_t^N, u(x, \mu_t^N, t)), \mu_t^N(x) \rangle \mathrm{d}t \\
&\quad + \frac{1}{N}\sum_{k=1}^N \phi'(x_t^k)\sigma(t, x_t^k, \mu_t^N, u(x_t^k, \mu_t^N, t))\mathrm{d}W_t^k
\end{aligned}$$

Assume that $\|\phi'(x)\| \cdot \|\sigma(t, x, m, u)\| \le C$, we have that

$$\frac{1}{N}\sum \phi'(x_k(t))\sigma(t, x_t^k, \mu_t^N, u(x_t^k, \mu_t^N, t))\mathrm{d}W_t^k \to 0 \quad \text{almost surely, as} \quad N \to \infty. \tag{3.39}$$

The proof of (3.39) follows the same steps we take to prove (2.17). Then we will have the limit of the empirical distribution μ satisfying

$$\mathrm{d}\langle\phi(x),\mu_t(x)\rangle = \langle\phi'(x)f(t,x,\mu_t,u(x,\mu_t,t)) + \frac{1}{2}\phi''(x)\sigma(t,x,\mu_t,u(x,\mu_t,t)),\mu_t(x)\rangle\mathrm{d}t \quad (3.40)$$

By integration by parts, we know from (3.40) that the measure $\mu_t(x)$ is the weak solution to the PDE:

$$\frac{\partial\mu_t(x)}{\partial t} = -\frac{\partial}{\partial x}(f(t,x,\mu_t,u(x,\mu_t,t))\mu_t(x)) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(\sigma^2(t,x,\mu_t,u(x,\mu_t,t))\mu_t(x)). \quad (3.41)$$

From the Fokker Planck Equation, $\mu_t(x)$ is the probability distribution of the process X_t , where X_t satisfies

$$\mathrm{d}X_t = f(t,X_t,\mu_t,u(X_t,\mu_t,t))\mathrm{d}t + \sigma(t,X_t,\mu_t,u(X_t,\mu_t,t))\mathrm{d}W(t). \quad (3.42)$$

In the study of mean field games, the process X_t is called the representative player, who characterizes the average behavior of the community. The mean field game theory studies the control problem of the representative player. For any pair $\{u_t,\mu_t\}$, we let x_t be the solution to the stochastic differential equation

$$\begin{aligned} \mathrm{d}x_t &= f(t,x_t,\mu_t,u_t)\mathrm{d}t + \sum_{j=1}^{d}\sigma^j(t,x_t,\mu_t,u_t)\mathrm{d}W_t^j, \quad (3.43) \\ x_0 &= \xi, \end{aligned}$$

and then we could associate to the pair $\{u_t,\mu_t\}$ a cost defined as

$$J(u,\mu) = E\Big[\int_0^T L(x_t,\mu_t,u_t)\mathrm{d}t + \phi(x_T,\mu_T)\Big]. \quad (3.44)$$

The objective is to find a law of control variable u_t such that μ_t is the probability distribution of x_t, $\forall t\in[0,T]$, and

$$J(u, \mu_t) \le J(v, \mu_t), \qquad \forall v. \tag{3.45}$$

3.3.2 Methods of Variation of Calculus

We assume that $\hat{u}_t$ is the optimal control for the mean field game (3.45) and $\hat{x}_t$ is the state variable of the system corresponding to the control $\hat{u}_t$. Define u_t^θ to be a perturbation of the optimal control $\hat{u}_t$:

$$u_t^\theta \triangleq \hat{u}_t + \theta v_t,$$

where v_t is arbitrary and $0 < \theta \ll 1$. Let

$$x_t^\theta \triangleq \hat{x}_t + \theta \tilde{x}_t$$

denote the state corresponding to u^θ. We have

$$\begin{aligned} \mathrm{d}\tilde{x}_t = &\Big(\partial_x f(t, \hat{x}_t, \mu_t, \hat{u}_t)\tilde{x}_t + \partial_u f(t, \hat{x}_t, \mu_t, \hat{u}_t) v_t\Big)\mathrm{d}t \\ &+ \sum_{j=1}^{d} \Big(\partial_x \sigma^j(t, \hat{x}_t, \mu_t, \hat{u}_t)\tilde{x}_t + \partial_u \sigma^j(t, \hat{x}_t, \mu_t, \hat{u}_t) v_t\Big)\mathrm{d}W_t, && (3.46) \\ \tilde{x}_0 = 0, &&& (3.47) \end{aligned}$$

and

$$\frac{\partial J(u_t^\theta, \mu_t)}{\partial \theta}\Big|_{\theta=0} = E\Big[\int_0^T \partial_x L(\hat{x}_t, \hat{u}_t, \mu_t)\tilde{x}_t + \partial_u L(\hat{x}_t, \hat{u}_t, \mu_t) v_t \mathrm{d}t + \partial_x \phi(\hat{x}_T, \mu_T)\tilde{x}_T\Big]. \tag{3.48}$$

We define the process $\{p_t, r_t^j\}$ to be the unique solution to the following backward stochastic differential equation

$$\begin{cases} \mathrm{d}p_t = \Big\{ - \Big(\dfrac{\partial L}{\partial x}(\hat{x}_t, \hat{u}_t, \mu_t)\Big)^T - \Big(\dfrac{\partial f}{\partial x}(t, \hat{x}_t, \hat{u}_t, \mu_t)\Big)^T p_t \\ \qquad\qquad - \displaystyle\sum_{j=1}^{d} \Big(\dfrac{\partial \sigma^j}{\partial x}(t, \hat{x}_t, \hat{u}_t, \mu_t)\Big)^T r_t^j \Big\}\mathrm{d}t + \sum_j r_t^j \mathrm{d}W_j, \quad t \in (0,T), \\ p_T = \dfrac{\partial \phi}{\partial x}^T (\hat{x}_T). \end{cases} \tag{3.49}$$

By Ito's formula we have

$$\partial_x \phi(\hat{x}_T, \mu_T)\tilde{x}_T = E\Big[\int_0^T p_t^T \partial_u f(t, \hat{x}_t, \hat{u}_t, \mu_t) v_t + \sum_j (r_t^j)^T \partial_u \sigma^j(t, \hat{x}_t, \hat{u}_t, \mu_t) v_t - \partial_x L(\hat{x}_t, \hat{u}_t, \mu_t)\tilde{x}_t \mathrm{d}t\Big]. \tag{3.50}$$

Therefore, we have

$$\begin{aligned} \frac{\partial J(u_t^\theta, \mu_t)}{\partial \theta}\Big|_{\theta=0} &= E\Big[\int_0^T \partial_u L(\hat{x}_t, \hat{u}_t, \mu_t) v_t + p_t^T \partial_u f(t, \hat{x}_t, \hat{u}_t, \mu_t) v_t + \sum_j (r_t^j)^T \partial_u \sigma^j(t, \hat{x}_t, \hat{u}_t, \mu_t) v_t \mathrm{d}t\Big] \\ &= E\Big[\int_0^T \Big(\partial_u L(\hat{x}_t, \hat{u}_t, \mu_t) + p_t^T \partial_u f(t, \hat{x}_t, \hat{u}_t, \mu_t) + \sum_j (r_t^j)^T \partial_u \sigma^j(t, \hat{x}_t, \hat{u}_t, \mu_t)\Big) v_t \mathrm{d}t\Big] \end{aligned} \tag{3.51}$$

Theorem 3.3.1. *The optimal control $\hat{u}_t$ of the mean field game satisfies*

$$\partial_u L(\hat{x}_t, \hat{u}_t, \mu_t) + p_t^T \partial_u f(t, \hat{x}_t, \hat{u}_t, \mu_t) + \sum_j (r_t^j)^T \partial_u \sigma^j(t, \hat{x}_t, \hat{u}_t, \mu_t) = 0 \tag{3.52}$$

where $\{p_t, r_t^j\}$ is the pair of the adjoint processes solves (3.49), and the measure μ_t satisfies the Fokker-Planck equation:

$$\frac{\partial \mu_t(x)}{\partial t} = -div(f(t, \hat{x}_t, \hat{u}_t, \mu_t)\mu_t) + \frac{1}{2}\sum_{j=1}^{d}\sum_{k,l=1}^{n} \frac{\partial^2}{\partial x^k x^l}\Big((\sigma^{kj}\sigma^{lj})(t, \hat{x}_t, \hat{u}_t, \mu_t)\mu_t\Big). \tag{3.53}$$

3.4 Solution to Multi-Objective Problem and the Representative Player

In the last section, we construct the representative player by the empirical distribution of a large interacting system. Intuitively, the evolution of representative player should represent averaged evolution of the community. In this section, we will revisit the inter-bank lending and borrowing model and solve the problem of the representative bank. Recall that the process x^i, which stands for the reserve assets of bank i, has the following dynamics:

$$\mathrm{d}x_t^i = (a(\bar{x}_t - x_t^i) + u_t^i)\mathrm{d}t + \sigma \mathrm{d}W_t^i, \tag{3.54}$$

Bank i controls u_t^i, which stands for the rate of lending to (if $u_t^i < 0$) and borrowing from (if $u_t^i > 0$) the central bank, in order to minimize

$$J^i(u^1, \dots, u^N) = E\Big(\int_0^T \frac{1}{2}(u_t^i)^2 - qu_t^i(\bar{x}_t - x_t^i) + \frac{\varepsilon}{2}(\bar{x}_t - x_t^i)^2\mathrm{d}t + \frac{c}{2}(\bar{x}_T - x_T^i)^2\Big). \tag{3.55}$$

From the discussion of empirical distribution and representative player, we know that the reserve assets x_t of the representative bank of the system (3.54) should satisfy

$$\mathrm{d}x_t = (a(m_t - x_t) + u_t)\mathrm{d}t + \sigma \mathrm{d}W_t, \tag{3.56}$$

where $m_t = \int \xi \mu_t(\xi)\mathrm{d}\xi$, and $\mu_t(\cdot)$ is the probability distribution of the process x_t. By Theorem 3.3.1 in the last section, the optimal control u_t could be solved as:

$$u_t = q(m_t - x_t) - p \tag{3.57}$$

where p is the adjoint variable satisfying

$$
\begin{aligned}
-\mathrm{d}p &= q^2(m_t - x_t) - (q+a)p + \varepsilon(x_t - m_t) + r_t \mathrm{d}W_t, \qquad (3.58)\\
p(T) &= c(x_T - m_T).
\end{aligned}
$$

We assume that the adjoint variable has the form

$$
p(t) = \tilde{\eta}(t)(x_t - m_t). \qquad (3.59)
$$

Using Ito's formula to the equation (3.59), we will have

$$
\mathrm{d}p = \Big(\dot{\tilde{\eta}}(t)(x_t - m_t) + \tilde{\eta}((a+q)(m_t - x_t) - \tilde{\eta}(x_t - m_t) - \dot{m}_t)\Big)\mathrm{d}t + \tilde{\eta}(t)\sigma \mathrm{d}W_t. \qquad (3.60)
$$

In equation (3.40) from last section, taking $\phi(x) = x$, we will have

$$
\dot{m}_t = \langle (a+q)(m_t - x) - \tilde{\eta}(t)(x - m_t), \mu_t(x) \rangle = 0. \qquad (3.61)
$$

From (3.58,3.60,3.61), we could determine $\tilde{\eta}(t)$, which solves

$$
\begin{aligned}
\dot{\tilde{\eta}}(t) &= \tilde{\eta}(t)^2 + 2(a+q)\tilde{\eta}(t) + (q^2 - \varepsilon) \qquad (3.62)\\
\tilde{\eta}(T) &= c.
\end{aligned}
$$

The optimal strategy for the representative bank could be written as

$$
\hat{u}_t = (q + \tilde{\eta}(t))(m_t - x_t). \qquad (3.63)
$$

Recall that in the multi-objective problem, the optimal strategy for the kth agent is

$$\hat{u}_t^k = (q + \eta)(\bar{x}_t - x_t^k). \tag{3.64}$$

The function $\tilde{\eta}$ and η appearing in the two formulas are almost identical since they solve the same ODE with identical terminal condition. So the representative bank behaves the same as the banks in the multi-objective problem.

Remark 3.4.1. The ODE satisfied by $\tilde{\eta}$ differs with the ODE of η by a $O(\frac{1}{N})$ term. We neglect this difference by assuming N is sufficiently large and the ODE is well posed.

Remark 3.4.2. The result $\dot{m}_t = 0$ is not surprising. In the multi-objective problem, we could show that $\forall t$, $\bar{x}_t = \bar{x}_0$ almost surely, if $N \to \infty$. Since the dynamics of $\bar{x}_t$ could be written as

$$\mathrm{d}\bar{x}_t = \frac{1}{N} \sum_{k=1}^{N} \mathrm{d}W_t^k, \tag{3.65}$$

which has the solution

$$\bar{x}_t = \bar{x}_0 + \frac{1}{N} \sum_{k=1}^{N} W_t^k. \tag{3.66}$$

By the law of large numbers we have $\bar{x}_t = \bar{x}_0$ almost surely if $N \to \infty$.

3.5 Mean Field Game with a Dominating Player

Let $x_0(t) \in \mathbb{R}$ and $x_k(t) \in \mathbb{R}$ denote the state variables for the dominating player and the minor players, respectively. Suppose the dynamics for $x_0(t)$ and $x_k(t)$ are given by the following stochastic differential equations,

$$\mathrm{d}x_0(t) = \Big[A_0 x_0(t) + B_0 \frac{1}{N}\sum_{j=1}^{N} x_j(t) + C_0 u_0(t)\Big]\mathrm{d}t + \sigma_0 \mathrm{d}W_0(t), \tag{3.67}$$

$$\mathrm{d}x_k(t) = \Big[A x_k(t) + B \frac{1}{N}\sum_{j=1}^{N} x_j(t) + C u_k(t) + D x_0(t)\Big]\mathrm{d}t + \sigma \mathrm{d}W_k(t). \tag{3.68}$$

Suppose the cost functional of the dominating player and the cost functionals of the minor players are given as follows:

$$\begin{aligned} J_0 = E\Big[\int_0^T \frac{Q_0}{2}\Big(x_0(t) - \alpha_0 \frac{1}{N}\sum x_j(t) - \zeta_0(t)\Big)^2 + \frac{R_0}{2} u_0^2(t)\mathrm{d}t \\ + \frac{\bar{Q}_0}{2}\Big(x_0(T) - \bar{\alpha}_0 \frac{1}{N}\sum x_j(T) - \bar{\zeta}_0\Big)^2\Big] \end{aligned} \tag{3.69}$$

$$\begin{aligned} J_k = E\Big[\int_0^T \frac{Q}{2}\Big(x_k(t) - \alpha \frac{1}{N}\sum x_j(t) - \beta x_0(t) - \zeta(t)\Big)^2 + \frac{R}{2} u_k^2(t)\mathrm{d}t \\ + \frac{\bar{Q}}{2}\Big(x_k(T) - \bar{\alpha} \frac{1}{N}\sum x_j(T) - \bar{\beta} x_0(T) - \bar{\zeta}\Big)^2\Big] \end{aligned} \tag{3.70}$$

3.5.1 Representative Player

Now we investigate the evolution of the empirical distribution $\mu^N(x,t)$ for the minor players, where $\mu^N(x,t) = \frac{1}{N}\sum_{k=1}^{N} \delta_{x_k(t)}(x)$. Assuming that $x_0(t)$ is a known process, and that the control $u_k(t)$ for the player k has the homogeneous feedback form $u_k(t) = u(x_k(t), \frac{1}{N}\sum_j x_j(t), t)$ we have

$$
\begin{aligned}
&\mathrm{d}\langle f(x), \mu^N(x,t)\rangle = \frac{1}{N}\sum \mathrm{d}f(x_k(t)) \\
&= \frac{1}{N} f'(x_k(t))\Big[Ax_k(t) + B\frac{1}{N}\sum x_j(t) + Cu(x_k(t), \langle \xi, \mu^N(\xi,t)\rangle, t) + Dx_0(t)\Big]\mathrm{d}t \\
&\quad + \frac{1}{N}\sum f''(x_k(t))\frac{1}{2}\sigma^2 \mathrm{d}t + \frac{1}{N}\sum f'(x_k(t))\sigma \mathrm{d}W_k(t) \\
&= \langle f'(x)[Ax + B\langle \xi, \mu^N(\xi,t)\rangle + Cu(x, \langle \xi, \mu^N(\xi,t)\rangle, t) + Dx_0(t)], \mu^N(x,t)\rangle \mathrm{d}t \\
&\quad + \langle f''(x)\frac{1}{2}\sigma^2, \mu^N(x,t)\rangle \mathrm{d}t + \frac{1}{N}\sum f'(x_k(t))\sigma \mathrm{d}W_k(t). \qquad (3.71)
\end{aligned}
$$

Assuming that $\|f'(x)\| \leq C$, we have that

$$
\frac{1}{N}\sum f'(x_k(t))\sigma \mathrm{d}W_k(t) \to 0 \quad \text{almost surely, as} \quad N \to \infty. \qquad (3.72)
$$

The proof of (3.39) follows the same steps we take to prove (2.17). Then we will have the limit of the empirical distribution μ satisfying

$$
\begin{aligned}
&\mathrm{d}\langle f(x), \mu(x,t)\rangle \qquad (3.73) \\
&= \langle f'(x)[Ax + B\langle \xi, \mu(\xi,t)\rangle + Cu(x, \langle \xi, \mu(\xi,t)\rangle, t) + Dx_0(t)], \mu(x,t)\rangle \mathrm{d}t \\
&\quad + \langle f''(x)\frac{1}{2}\sigma^2, \mu(x,t)\rangle \mathrm{d}t.
\end{aligned}
$$

From the Fokker Planck Equation, $\mu(x,t)$ is the conditional distribution of the process $X(t)$ under the filtration $\mathscr{F}_t = \sigma\{W(s), s \leq t\}$, where $X(t)$ satisfies

$$
\mathrm{d}X(t) = [AX(t) + Bz(t) + Cu(X(t), z(t), t) + Dx_0(t)]\mathrm{d}t + \sigma \mathrm{d}W(t), \qquad (3.74)
$$

where $z(t) = \langle x, \mu(x,t)\rangle$. By taking $f(x) = x$ in equation (3.73), the dynamic of $z(t)$ is given as

$$\mathrm{d}z(t) = [(A+B)z(t) + C\langle u(x,z(t),t), \mu(x,t)\rangle + Dx_0(t)]\mathrm{d}t. \tag{3.75}$$

3.5.2 Optimal Control for the Representative Player and the Dominating Player

Instead of considering N minor players' game, we will just look at the process $X(t)$ as a representative player with a cost function

$$\begin{aligned}\hat{J}(u) = E\Big[\int_0^T \frac{Q}{2}\Big(X(t) - \alpha z(t) - \beta x_0(t) - \zeta(t)\Big)^2 + \frac{R}{2}u^2(t)\mathrm{d}t \\ + \frac{\bar{Q}}{2}\Big(X(T) - \bar{\alpha}z(T) - \bar{\beta}x_0(T) - \bar{\zeta}\Big)^2\Big]\end{aligned} \tag{3.76}$$

By Theorem 3.3.1, the optimal control of the representative player could be computed using equation (3.52), which implies $\hat{u}(t) = -R^{-1}Cn(t)$, where $n(t)$ is the adjoint variable which satisfies the backward stochastic equation

$$\begin{aligned}-\mathrm{d}n(t) &= [An(t) + Q(X(t) - \alpha z(t) - \beta x_0(t) - \zeta(t))]\mathrm{d}t \\ &\quad - Z_{n0}(t)\mathrm{d}W_0 - Z_n(t)\mathrm{d}W, \qquad (3.77)\\ n(T) &= \bar{Q}(X(T) - \bar{\alpha}z(T) - \bar{\beta}x_0(T) - \bar{\zeta}). \qquad (3.78)\end{aligned}$$

By assuming $n(t) = P_tX(t) + g(t)$, and differentiating $n(t)$, we derive that

$$\begin{aligned}&\dot{P}_t + 2P_tA - P_t^2C^2R^{-1} + Q = 0, \qquad (3.79)\\ &P_T = \bar{Q}, \qquad (3.80)\end{aligned}$$

and

$$
\begin{aligned}
-\mathrm{d}g(t) &= \big[(A - C^2R^{-1}P_t)g(t) + (P_tB - Q\alpha)z(t) + (P_tD - Q\beta)x_0(t) - Q\zeta\big]\mathrm{d}t \\
&\qquad -Z_{n0}(t)\mathrm{d}W_0, &(3.81) \\
g(T) &= \bar{Q}(-\bar{\alpha}z(T) - \bar{\beta}x_0(T) - \bar{\zeta}). &(3.82)
\end{aligned}
$$

Therefore, the optimal control has the feedback form

$$
\hat{u}(t) = u(X(t), z(t), t) = -R^{-1}C(P_tX(t) + g(t)). \tag{3.83}
$$

Substituting (3.83) into (3.75) we have

$$
\mathrm{d}z(t) = \big[(A+B)z(t) - R^{-1}C^2(P_tz(t) + g(t)) + Dx_0\big]\mathrm{d}t. \tag{3.84}
$$

Now, the problem becomes minimizing the cost functional for the dominating player:

$$
\begin{aligned}
\hat{J}_0(u_0) = E\Big[\int_0^T \frac{Q_0}{2}\Big(x_0(t) - \alpha_0 z(t) - \zeta_0(t)\Big)^2 + \frac{R_0}{2}u_0^2(t)\mathrm{d}t \qquad &(3.85) \\
+\frac{\bar{Q}_0}{2}\Big(x_0(T) - \bar{\alpha}_0 z(T) - \bar{\zeta}_0\Big)^2\Big]
\end{aligned}
$$

subject to

$$
\begin{cases}
\mathrm{d}x_0(t) = \big[A_0x_0(t) + B_0z(t) + C_0u_0(t)\big]\mathrm{d}t + \sigma_0\mathrm{d}W_0(t), \\
\mathrm{d}z(t) = \big[(A+B)z(t) - R^{-1}C^2(P_tz(t) + g(t)) + Dx_0\big]\mathrm{d}t, \\
-\mathrm{d}g(t) = \big[(A - C^2R^{-1}P_t)g(t) + (P_tB - Q\alpha)z(t) + (P_tD - Q\beta)x_0(t) - Q\zeta\big]\mathrm{d}t \\
\qquad\qquad -Z_{n0}(t)\mathrm{d}W_0 - Z_n(t)\mathrm{d}W, \\
g(T) = \bar{Q}(-\bar{\alpha}z(T) - \bar{\beta}x_0(T) - \bar{\zeta}).
\end{cases} \tag{3.86}
$$

We consider the following general forward backward optimal control problem:

$$\text{Minimize} \quad J(u) = E\Big[\int_0^T L(x(t), y(t), u(t))\mathrm{d}t + \Psi(x(T))\Big] \tag{3.87}$$

subject to

$$\begin{cases} \mathrm{d}x(t) = f(x(t), y(t), u(t), t)\mathrm{d}t + \sigma \mathrm{d}W, & x(0) = x_0, \\ -\mathrm{d}y(t) = g(x(t), y(t), u(t), t)\mathrm{d}t - Z_y(t)\mathrm{d}W, & y(T) = h(x(T)). \end{cases} \tag{3.88}$$

Let us make a perturbation for the control $u(t) \to u(t) + \theta\tilde{u}(t)$ for small θ, then we will have $\{x(t) + \theta\tilde{x}(t), y(t) + \theta\tilde{y}(t)\}$ as the solution to the system (3.88) corresponding to the control $u(t) + \theta\tilde{u}(t)$. And we have

$$\begin{cases} \mathrm{d}\tilde{x}(t) = f_x(x, y, u, t)\tilde{x}(t) + f_y(x, y, u, t)\tilde{y}(t) + f_u(x, y, u, t)\tilde{u}(t), \\ -\mathrm{d}\tilde{y}(t) = g_x(x, y, u, t)\tilde{x}(t) + g_y(x, y, u, t)\tilde{y}(t) + g_u(x, y, u, t)\tilde{u}(t) - \tilde{Z}_y(t)\mathrm{d}W, \end{cases} \tag{3.89}$$

with boundary condition

$$\begin{cases} \tilde{x}(0) = 0, \\ \tilde{y}(T) = h_x(x(T))\tilde{x}(T) \end{cases} \tag{3.90}$$

Now,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\theta} J(u + \theta\tilde{u}) = E\Big[\int_0^T & L_x(x, y, u, t)\tilde{x}(t) + L_y(x, y, u, t)\tilde{y}(t) + L_u(x, y, u, t)\tilde{u}(t)\mathrm{d}t \\ & + \Psi_x(x(T))\tilde{x}(T)\Big] \end{aligned} \tag{3.91}$$

We consider the adjoint system

$$
\begin{cases}
-\mathrm{d}p(t) = \Big[f_x^T(x,y,u,t)p(t) + g_x^T(x,y,u,t)r(t) + L_x^T(x,y,u,t)\Big]\mathrm{d}t - Z_p(t)\mathrm{d}W, \\
\mathrm{d}r(t) = \Big[f_y^T(x,y,u,t)p(t) + g_y^T(x,y,u,t)r(t) + L_y^T(x,y,u,t)\Big]\mathrm{d}t \\
p(T) = \Psi_x^T(x(T)) + h_x^T(x(T))r(T) \\
r(0) = 0.
\end{cases} \tag{3.92}
$$

We compute that

$$
\begin{aligned}
\mathrm{d}(p^T\tilde{x} - r^T\tilde{y}) &= p^T(f_x\tilde{x} + f_y\tilde{y} + f_u\tilde{u}) - (p^T f_x + r^T g_x + L_x)\tilde{x} \\
&\quad + r^T(g_x\tilde{x} + g_y\tilde{y} + g_u\tilde{u}) - (p^T f_y + r^T g_y + L_y)\tilde{y} + (\dots)\mathrm{d}W \\
&= -L_x\tilde{x} - L_y\tilde{y} + p^T f_u\tilde{u} + r^T g_u\tilde{u} + (\dots)\mathrm{d}W.
\end{aligned} \tag{3.93}
$$

Integrating the above equation and taking expectation we have

$$
E\Big[\Psi_x(x(T))\tilde{x}(T)\Big] = E\Big[\int_0^T \Big(-L_x\tilde{x} - L_y\tilde{y} + p^T f_u\tilde{u} + r^T g_u\tilde{u}\Big)\mathrm{d}t\Big]. \tag{3.94}
$$

Then, we have

$$
\frac{\mathrm{d}}{\mathrm{d}\theta}J(u+\theta\tilde{u}) = E\Big[\int_0^T \Big(p^T f_u + r^T g_u + L_u\Big)\tilde{u}\mathrm{d}t\Big]. \tag{3.95}
$$

Since $\tilde{u}(t)$ is arbitrary, the optimal control $u(t)$ should satisfy

$$
p^T f_u(x,y,u,t) + r^T g_u(x,y,u,t) + L_u(x,y,u,t) = 0. \tag{3.96}
$$

Recall the system (3.86), the optimal control for the dominating player $\hat{u}_0$ should satisfy

$$p(t)C_0 + R_0\hat{u}_0 = 0, \tag{3.97}$$

which gives $\hat{u}_0(t) = -R^{-1}C_0 p(t)$. Then, to solve the dominating player game, we have the following forward backward system:

$$\begin{cases} \mathrm{d}x_0(t) = \Big[A_0 x_0(t) + B_0 z(t) - C_0^2 R_0^{-1} p(t)\Big]\mathrm{d}t + \sigma_0 \mathrm{d}W_0(t), \\ \mathrm{d}z(t) = \big[(A+B)z(t) - R^{-1}C^2(P_t z(t) + g(t)) + Dx_0\big]\mathrm{d}t, \\ -\mathrm{d}g(t) = \big[(A - C^2R^{-1}P_t)g(t) + (P_tB - Q\alpha)z(t) + (P_tD - Q\beta)x_0(t) - Q\zeta\big]\mathrm{d}t \\ \qquad\qquad -Z_{n0}(t)\mathrm{d}W_0, \\ -\mathrm{d}p(t) = \Big[A_0 p(t) + Dq + (P_tD - Q\beta)r(t) + Q_0(x_0(t) - \alpha_0 z(t) - \zeta_0(t))\Big]\mathrm{d}t \\ \qquad\qquad -Z_{p0}\mathrm{d}W_0 \\ -\mathrm{d}q(t) = \Big[B_0 p(t) + (A + B - R^{-1}C^2P_t)q(t) + (P_tB - Q\alpha)r(t) \\ \qquad\qquad -\alpha_0 Q_0(x_0(t) - \alpha_0 z(t) - \zeta_0(t))\Big] - Z_{q0}\mathrm{d}W_0 \\ \mathrm{d}r(t) = -R^{-1}C^2 q(t) + (A - C^2R^{-1}P_t)r(t), \\ g(T) = \bar{Q}(-\bar{\alpha}z(T) - \bar{\beta}x_0(T) - \bar{\zeta}). \\ p(T) = \bar{Q}_0(x_0(T) - \bar{\alpha}_0 z(T) - \bar{\zeta}_0) - \bar{\beta}\bar{Q}r(T) \\ q(T) = -\bar{\alpha}_0\bar{Q}_0(x_0(T) - \bar{\alpha}_0 z(T) - \bar{\zeta}_0) - \bar{\alpha}\bar{Q}r(T) \end{cases} \tag{3.98}$$

Ma, Protter and Yong have some work on the explicit scheme for forward backward stochastic equations. We rearrange the above equations as follows:

$$\begin{cases} \mathrm{d}\mathbf{x} = (\mathcal{A}\mathbf{x} - \mathcal{B}\mathbf{p})\mathrm{d}t + \sigma\mathrm{d}W, \\ -\mathrm{d}\mathbf{p} = (\mathcal{C}\mathbf{x} + \mathcal{D}\mathbf{p} + \mathbf{k})\mathrm{d}t - \mathbf{Z}\mathrm{d}W, \\ \mathbf{p}(T) = \mathcal{Q}\mathbf{x}(T) + \mathbf{h}, \end{cases} \tag{3.99}$$

where

$$\mathbf{x} = \begin{bmatrix} x_0 \\ z \\ r \end{bmatrix}, \qquad \mathbf{p} = \begin{bmatrix} p \\ q \\ g \end{bmatrix}, \qquad \mathcal{A} = \begin{bmatrix} A_0 & B_0 & 0 \\ D & A + B - R^{-1}C^2P_t & 0 \\ 0 & 0 & A - C^2R^{-1}P_t \end{bmatrix},$$

$$\mathcal{B} = \begin{bmatrix} C_0^2R_0^{-1} & 0 & 0 \\ 0 & 0 & R^{-1}C^2 \\ 0 & R^{-1}C^2 & 0 \end{bmatrix}, \qquad \mathcal{C} = \begin{bmatrix} Q_0 & -\alpha_0Q_0 & P_tD - Q\beta \\ -\alpha_0Q_0 & \alpha_0^2Q_0 & P_tB - Q\alpha \\ P_tD - Q\beta & P_tB - Q\alpha & 0 \end{bmatrix},$$

$$\mathcal{D} = \begin{bmatrix} A_0 & D & 0 \\ B_0 & A + B - R^{-1}C^2P_t & 0 \\ 0 & 0 & A - C^2R^{-1}P_t \end{bmatrix}, \qquad \mathbf{k} = \begin{bmatrix} -Q_0\zeta_0 \\ \alpha_0Q_0\zeta_0 \\ -Q\zeta \end{bmatrix}, \qquad \mathbf{h} = \begin{bmatrix} -\bar{Q}_0\bar{\zeta}_0 \\ \bar{\alpha}_0\bar{Q}_0\bar{\zeta}_0 \\ -\bar{Q}\bar{\zeta} \end{bmatrix},$$

$$\mathcal{Q} = \begin{bmatrix} \bar{Q}_0 & -\bar{Q}_0\bar{\alpha}_0 & -\bar{\beta}\bar{Q} \\ -\bar{\alpha}_0\bar{Q}_0 & \bar{\alpha}_0^2\bar{Q}_0 & -\bar{\alpha}\bar{Q} \\ -\bar{Q}\bar{\beta} & -\bar{Q}\bar{\alpha} & 0 \end{bmatrix}, \qquad \sigma = \begin{bmatrix} \sigma_0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \mathbf{Z} = \begin{bmatrix} Z_{p0} & Z_p \\ Z_{q0} & Z_q \\ Z_{n0} & Z_n \end{bmatrix},$$

To solve the system (3.99), we assume $\mathbf{p} = S_t x + \eta(t)$. We could derive the evolution for S_t and $\eta(t)$, which satisfy

$$\begin{cases} \dot{S}_t + S_t\mathcal{A} + \mathcal{D}S_t - S_t\mathcal{B}S_t + \mathcal{C} = 0, \\ \dot{\eta} - S_t\mathcal{B}\eta + \mathcal{D}\eta + \mathbf{k} = 0, \\ S_T = \mathcal{Q}, \quad \eta(T) = \mathbf{h}. \end{cases} \tag{3.100}$$

Then, the forward backward system becomes a forward SDE:

$$\mathrm{d}\mathbf{x} = (\mathcal{A}\mathbf{x} - \mathcal{B}(S_t\mathbf{x} + \eta_t))\mathrm{d}t + \sigma\mathrm{d}W. \tag{3.101}$$

3.5.3 Numerical Experiment

Consider the system

$$\begin{cases} \mathrm{d}x_0 = (A_0 x_0 + B_0 z + C_0 u_0)\mathrm{d}t + \sigma_0 \mathrm{d}W_0 \\ \mathrm{d}x = (Ax + Bz + Cu + Dx_0)\mathrm{d}t + \sigma \mathrm{d}W \end{cases} \tag{3.102}$$

with the following set of parameters

A_0	B_0	C_0	σ_0	A	B	C	D	σ
0.1	-0.05	1	0.2	0.2	-0.05	1	0.01	0.3

and the cost functional

$$J_0 = E\Big\{ \int_0^T \frac{1}{2}(x_0(t) - 10z(t))^2 + \frac{1}{2}u_0(t)^2\mathrm{d}t + \frac{1}{2}(x_0(T) - 10z(T))^2 \Big\} \tag{3.103}$$

$$J = E\Big\{ \int_0^T \frac{1}{2}(x(t) - z(t))^2 + \frac{1}{2}u(t)^2\mathrm{d}t + \frac{1}{2}(x(T) - z(T))^2 \Big\} \tag{3.104}$$

From the cost functional, we could see that the dominating player will keep track of the average of the minor players. He thinks 10 times as large as the average is the good size for him to remain the dominant position, and to avoid the market turning monopolistic. The minor players intend to stay close to the average for a safe environment of growth. The following table shows the cost of the representative player and the dominating player under different controls.

Table 3.1: List of Cost of the Representative Agent

u	u_0	J
0	0	1.947
0.1	0	2.014
-0.1	0	1.910
-0.2	0	2.064
$\hat{u}$	0	0.271

Table 3.2: List of Cost of the Dominating Agent

u	u_0	J_0
$\hat{u}$	0	243.44
$\hat{u}$	0.1	226.56
$\hat{u}$	0.2	211.09
$\hat{u}$	$\hat{u}_0$	22.13

Remark 3.5.1. We could see from Table (3.1) that in the scenario when the dominating agent takes a constant control strategy, the cost of the representative agent is minimized by taking the optimal control $\hat{u}$. The Table (3.2) displays that in the scenario when the representative agent sticks to his optimal control $\hat{u}$, the dominating player could minimize his cost J_0 by taking his optimal strategy $\hat{u}_0$.

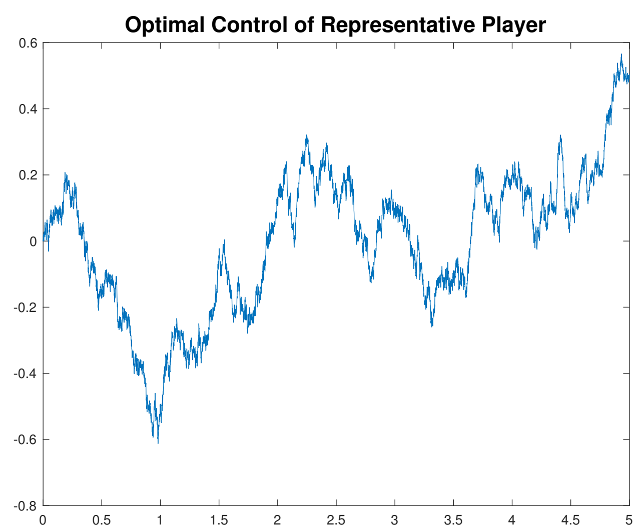

Figure 3.1: Path/State of Optimal Control of Repr. Player:
$\hat{u}(t) = -\frac{C(P_t x(t)+g(t))}{R}$

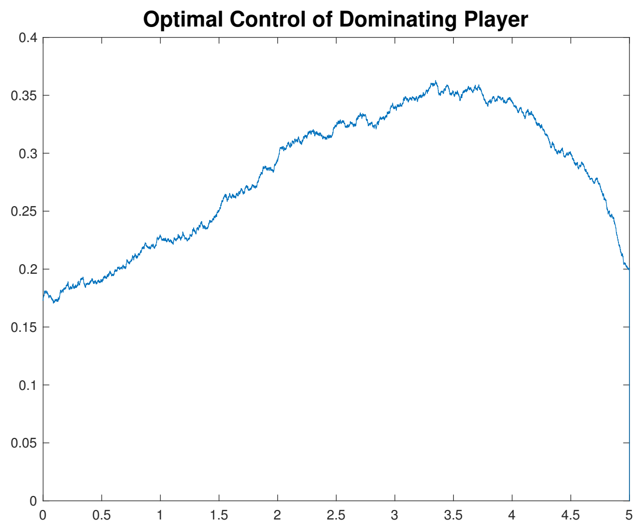

Figure 3.2: Path/State of Optimal Control of Dominating Player: $\hat{u}_0(t) = -R^{-1}C_0 p(t)$

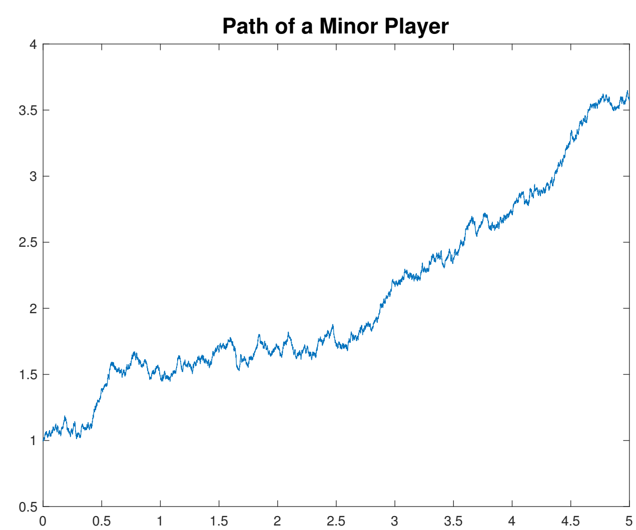

Figure 3.3: Optimal Path/State of Representative Player

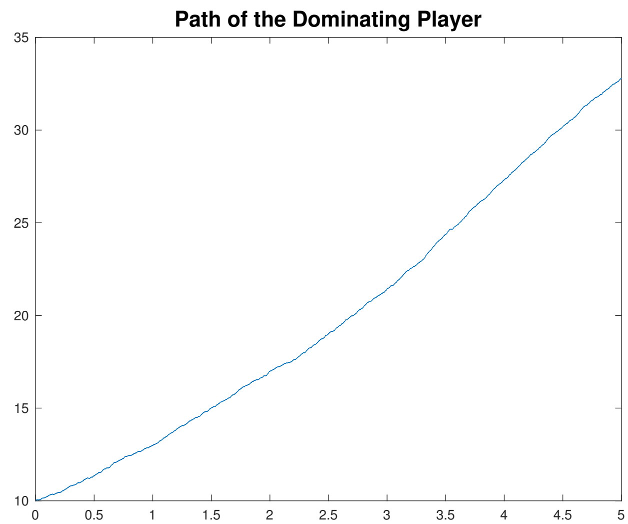

Figure 3.4: Optimal Path/State of Dominating Player

3.6 Impulsive Mean Field Control

In this section we will discuss the topic of impulsive mean field control. In case of the multi-banks game, we consider the the banks are controlling their portfolio to optimize their payoffs. Then fixed assets could be modeled as impulsive control, where changes are made only at discrete times. We will study the necessary conditions for the optimal continuous control and impulsive control of mean field control problems.

3.6.1 Mean Field Type Control

Before we discuss the impulsive mean field control problem, consider the process $x_u(t)$ whose evolution satisfies the following SDE:

$$\mathrm{d}x_u(t) = f(t, x_u(t), \mu_u(t,\cdot), u(t))\mathrm{d}t + \sum_{j=1}^{d} \sigma(t, x_u(t), \mu_u(t,\cdot), u(t))\mathrm{d}W^j, \tag{3.105}$$

where $\mu_u(t,\cdot)$ is the probability distribution of $x_u(t)$. We use the notation x_u, μ_u to identify the dependence of the states and the distributions on the control process $u(t)$. Then we define the cost:

$$J(u) = E\Big[\int_0^T L(x_u(t), \mu_u(t,\cdot), u(t))\mathrm{d}t + \phi(x_u(T), \mu_u(T,\cdot))\Big]. \tag{3.106}$$

The mean field type control problem is defined as follows: find a control process $u(t)$, such that $J(u) \leq J(v)$, $\forall v$.

Remark 3.6.1. It is important to distinguish between mean field type control and mean field games(Section 3.3). For the mean field games, the state process satisfies (3.43) and the cost is defined as (3.45). Notice that the cost (3.45) depends on the control u and measure m, because the measure m is known to be external to (3.43). In the mean field type control problem, the measure μ_u in (3.105) is required to be the probability of the process x_t defined by (3.105), so the cost (3.106) only depends on the control u.

We could write the cost (3.106) as follows:

$$\begin{aligned} J(u) &= E\Big[\int_0^T L(x_u(t), \mu_u(t,\cdot), u(t))\mathrm{d}t + \phi(x_u(T), \mu_u(T,\cdot))\Big] \\ &= \int_0^T \int_{\mathbb{R}^n} L(x, \mu_u(t,\cdot), u(t))\mu_u(t,x)\mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}^n} \phi(x, \mu_u(T,\cdot))\mu_u(T,x)\mathrm{d}x. \end{aligned} \tag{3.107}$$

We approach the problem using dynamic programming. Consider the family of control problems indexed by initial condition (t,μ):

$$\begin{cases} \mathrm{d}x_u = f(s, x_u, \mu_u(s,\cdot), u(s))\mathrm{d}s + \sum_j \sigma^j(s, x_u, \mu_u(s,\cdot), u(s))\mathrm{d}W_s^j, & s \in (t,T) \\ \mu_u(t,x) = \mu(x), \end{cases} \tag{3.108}$$

with cost functional

$$V(t,\mu) = \inf_{u_s} \Big\{E\Big[\int_t^T L(x_u(s), \mu_u(s,\cdot), u_s)\mathrm{d}s + \phi(x_u(T), \mu_u(T,\cdot))\Big]\Big\}. \tag{3.109}$$

Let

$$v(s) \triangleq \begin{cases} \bar{v}, & t \le s < t+\varepsilon \\ \text{optimal control}, & t+\varepsilon \le s < T \end{cases} \tag{3.110}$$

Then using the dynamic programming approach we have

$$V(t,\mu) \le E\Big\{ \int_t^{t+\varepsilon} L(x_v(s), \mu_v(s,\cdot), \bar{v})\mathrm{d}s \Big\} + V(t+\varepsilon, \mu_v(t+\varepsilon,\cdot)) \tag{3.111}$$

where the state $x_v(t)$ satisfies

$$\mathrm{d}x_v = f(s, x_v, \mu_v(s,\cdot), \bar{v})\mathrm{d}s + \sigma(s, x_v, \mu_v(s,\cdot), \bar{v})\mathrm{d}W_s,$$

in the interval $s \in (t, t+\varepsilon)$. Keep in mind that the Fokker Planck equation in time interval $(t, t+\varepsilon)$ has the form

$$\frac{\partial \mu_v(t,x)}{\partial t} = -\mathrm{div}(f(t,x,\mu_v(t,\cdot),\bar{v})\mu_v(t,x)) + \frac{1}{2}\sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j}\Big(a_{ij}(t,x,\mu_v(t,\cdot),\bar{v})\mu_v(t,x)\Big). \tag{3.112}$$

where $a = \sigma\sigma^T$. Then, $\mu_v(t+\varepsilon,\cdot)$ could be approximated by

$$\begin{aligned} \mu_v(t+\varepsilon,x) = \mu(x) + \Big[&- \mathrm{div}(f(t,x,\mu(\cdot),\bar{v})\mu(x)) \\ &+ \frac{1}{2}\sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j}\Big(a_{ij}(t,x,\mu(\cdot),\bar{v})\mu(x)\Big)\Big]\varepsilon + o(\varepsilon), \end{aligned} \tag{3.113}$$

Next,

$$
\begin{aligned}
& \frac{1}{\varepsilon}\Big(V(t+\varepsilon,\mu_v(t+\varepsilon,\cdot)) - V(t,\mu_v(t,\cdot))\Big) \\
= \quad & \frac{1}{\varepsilon}\Big(V(t+\varepsilon,\mu_v(t+\varepsilon,\cdot)) - V(t+\varepsilon,\mu_v(t,\cdot)) + V(t+\varepsilon,\mu(\cdot)) - V(t,\mu(\cdot))\Big) \\
= \quad & \int_{\mathbb{R}^n} \frac{\partial V}{\partial m}(t,\mu,x)(\mu_v(t+\varepsilon,x) - \mu_v(t,x))\mathrm{d}x + \frac{\partial V}{\partial t}(\mu,t) + O(\varepsilon) \\
= \quad & \int_{\mathbb{R}^n} \frac{\partial V}{\partial m}(t,\mu,x)\Big[-\mathrm{div}(f(t,x,\mu,\bar{v})\mu(x)) + \sum_{i,j}\frac{\partial^2}{\partial x_i \partial x_j}\Big(\frac{1}{2}a_{ij}(t,x,\mu,\bar{v})\mu(x)\Big)\Big]\mathrm{d}x \\
& + \frac{\partial V}{\partial t}(t,\mu) + O(\varepsilon), \\
= \quad & \int_{\mathbb{R}^d} \frac{\partial}{\partial x}\Big(\frac{\partial V}{\partial m}(t,\mu,x)\Big) f(t,x,\mu,\bar{v})\mu(x) + \frac{1}{2}\sum_{i,j}\frac{\partial^2}{\partial x_i \partial x_j}\Big(\frac{\partial V}{\partial m}(t,\mu,x)\Big)a_{ij}(t,x,\mu,\bar{v})\mu(x)\mathrm{d}x \\
& + \frac{\partial V}{\partial t}(t,\mu) + O(\varepsilon). \qquad (3.114)
\end{aligned}
$$

Then, from (3.111, 3.114) and let $\varepsilon \to 0$ we have

$$
\begin{aligned}
\frac{\partial V}{\partial t}(t,\mu) &+ \int_{\mathbb{R}^d} L(x,\mu,\bar{v})\mu(x)\mathrm{d}x + \int_{\mathbb{R}^d} \frac{\partial}{\partial x}\Big(\frac{\partial V}{\partial m}(t,\mu,x)\Big) f(t,x,\mu,\bar{v})\mu(x)\mathrm{d}x \\
&+ \int_{\mathbb{R}^d} \frac{1}{2}\sum_{i,j}\frac{\partial^2}{\partial x_i \partial x_j}\Big(\frac{\partial V}{\partial m}(t,\mu,x)\Big)a_{ij}(t,x,\mu,\bar{v})\mu(x)\mathrm{d}x \geq 0. \qquad (3.115)
\end{aligned}
$$

Equality will hold when optimal control $\hat{v}$ is chosen, which minimizes the left hand side of (3.115). The optimal control has the feedback form $\hat{v}(t,x,\mu,\nabla\frac{\partial V}{\partial m},\nabla^2\frac{\partial V}{\partial m})$, where

$$
\hat{v}(t,x,\mu,p,P) = \arg\min_{v}\Big\{L(x,\mu,v) + p^T f(t,x,\mu,v) + \frac{1}{2}\sum_{i,j}P_{ij}a_{ij}(t,x,\mu,v)\Big\}. \qquad (3.116)
$$

Then, we have

$$
\begin{aligned}
&\frac{\partial V}{\partial t}(t,\mu) + \int_{\mathbb{R}^d} L(x,\mu,\hat{v}(t,x,\mu,\nabla\frac{\partial V}{\partial m},\nabla^2\frac{\partial V}{\partial m}))\mu(x)\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^d} \frac{\partial}{\partial x}\Big(\frac{\partial V}{\partial m}(t,\mu,x)\Big) f(t,x,\mu,\hat{v}(t,x,\mu,\nabla\frac{\partial V}{\partial m},\nabla^2\frac{\partial V}{\partial m}))\mu(x)\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^d} \frac{1}{2}\sum_{i,j}\frac{\partial^2}{\partial x_i\partial x_j}\Big(\frac{\partial V}{\partial m}(t,\mu,x)\Big) a_{ij}(t,x,\mu,\hat{v}(t,x,\mu,\nabla\frac{\partial V}{\partial m},\nabla^2\frac{\partial V}{\partial m}))\mu(x)\mathrm{d}x = 0.
\end{aligned}
\tag{3.117}
$$

Define $U(t,\mu,x) = \dfrac{\partial V}{\partial m}(t,\mu)$, and taking derivative with respect to μ in equation (3.117), we have

$$
\begin{aligned}
&\frac{\partial U}{\partial t}(t,\mu,x) + L(x,\mu,\hat{v}(t,x,\mu,\nabla U,\nabla^2 U)) \\
&\quad + \int_{\mathbb{R}^d} \frac{\partial L}{\partial m}(\xi,\mu,\hat{v}(t,\xi,\mu,\nabla U,\nabla^2 U),x)\mu(\xi)\mathrm{d}\xi \\
&\quad + \frac{\partial}{\partial x}\Big(U(t,\mu,x)\Big) f(t,x,\mu,\hat{v}(t,x,\mu,\nabla U,\nabla^2 U)) \\
&\quad + \int_{\mathbb{R}^n} \frac{\partial}{\partial x}\Big(U(\mu,t,\xi)\Big)\frac{\partial f}{\partial m}(t,\xi,\mu,\hat{v}(t,\xi,\mu,\nabla U,\nabla^2 U),x)\mu(\xi)\mathrm{d}\xi \\
&\quad - \int_{\mathbb{R}^n} \frac{\partial U}{\partial m}(t,\mu,\xi,x)\nabla\cdot\Big(f(t,\xi,\mu,\hat{v}(t,\xi,\mu,\nabla U,\nabla^2 U))\mu(\xi)\Big)\mathrm{d}\xi \\
&\quad + \sum_{i,j}\frac{1}{2}a_{ij}(t,x,\mu,\hat{v}(t,x,\mu,\nabla U,\nabla^2 U))\frac{\partial^2}{\partial x_i\partial x_j}U(t,\mu,x) \\
&\quad + \int_{\mathbb{R}^n} \frac{\partial U}{\partial m}(t,\mu,\xi,x)\sum_{i,j}\frac{\partial^2}{\partial x_i\partial x_j}\Big(\frac{1}{2}a_{ij}(t,\xi,\mu,\hat{v}(t,\xi,\mu,\nabla U,\nabla^2 U))\mu(\xi)\Big)\mathrm{d}\xi \\
&\quad + \int_{\mathbb{R}^d} \frac{1}{2}\sum_{i,j}\frac{\partial^2}{\partial x_i\partial x_j}\Big(U(t,\mu,\xi)\Big)\frac{\partial a_{ij}}{\partial m}(t,\xi,\mu,\hat{v}(t,\xi,\mu,\nabla U,\nabla^2 U),x)\mu(\xi)\mathrm{d}\xi = 0.
\end{aligned}
\tag{3.118}
$$

Let $\mu_{\hat{v}}(s,x)$ solve

$$\begin{aligned}\frac{\partial \mu_{\hat{v}}(t,x)}{\partial t} &= -\nabla\cdot\Big(f(t,x,\mu_{\hat{v}}(t,\cdot),\hat{v}(t,x,\mu_{\hat{v}}(t,\cdot),\nabla U,\nabla^2 U))\mu_{\hat{v}}(t,x)\Big) \qquad (3.119)\\ &\quad+\sum_{i,j}\frac{\partial^2}{\partial x_i\partial x_j}\Big(\frac{1}{2}a_{ij}(t,x,\mu_{\hat{v}}(t,\cdot),\hat{v}(t,x,\mu_{\hat{v}}(t,\cdot),\nabla U,\nabla^2 U))\mu_{\hat{v}}(t,x)\Big),\quad t\in(0,T),\\ \mu_{\hat{v}}(0,x) &= \mu_0(x). \qquad (3.120)\end{aligned}$$

We define $\hat{U}(t,x) = U(t,\mu_{\hat{v}}(t,\cdot),x)$, then we compute

$$\begin{aligned}\frac{\partial \hat{U}(t,x)}{\partial t} &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}(U(t+\varepsilon,\mu_{\hat{v}}(t+\varepsilon,\cdot),x) - U(t,\mu_{\hat{v}}(t,\cdot),x))\\ &= \frac{\partial U}{\partial t}(t,\mu_{\hat{v}}(t,\cdot),x) + \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{\mathbb{R}^d}\frac{\partial U}{\partial m}(t,\mu_{\hat{v}}(t,\cdot),x,\xi)(\mu_{\hat{v}}(t+\varepsilon,\xi)-\mu_{\hat{v}}(t,\xi))\mathrm{d}\xi\\ &= \frac{\partial U}{\partial t}(t,\mu_{\hat{v}}(t,\cdot),x) - \int_{\mathbb{R}^n}\frac{\partial U}{\partial m}(t,\mu_{\hat{v}},x,\xi)\nabla\cdot\Big(f(t,\xi,\mu_{\hat{v}},\hat{v}(t,\xi,\mu_{\hat{v}},\nabla U,\nabla^2 U))\mu_{\hat{v}}(t,\xi)\Big)\mathrm{d}\xi\\ &\quad+\int_{\mathbb{R}^n}\frac{\partial U}{\partial m}(t,\mu_{\hat{v}},x,\xi)\sum_{i,j}\frac{\partial^2}{\partial x_i\partial x_j}\Big(\frac{1}{2}a_{ij}(t,\xi,\mu_{\hat{v}},\hat{v}(t,\xi,\mu_{\hat{v}},\nabla U,\nabla^2 U))\mu_{\hat{v}}(t,\xi)\Big)\mathrm{d}\xi\end{aligned}\tag{3.121}$$

Under sufficient smoothness we have

$$\frac{\partial^2 V}{\partial m^2}(t,\mu,x,\xi) = \frac{\partial^2 V}{\partial m^2}(t,\mu,\xi,x) \implies \frac{\partial U}{\partial m}(t,\mu,x,\xi) = \frac{\partial U}{\partial m}(t,\mu,\xi,x).\tag{3.122}$$

Comparing (3.121) and (3.118), we have

$$
\begin{aligned}
&\frac{\partial \hat{U}}{\partial t}(t,x) + L(x, \mu_{\hat{v}}(t,\cdot), \hat{v}(t,x,\mu_{\hat{v}}(t,\cdot), \nabla \hat{U}, \nabla^2 \hat{U})) \\
&\quad + \frac{\partial}{\partial x}\Big(\hat{U}(t, \mu_{\hat{v}}(t,\cdot), x)\Big) f(t,x,\mu_{\hat{v}}(t,\cdot), \hat{v}(t,x,\mu_{\hat{v}}(t,\cdot), \nabla \hat{U}, \nabla^2 \hat{U})) \\
&\quad + \sum_{i,j} \frac{1}{2} a_{ij}(t,x,\mu_{\hat{v}}(t,\cdot), \hat{v}(t,x,\mu_{\hat{v}}(t,\cdot), \nabla \hat{U}, \nabla^2 \hat{U})) \frac{\partial^2}{\partial x_i \partial x_j} \hat{U}(t,x) \\
&\quad + \int_{\mathbb{R}^d} \frac{\partial L}{\partial m}(\xi, \mu_{\hat{v}}(t,\cdot), \hat{v}(t,\xi,\mu_{\hat{v}}(t,\cdot), \nabla \hat{U}, \nabla^2 \hat{U}), x) \mu_{\hat{v}}(t,\xi) \mathrm{d}\xi \\
&\quad + \int_{\mathbb{R}^n} \frac{\partial}{\partial x}\Big(\hat{U}(t,\xi)\Big) \frac{\partial f}{\partial m}(t,\xi,\mu_{\hat{v}}(t,\cdot), \hat{v}(t,\xi,\mu_{\hat{v}}(t,\cdot), \nabla \hat{U}, \nabla^2 \hat{U}), x) \mu_{\hat{v}}(t,\xi) \mathrm{d}\xi \\
&\quad + \int_{\mathbb{R}^d} \frac{1}{2} \sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j}\Big(\hat{U}(t,\xi)\Big) \frac{\partial a_{ij}}{\partial m}(t,\xi,\mu_{\hat{v}}(t,\cdot), \hat{v}(t,\xi,\mu_{\hat{v}}(t,\cdot), \nabla \hat{U}, \nabla^2 \hat{U}), x) \mu_{\hat{v}}(t,\xi) \mathrm{d}\xi = 0.
\end{aligned} \tag{3.123}
$$

By defining

$$
\begin{aligned}
H(t,x,\mu,P,Q) = {} & L(x,\mu,\hat{v}(t,x,\mu,P,Q)) + p^T f(t,x,\mu,\hat{v}(t,x,\mu,P,Q)) \\
& + \frac{1}{2} \sum_{i,j} Q_{ij} a_{ij}(t,x,\mu,\hat{v}(t,x,\mu,P,Q)),
\end{aligned} \tag{3.124}
$$

equation (3.123) will be reduced to

$$
\frac{\partial \hat{U}}{\partial t}(t,x) + H(t,x,\mu_{\hat{v}}(t,\cdot), \nabla \hat{U}, \nabla^2 \hat{U}) + \int_{\mathbb{R}^d} \frac{\partial H}{\partial m}(t,x,\mu_{\hat{v}}(t,\cdot), \nabla \hat{U}, \nabla^2 \hat{U}, \xi) \mu_{\hat{v}}(t,\xi) \mathrm{d}\xi = 0. \tag{3.125}
$$

For the terminal condition of $\hat{U}(t,x)$, we have

$$\begin{aligned}\hat{U}(T,x) &= \frac{\partial}{\partial m}V(T,\mu_{\hat{v}}(T,\cdot),x) \\ &= \frac{\partial}{\partial m}\int_{\mathbb{R}^d}\phi(y,\mu_{\hat{v}}(T,\cdot))\mu_{\hat{v}}(t,y)\mathrm{d}y \\ &= \phi(x,\mu_{\hat{v}}(T,\cdot)) + \int_{\mathbb{R}^d}\frac{\partial\phi}{\partial m}(y,\mu_{\hat{v}}(T,\cdot),x)\mu_{\hat{v}}(T,y)\mathrm{d}y.\end{aligned} \tag{3.126}$$

To sum up the mean field type control problem, we have to solve the coupled forward backward PDEs:

$$\begin{aligned}\frac{\partial\mu_{\hat{v}}(t,x)}{\partial t} &= -\nabla\cdot\Big(f(t,x,\mu_{\hat{v}}(t,\cdot),\hat{v}(t,x,\mu_{\hat{v}}(t,\cdot),\nabla\hat{U},\nabla^2\hat{U}))\mu_{\hat{v}}(t,x)\Big) \\ &\quad + \sum_{i,j}\frac{\partial^2}{\partial x_i\partial x_j}\Big(\frac{1}{2}a_{ij}(t,x,\mu_{\hat{v}}(t,\cdot),\hat{v}(t,x,\mu_{\hat{v}}(t,\cdot),\nabla\hat{U},\nabla^2\hat{U}))\mu_{\hat{v}}(t,x)\Big),\end{aligned} \tag{3.127}$$

$$\frac{\partial\hat{U}}{\partial t}(t,x) + H(t,x,\mu_{\hat{v}}(t,\cdot),\nabla\hat{U},\nabla^2\hat{U}) + \int_{\mathbb{R}^d}\frac{\partial H}{\partial m}(t,x,\mu_{\hat{v}}(t,\cdot),\nabla\hat{U},\nabla^2\hat{U},\xi)\mu_{\hat{v}}(t,\xi)\mathrm{d}\xi = 0. \tag{3.128}$$

with boundary conditions

$$\mu_{\hat{v}}(t,x) = \mu_0(x), \tag{3.129}$$

$$\hat{U}(T,x) = \phi(x,\mu_{\hat{v}}(T,\cdot)) + \int_{\mathbb{R}^d}\frac{\partial\phi}{\partial m}(y,\mu_{\hat{v}}(T,\cdot),x)\mu_{\hat{v}}(T,y)\mathrm{d}y. \tag{3.130}$$

3.6.2 Impulsive Mean Field Control

We consider the system

$$\begin{cases}\mathrm{d}x(t) = f(t,x(t),\mu(t,\cdot),u_t)\mathrm{d}t + \sigma(t,x(t),\mu(t,\cdot),u_t)\mathrm{d}W_t, & t\in(0,\tau)\cup(\tau,T), \\ x(\tau^+) = g(x(\tau^-),c),\end{cases} \tag{3.131}$$

where $c\in\mathbb{R}^l$ is the impulse control, and $g(x,c)$ defined on $\mathbb{R}^d\times\mathbb{R}^l$ represents the jump condition of the state when the impulse is applied. The problem is to find a pair $\{u_t, c\}$ in order to minimize the cost functional

$$J(u_t, c) = E\Big\{ \int_0^T L(x(t), \mu(t, \cdot), u_t)\mathrm{d}t + \psi(x(\tau^-), c) + \phi(x(T), \mu(T, \cdot)) \Big\}. \tag{3.132}$$

In mean field control problems, the probability distribution $\mu(t, \cdot)$ plays an important role. Suppose the impulse control is of the form $c = c(x(\tau^-))$. Then this impulsive control does not only gives the jump condition of the state, but it also gives the change in the distribution. Let $h(x)$ be an arbitrary smooth function defined on $\mathbb{R}^d$, then we compute

$$Eh(x(\tau^+)) = \int_{\mathbb{R}^d} h(x)\mu(\tau^+, x)\mathrm{d}x. \tag{3.133}$$

On the other hand we have

$$Eh(x(\tau^+)) = Eh(g(x(\tau^-), c(x(\tau^-)))) = \int_{\mathbb{R}^d} h(g(y, c(y)))\mu(\tau^-, y)\mathrm{d}y. \tag{3.134}$$

Assuming an implicit function $y = y(x)$ is defined by $x = g(y, c(y))$,

$$\int_{\mathbb{R}^d} h(g(y, c(y)))\mu(\tau^-, y)\mathrm{d}y = \int_{\mathbb{R}^d} h(x)\mu(\tau^-, y(x))|y_x(x)|\mathrm{d}x, \tag{3.135}$$

which implies

$$\mu(\tau^+, x) = \mu(\tau^-, y(x))|y_x(x)|, \tag{3.136}$$

where $|y_x(x)|$ is the determinant of the Jacobian matrix.

Remark 3.6.2. In the jump condition, the function $y(x)$ plays a central role in the measure jump, which depends on the choice of the feedback impulse strategy $c(x)$.

As we did in the previous section, we consider a family of control problems indexed by initial condition (t, μ):

$$\text{if } t \geq \tau, \quad \begin{cases} \mathrm{d}x = f(s, x, \mu(s, \cdot), v_s)\mathrm{d}s + \sigma(s, x, \mu(s, \cdot), v_s)\mathrm{d}W_s, \quad s \in (t, T) \\ \mu(t, \cdot) = \mu(\cdot), \end{cases} \tag{3.137}$$

$$\text{if } t < \tau, \quad \begin{cases} \mathrm{d}x = f(s, x, \mu(s, \cdot), v_s)\mathrm{d}s + \sigma(s, x, \mu(s, \cdot), v_s)\mathrm{d}W_s, \quad s \in (t, \tau) \cup (\tau, T) \\ \mu(t, \cdot) = \mu(\cdot), \\ x(\tau^+) = g(x(\tau^-), c), \end{cases} \tag{3.138}$$

with cost functional

$$V(t, \mu) = \inf_{u_s, c} \left\{ E\Big[\int_t^T L(x(s), \mu(s, \cdot), u_s) + \psi(x(\tau^-), c)\delta(s - \tau)\mathrm{d}s + \phi(x(T), \mu(T, \cdot))\Big] \right\}. \tag{3.139}$$

Keep in mind that the flow of the measure $\mu(s, \cdot)$ depends on the control $\{u_s, c\}$. Particularly, the jump of measure from $\mu(\tau^-, \cdot)$ to $\mu(\tau^+, \cdot)$ is determined by the impulse control $c(x)$. We use the notation $\mu_c(\tau^+, \cdot)$ to specify the dependency of the measure on the impulse.

We compute

$$\begin{aligned} V(\tau^-, \mu(\tau^-, \cdot)) &= \inf_{u_s, c} \left\{ E\Big[\psi(x(\tau^-), c) + \int_{\tau^+}^T L(x(s), \mu(s, \cdot), u_s)\mathrm{d}s + \phi(x(T), \mu(T, \cdot))\Big] \right\} \\ &= \inf_c \left\{ \int_{\mathbb{R}^d} \psi(x, c(x))\mu(\tau^-, x)\mathrm{d}x + V(\tau^+, \mu_c(\tau^+, \cdot)) \right\} \end{aligned} \tag{3.140}$$

Suppose $c = \hat{c}(x)$ is the optimal impulse strategy, then it is true that

$$\frac{\mathrm{d}}{\mathrm{d}\theta} \Big\{ \int_{\mathbb{R}^d} \psi(x, \hat{c}(x) + \theta c(x))\mu(\tau^-, x)\mathrm{d}x + V(\tau^+, \mu_{\hat{c}+\theta c}(\tau^+, \cdot)) \Big\}\Big|_{\theta=0} = 0, \tag{3.141}$$

for any function $c(x)$. The derivative (3.141) is computed by

$$\lim_{\theta\to 0}\frac{1}{\theta}\int_{\mathbb{R}^d}\Big(\psi(x,\hat{c}(x)+\theta c(x))-\psi(x,\hat{c}(x))\Big)\mu(\tau^-,x)\mathrm{d}x \qquad (3.142)$$
$$+\lim_{\theta\to 0}\frac{1}{\theta}\Big\{V(\tau^+,\mu_{\hat{c}+\theta c}(\tau^+,\cdot))-V(\tau^+,\mu_{\hat{c}}(\tau^+,\cdot))\Big\}$$

It is clear that

$$\lim_{\theta\to 0}\frac{1}{\theta}\int_{\mathbb{R}^d}\Big(\psi(x,\hat{c}(x)+\theta c(x))-\psi(x,\hat{c}(x))\Big)\mu(\tau^-,x)\mathrm{d}x=\int_{\mathbb{R}^d}\frac{\partial\psi}{\partial c}(x,\hat{c}(x))c(x)\mu(\tau^-,x)\mathrm{d}x. \qquad (3.143)$$

For any fixed function $c(x)$, the implicit function defined by $x = g(y, \hat{c}(y) + \theta c(y))$ depends on θ. Therefore, we will use the notation $y = y(x, \theta)$. We consider the one dimensional case for computing the second term of (3.147):

$$
\begin{aligned}
&\lim_{\theta\to 0}\frac{1}{\theta}\Big(V(\tau^+,\mu_{\hat{c}+\theta c}(\tau^+,\cdot))-V(\tau^+,\mu_{\hat{c}}(\tau^+,\cdot))\Big) \qquad (3.144)\\
&=\lim_{\theta\to 0}\frac{1}{\theta}\int_{\mathbb{R}}\frac{\partial V}{\partial m}(\tau^+,\mu_{\hat{c}}(\tau^+,\cdot),x)\Big[\mu_{\hat{c}+\theta c}(\tau^+,x)-\mu_{\hat{c}}(\tau^+,x)\Big]\mathrm{d}x\\
&=\lim_{\theta\to 0}\frac{1}{\theta}\int_{\mathbb{R}}\frac{\partial V}{\partial m}(\tau^+,\mu_{\hat{c}}(\tau^+,\cdot),x)\Big[\mu(\tau^-,y(x,\theta))y_x(x,\theta)-\mu(\tau^-,y(x,0))y_x(x,0)\Big]\mathrm{d}x\\
&=\int_{\mathbb{R}}\frac{\partial V}{\partial m}(\tau^+,\mu_{\hat{c}}(\tau^+,\cdot),x)\Big[\mu(\tau^-,y(x,0))y_{x\theta}(x,0)+\mu_y(\tau^-,y(x,0))y_\theta(x,0)y_x(x,0)\Big]\mathrm{d}x\\
&=\int_{\mathbb{R}}\frac{\partial V}{\partial m}(\tau^+,\mu_{\hat{c}}(\tau^+,\cdot),x)\mu(\tau^-,y(x,0))y_{x\theta}(x,0)-\frac{\partial}{\partial x}\Big(\frac{\partial V}{\partial m}(\mu_{\hat{c}}(\tau^+,\cdot),x)y_\theta(x,0)\Big)\mu(\tau^-,y(x,0))\mathrm{d}x\\
&=\int_{\mathbb{R}}-\frac{\partial}{\partial x}\Big(\frac{\partial V}{\partial m}(\tau^+,\mu_{\hat{c}}(\tau^+,\cdot),x)\Big)y_\theta(x,0)\mu(\tau^-,y(x,0))\mathrm{d}x\\
&=\int_{\mathbb{R}}-\frac{\partial U}{\partial x}(\tau^+,\mu_{\hat{c}}(\tau^+,\cdot),x)\Big(-\frac{g_c c}{g_y+g_c\hat{c}'}\Big)\mu(\tau^-,y(x,0))\mathrm{d}x\\
&=\int_{\mathbb{R}}-\frac{\partial U}{\partial x}(\tau^+,\mu_{\hat{c}}(\tau^+,\cdot),g(y,\hat{c}(y)))\Big(-\frac{g_c c}{g_y+g_c\hat{c}'}\Big)\mu(\tau^-,y)(g_y+g_c\hat{c}')\mathrm{d}y\\
&=\int_{\mathbb{R}}\frac{\partial U}{\partial x}(\tau^+,\mu_{\hat{c}}(\tau^+,\cdot),g(x,\hat{c}(x)))g_c(x,\hat{c}(x))c(x)\mu(\tau^-,x)\mathrm{d}x,
\end{aligned}
$$

where we used the fact that

$$x=g(y(x,\theta),\hat{c}(y(x,\theta))+\theta c(y(x,\theta))), \qquad (3.145)$$

differentiate with respect to θ,

$$\implies\ 0=g_y y_\theta+g_c(\hat{c}'y_\theta+c+\theta c'y_\theta)\ \implies\ y_\theta(x,0)=-\frac{(g_c c)}{g_y+g_c\hat{c}'}. \qquad (3.146)$$

Therefore the equation (3.141) could be written as

$$\int_{\mathbb{R}}\Big[\frac{\partial\psi}{\partial c}(x,\hat{c}(x))+\frac{\partial U}{\partial x}(\tau^+,\mu_{\hat{c}}(\tau^+,\cdot),g(x,\hat{c}(x)))g_c(x,\hat{c}(x))\Big]c(x)\mu(\tau^-,x)\mathrm{d}x=0. \qquad (3.147)$$

Since $c(x)$ is an arbitrary function, (3.147) implies that

$$\frac{\partial \psi}{\partial c}(x, \hat{c}(x)) + \frac{\partial U}{\partial x}(\tau^+, \mu_{\hat{c}}(\tau^+, \cdot), g(x, \hat{c}(x)))g_c(x, \hat{c}(x)) = 0. \tag{3.148}$$

With $\hat{U}(t, x)$ defined as $U(t, \mu(t, \cdot), x)$, we have

$$\frac{\partial \psi}{\partial c}(x, \hat{c}(x)) + \frac{\partial \hat{U}}{\partial x}(\tau^+, g(x, \hat{c}(x)))g_c(x, \hat{c}(x)) = 0. \tag{3.149}$$

As long as we obtained the optimal impulse from (3.149), the equation (3.140) can be written as

$$V(\tau^-, \mu(\tau^-, \cdot)) = \int_{\mathbb{R}} \psi(x, \hat{c}(x))\mu(t^-, x)\mathrm{d}x + V(\tau^+, \mu_{\hat{c}}(\tau^+, \cdot)). \tag{3.150}$$

Taking the derivative of the equation (3.150) with respect to μ in the direction of $\tilde{\mu}$, the left hand side equals

$$\int_{\mathbb{R}} \frac{\partial V}{\partial m}(\tau^-, \mu(\tau^-, \cdot), x)\tilde{\mu}(x)\mathrm{d}x = \int_{\mathbb{R}} \hat{U}(\tau^-, x)\tilde{\mu}(x)\mathrm{d}x, \tag{3.151}$$

and the derivative of the right hand side splits into two parts, where the first term equals

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}\theta}\Big(\int_{\mathbb{R}} \psi(x, \hat{c}(x))(\mu(t^-, x) + \theta\tilde{\mu}(x))\mathrm{d}x\Big)\Big|_{\theta=0} \\ &= \int_{\mathbb{R}} \psi(x, \hat{c}(x))\tilde{\mu}(x)\mathrm{d}x, \end{aligned} \tag{3.152}$$

while the second term equals

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\theta}\Big(V(\tau^+,(\mu+\theta\tilde{\mu})_{\hat{c}}(\tau^+,\cdot))\Big)\Big|_{\theta=0} \qquad (3.153)\\
&=\frac{\mathrm{d}}{\mathrm{d}\theta}\Big(V(\tau^+,(\mu+\theta\tilde{\mu})(\tau^-,y(\cdot))|y_x|)\Big)\Big|_{\theta=0}\\
&=\frac{\mathrm{d}}{\mathrm{d}\theta}\Big(V(\tau^+,\mu_{\hat{c}}(\tau^+,\cdot)+\theta\tilde{\mu}(\tau^-,y(\cdot))|y_x|)\Big)\Big|_{\theta=0}\\
&=\int_{\mathbb{R}}\frac{\partial V}{\partial m}(\tau^+,\mu_{\hat{c}}(\tau^+,\cdot),x)\tilde{\mu}(\tau^-,y(x))|y_x|\mathrm{d}x\\
&=\int_{\mathbb{R}}\frac{\partial V}{\partial m}(\tau^+,\mu_{\hat{c}}(\tau^+,\cdot),g(y,\hat{c}(y)))\tilde{\mu}(\tau^-,y)\mathrm{d}y\\
&=\int_{\mathbb{R}}\hat{U}(\tau^+,g(y,\hat{c}(y)))\tilde{\mu}(\tau^-,y)\mathrm{d}y
\end{aligned}
$$

Since the direction $\tilde{\mu}$ could be arbitrary, it gives that

$$
\hat{U}(\tau^-,x)=\psi(x,\hat{c}(x))+\hat{U}(\tau^+,g(x,\hat{c}(x))). \qquad (3.154)
$$

Therefore we summarize the necessary condition for the impulse mean field control problem (3.131, 3.132) as

Theorem 3.6.3. *The optimal control $\{\hat{v},\hat{c}\}$ minimizing the cost (3.132) should satisfy*

$$
\hat{v}=arg\min_{v}\Big\{L(x,\mu,v)+\frac{\partial\hat{U}}{\partial x}f(t,x,\mu,v)+\frac{1}{2}\sum_{i,j}\frac{\partial^2\hat{U}}{\partial x_i\partial x_j}a_{ij}(t,x,\mu,v)\Big\}, \qquad (3.155)
$$

$$
\frac{\partial\psi}{\partial c}(x,\hat{c}(x))+\frac{\partial\hat{U}}{\partial x}(\tau^+,g(x,\hat{c}(x)))g_c(x,\hat{c}(x))=0, \qquad (3.156)
$$

where $\mu(t,x)$ and $\hat{U}(t,x)$ satisfy

$$\frac{\partial \mu(t,x)}{\partial t} = -\nabla \cdot \Big(f(t,x,\mu(t,\cdot),\hat{v}(t,x,\mu(t,\cdot),\nabla\hat{U},\nabla^2\hat{U}))\mu(t,x) \Big)$$
$$+ \sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} \Big(\frac{1}{2} a_{ij}(t,x,\mu(t,\cdot),\hat{v}(t,x,\mu(t,\cdot),\nabla\hat{U},\nabla^2\hat{U}))\mu(t,x) \Big), \tag{3.157}$$

$$\frac{\partial \hat{U}}{\partial t}(t,x) + H(t,x,\mu(t,\cdot),\nabla\hat{U},\nabla^2\hat{U}) + \int_{\mathbb{R}^d} \frac{\partial H}{\partial m}(t,x,\mu(t,\cdot),\nabla\hat{U},\nabla^2\hat{U},\xi)\mu(t,\xi)\mathrm{d}\xi = 0. \tag{3.158}$$

with jump conditions

$$\mu(\tau^+,x) = \mu(\tau^-,y(x))|y_x(x)|, \tag{3.159}$$

$$\hat{U}(\tau^-,x) = \psi(x,\hat{c}(x)) + \hat{U}(\tau^+,g(x,\hat{c}(x))), \tag{3.160}$$

where $y(x)$ is defined by the implicit function $x = g(y,c(y))$.

3.7 Numerical Result

Example 3.7.1. We consider the problem: find continuous control $u(t)$ and impulsive control c_k to minimize

$$J = E\Big[\Big(\int_{t_0}^{T} \frac{1}{2}(X(t)^2 + u(t)^2)\mathrm{d}t + \frac{a}{2}c^2\Big) + \frac{1}{2}X(T)^2\Big], \tag{3.161}$$

where $X(t)$ is the process satisfying

$$\begin{cases} \mathrm{d}X = \alpha EX \cdot (1-u)\mathrm{d}t + \sigma \mathrm{d}W, & t \in (t_0,\tau)\cup(\tau,T) \\ X(\tau^+) = X(\tau^-) - c_k(X(\tau^-)). \end{cases} \tag{3.162}$$

By Theorem 3.6.3, the feedback control $u(x)$ has the form

$$u(x) = \alpha EX \frac{\partial V(x,t)}{\partial x}. \tag{3.163}$$

The control problem becomes the forward backward PDE

$$\begin{cases} \frac{\partial V(x,t)}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial x^2} + \frac{\partial V}{\partial x}\alpha EX(1 - \alpha EX\frac{\partial V}{\partial x}) + \frac{1}{2}(x^2 + (\alpha EX\frac{\partial V}{\partial x})^2) \\ \qquad +x\int_{\mathbb{R}} \alpha \frac{\partial V}{\partial x}(\xi,t)\Big(1 - \alpha EX\frac{\partial V}{\partial x}(\xi,t)\Big)m(t,\xi)\mathrm{d}\xi = 0, \qquad t \in (t_0,\tau)\cup(\tau,T), \\ \frac{\partial m(t,x)}{\partial t} = -\frac{\partial}{\partial x}\Big(\alpha EX\cdot(1 - \alpha EX\frac{\partial V}{\partial x}m(t,x))\Big) + \frac{\sigma^2}{2}\frac{\partial^2 m(t,x)}{\partial x^2}, \end{cases} \tag{3.164}$$

with initial condition $m(t_0,x) = m_0(x)$ and terminal condition $V(x,t_N) = \frac{1}{2}x^2$. The impulsive control $c(x)$ satisfies

$$\frac{\partial V}{\partial x}(x - c(x), \tau+) = ac(x). \tag{3.165}$$

The jump of value function $V(x,t)$ and the density $m(t,x)$ at time τ will satisfy

$$V(x,\tau^-) = V(x - c(x),\tau^+) + \tfrac{a}{2}c(x)^2 \tag{3.166}$$

$$m(\tau^+,x) = m(\tau^-,z(x))|z_x(x)|, \tag{3.167}$$

where $y = z(x)$ is the implicit function of $x = y - c(y)$.

Choose $a = 30$, $\alpha = 0.1$, $\sigma = 1$, $m_0(x) = \frac{1}{\sqrt{2\pi \cdot (0.2)^2}} \exp(-\frac{(x-6)^2}{2 \cdot (0.2)^2})$, and $t_0 = 0$, $\tau = 0.5$, $T = 1$.

Table 3.3: List of Cost Tested by Varying Controls

u	c	J
0.0	0.0	42.6538
0.0	0.1	42.5056
0.1	0.0	42.0284
0.5	0.0	39.7232
1.0	0.0	37.2850
2.0	0.0	33.8410
3.0	0.0	32.1914
4.0	0.0	32.2042
3.5	0.0	31.9975
3.4	0.0	32.0037
3.6	0.0	32.0037
3.5	0.05	31.3606
3.5	0.04	31.3095
3.5	0.03	31.3477
$\hat{u}$	$\hat{c}$	30.8517

Remark 3.7.2. In this table, we test the system by using different constant continuous control u and different impulse c, and list the cost corresponding to the control pair. The last row of the table gives the cost under the optimal control. The evolution of the distribution under the optimal control is shown in Figure 3.5 and the jump of the distribution is shown in Figure 3.6. A particular path of optimal control and the corresponding trajectory are shown in Figure 3.8 and Figure 3.7. In Figure 3.10 and Figure 3.9, we give the expected value of optimal control and the expected path of the optimal trajectory.

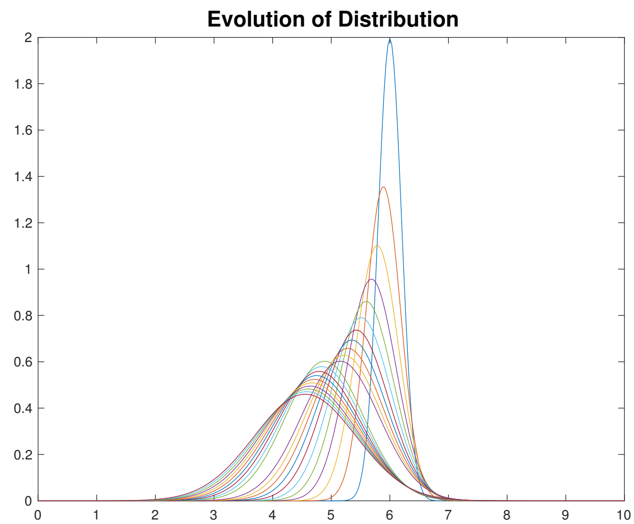

Figure 3.5: Evolution of Distribution

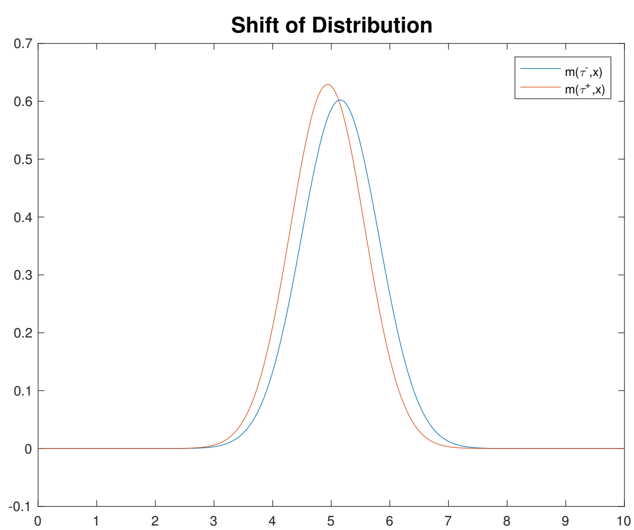

Figure 3.6: Shift of Distribution

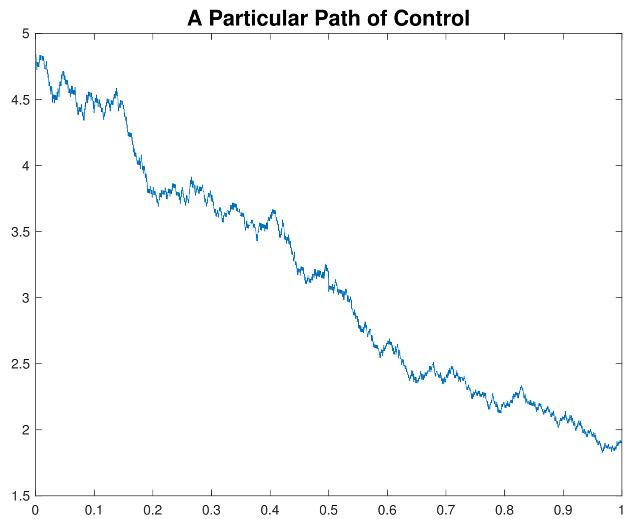

Figure 3.7: A Particular Path of Optimal Control: $\hat{u}(t) = \alpha E X_t \frac{\partial V(X_t,t)}{\partial x}$

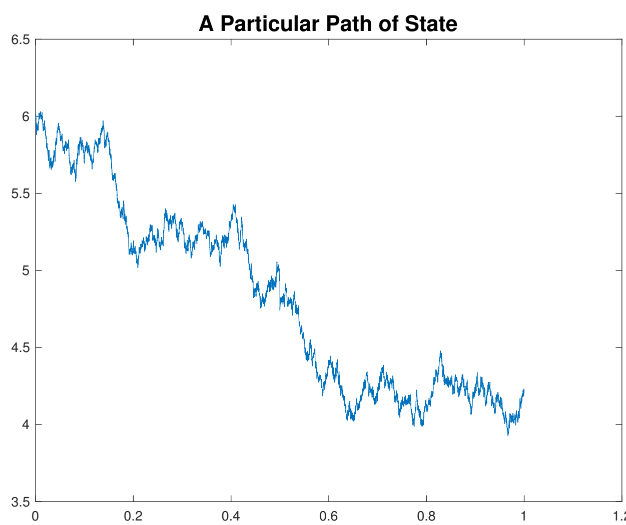

Figure 3.8: An Optimal Path of State

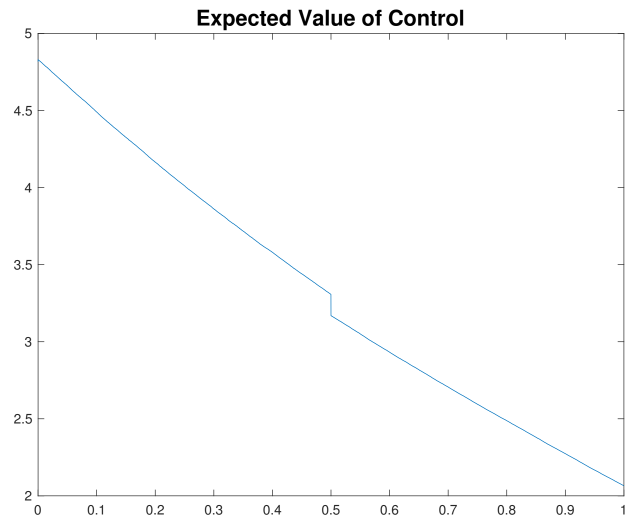

Figure 3.9: Expected Optimal Control

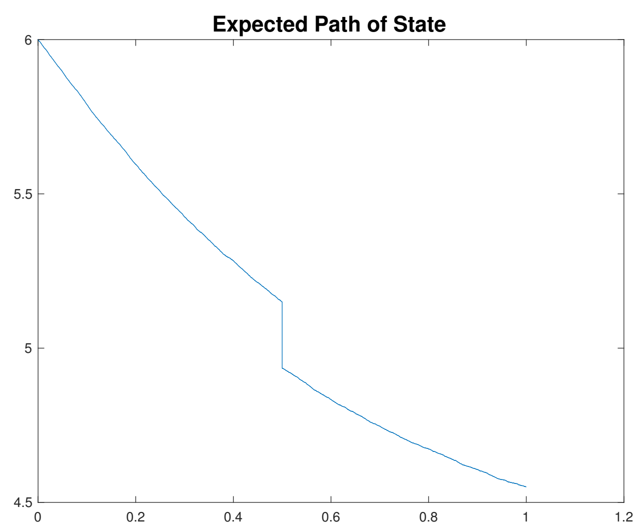

Figure 3.10: Expected Optimal Trajectory: $m(t) = EX_t$

Example 3.7.3. Now we introduce an impulse to the bank model. Consider the process

$$\mathrm{d}x(t) = [a(m(t) - x(t)) + u_t]\mathrm{d}t + \sigma \mathrm{d}W_t \tag{3.168}$$

and the impulse is applied to the system at time τ in the following manner:

$$x(\tau^+) = x(\tau^-) - c. \tag{3.169}$$

We consider the problem of finding the control $\{u(\cdot), c\}$ to minimize the following cost functional:

$$\begin{aligned} J(u_t, c) = E\Big[\int_0^T \frac{1}{2}u_t^2 - qu_t(m_u(t) - x_u(t)) + \frac{\varepsilon}{2}(m_u(t) - x_u(t))^2 \mathrm{d}t \qquad (3.170)\\ +\frac{1}{2}\beta c^2 + \frac{\alpha}{2}(m_u(T) - x_u(T))^2\Big]. \end{aligned}$$

By Theorem 3.6.3, we know that the problem is translated into solving the system of the FBPDEs

$$\frac{\partial \mu}{\partial t} = -\partial_x[((m(t) - x)(a+q) - \partial_x \hat{U})\mu] + \frac{1}{2}\sigma^2 \partial_{xx}\mu \tag{3.171}$$

$$\begin{aligned} \frac{\partial \hat{U}}{\partial t} + (a+q)\partial_x \hat{U}(m(t) - x) + \frac{1}{2}(\varepsilon - q^2)(m(t) - x)^2 - \frac{1}{2}(\partial_x \hat{U})^2 + \frac{1}{2}\sigma^2 \partial_{xx}\hat{U} \quad (3.172)\\ + \int_{\mathbb{R}^d} [(a+2q)\partial_x \hat{U}(t,\xi) + (\varepsilon - q^2)(m(t) - \xi)]\mu(t,\xi)\mathrm{d}\xi = 0, \end{aligned}$$

in the interval $t \in (0,\tau) \cup (\tau, T)$. We already know that $\hat{U}(t,x) = (1/2)\eta(t)(m(t)-x)^2 + \zeta(t)$ satisfies (3.172) for $t \in (\tau, T)$, where $\eta(t), \zeta(t)$ solves

$$
\begin{cases} \dot{\eta}(t) = 2(a+q)\eta(t) + \eta(t)^2 - (\varepsilon - q^2), \\ \dot{\zeta}(t) = -\frac{1}{2}\sigma^2\eta(t), \end{cases} \qquad t \in (\tau, T), \tag{3.173}
$$

with terminal condition $\eta(T) = \alpha$, $\zeta(T) = 0$. The equation which optimal impulse satisfies is written as

$$
\beta c + \eta(\tau^+)(m(\tau^+) - (x - c)) = 0, \tag{3.174}
$$

where the optimal impulse could be obtained as

$$
\hat{c}(x) = -\frac{\eta(\tau^+)(m(\tau^+) - x)}{\beta + \eta(\tau^+)}. \tag{3.175}
$$

The function $y = y(x)$ defined by

$$
x = y - \hat{c}(y) \tag{3.176}
$$

could be solved out as

$$
y(x) = x - \frac{\eta(\tau^+)(m(\tau^+) - x)}{\beta}. \tag{3.177}
$$

Then the the jump condition of the value function could be written as:

$$
\begin{aligned} \hat{U}(\tau^-, x) &= \frac{1}{2}\beta\hat{c}^2(x) + \frac{1}{2}\eta(\tau^+)(m(\tau^+) - (x - \hat{c}(x))) + \zeta(\tau^+) \\ &= \frac{\beta\eta(\tau^+)}{2(\beta + \eta(\tau^+))}(m(\tau^+) - x)^2 + \zeta(\tau^+). \end{aligned} \tag{3.178}
$$

We expect $m(\tau^+) = m(\tau^-)$, such that $\hat{U}(\tau^-, x)$ has the form

$$\hat{U}(\tau^-, x) = \frac{1}{2}\eta(\tau^-)(m(\tau^-) - x)^2 + \zeta(\tau^-). \tag{3.179}$$

In fact

$$\begin{aligned} m(\tau^-) &= \int_{\mathbb{R}} y\mu(\tau^-, y)\mathrm{d}y \qquad (3.180) \\ &= \int_{\mathbb{R}} y(x)\mu(\tau^-, y(x))|y_x|\mathrm{d}x \\ &= \int_{\mathbb{R}} \Big(x - \frac{\eta(\tau^+)(m(\tau^+) - x)}{\beta}\Big)\mu(\tau^+, x)\mathrm{d}x \\ &= m(\tau^+). \end{aligned}$$

Therefore we conclude that $\hat{U}(t, x) = \frac{1}{2}\eta(t)(m(t) - x)^2 + \zeta(t)$ solves (3.172) , where $\eta(t)$ satisfies

$$\dot{\eta}(t) = 2(a + q)\eta(t) + \eta(t)^2 - (\varepsilon - q^2), \qquad t \in (0, \tau) \cup (\tau, T), \tag{3.181}$$

$$\eta(T) = \alpha; \tag{3.182}$$

$$\eta(\tau^-) = \frac{\beta\eta(\tau^+)}{\beta + \eta(\tau^+)} \tag{3.183}$$

and

$$\zeta(t) = \int_t^T \frac{1}{2}\sigma^2\eta(s)\mathrm{d}s. \tag{3.184}$$

Following is a table shows that how the control process u_t and the impulse c works to minimize the cost functional. The coefficient are chosen as: $a = 0.1$, $q = 0.1$, $\beta = 0.1$, $\alpha = 1$, $\varepsilon = 1$, $\sigma = 2$, $\mu_0(x) = \frac{1}{\sqrt{2\pi}}\exp\{-x^2/2\}$.

Table 3.4: Value of Cost Function

	$u_t = 0$, $c = 0$	$u_t = \hat{u}_t$, $c = 0$	$u_t = \hat{u}_t$, $c = \hat{c}$
$J(u_t, c)$	7.972162	3.839801	3.524391

3.8 Summary and Future Work

3.8.1 Summary

In this thesis three types of impulsive optimal control problems are considered. Our primary motivation is control of epidemics models. There are some works in the literature dealing with deterministic impulsive control problems. However, papers that deal with stochastic impulsive control problems are rare. Mean field control problems have recently been dealt with by many authors. To our knowledge impulsive mean field control has not been studied before. In this thesis, the three different types of impulsive control problems have been dealt with in a coherent manner. In each case necessary conditions to characterize the optimal control, and numerical examples to validate the necessary conditions are presented.

Stability properties of SIR models have been studied by many authors. We studied stability properties of both deterministic and stochastic multi-group SIR models. When dealing with migration, the positivity of the populations are ignored by many papers, and the validity of the Lyapunov function also needs justification. In the discussion of the stochastic version, we proved the law of large numbers in two particular forms:

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(t)\mathrm{d}B_t = 0, \quad \text{a.s.}$$
$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(X(t))\mathrm{d}t = \int_{\mathbb{R}} f(x)\nu(x)\mathrm{d}x, \quad \text{a.s.}$$

when the function $f(\cdot)$ and the process $X(t)$ satisfy some conditions. We applied impulsive optimal controls to SIR models, where migration restriction and vaccination are modeled as continuous and impulsive controls respectively.

Mean field controls often arise in studying large numbers of interacting agents. We studied inter-banks lending and borrowing model and we compared Carmona's solution [9] with the solution to the weighted multi-objective control problem and the solution of the representative bank. The solutions to the first two problems converges to the solution of the representative bank when the number of banks tends to infinity. When dealing with the control problem of the dominating player and the representative player, we developed a general conclusion of coupled forward backward optimal control problem. We applied the dominating player game to the inter-banks model and displayed the numerical results.

3.8.2 Future Work

Control problems where impulsive controls are applied at random times are very interesting and need to be considered. Unlike vaccination models, there are many problems where impulsive controls are not applied at known times. In some cases, the time to apply impulsive controls could be considered as a control variable. It will be necessary to find the best timing and optimal control to apply.

In future work we plan to consider epidemic models with large real data. The control strategy could be more than migration restriction and vaccination, and the cost function should be properly modeled.

In mean field problems we are interested in opinion dynamic in social networks. People have to balance seeking consensus and sticking to their own opinions and decisions as they relate to individuals within their own group as well as outside of their group. We also plan to study mean field problems to study problems in economics. Finally we have interest in studying these problems in the framework of variational problems in abstract spaces.

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