Abstract

Gagliardi, Daniel James. Relations between Characters of Lie Algebras and Symmetric Spaces. (Under the direction of Dr. Aloysius Helminck.)

Let Φ be an irreducible root system. The Classification Theorem, ([Hum72, Section 11.4]), then states that its Dynkin diagram must be one of A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, or G_2. This is fundamental to the study of finite-dimensional semisimple Lie algebras over algebraically closed fields. In [Hel88] A. G. Helminck established an analogous result for local symmetric spaces where he identified twenty-four graphical structures called involution or θ-diagrams. Implicit in each of these diagrams are two root systems $\Phi(\mathfrak{a})$ and $\Phi(\mathfrak{t})$ with $\mathfrak{a}$ a maximal torus in a local symmetric space $\mathfrak{p}$ and $\mathfrak{t} \supset \mathfrak{a}$ a maximal torus in the corresponding semisimple Lie algebra $\mathfrak{g}$. In Chapter 2 we describe $\Phi(\mathfrak{a})$ as the image of $\Phi(\mathfrak{t})$ under a projection π derived from an involution θ on $\Phi(\mathfrak{t})$. The weight lattices associated with $\Phi(\mathfrak{t})$ and $\Phi(\mathfrak{a})$ are denoted by $\Lambda_{\mathfrak{t}}$ and $\Lambda_{\mathfrak{a}}$, respectively. We consider a linear extension of π from $\Phi(\mathfrak{t})$ to the lattice $\Lambda_{\mathfrak{t}}$. It was shown, again in [Hel88], that $\pi(\Lambda_{\mathfrak{t}}) \subseteq \Lambda_{\mathfrak{a}}$ for cases where $\Phi(\mathfrak{a})$ is not of type BC_n. In this thesis we prove the converse of this result. For cases where $\Phi(\mathfrak{a})$ is of type BC_n it was shown in this same paper that $\pi(\Lambda_{\mathfrak{t}}) = \Lambda_{\mathfrak{a}} = R_{\mathfrak{a}}$. For these cases we offer a direct proof and for both cases provide explicit formulas for the characters of each in terms of the other.

RELATIONS BETWEEN CHARACTERS OF LIE ALGEBRAS AND SYMMETRIC SPACES

by

Daniel Gagliardi

a dissertation submitted to the graduate faculty of
north carolina state university
in partial fulfillment of the
requirements for the degree of
doctor of philosophy

department of mathematics

raleigh

March 26, 2003

approved by:

chair of advisory committee

To my wife who made this dream possible

And to my advisor who made it a reality

Biography

Daniel Gagliardi was born on Sept. 13, 1959 in Ossining, New York, USA. He received his elementary and secondary education from the Ossining Public School system. He received his Bachelor of Arts degree in Mathematics from Purchase College in 1984 and worked as a computer programmer at IBM until 1992. Daniel then went on to matriculate with Master of Science degree in Mathematics from New Mexico State University in 1993. In the fall of 1997, he entered the Ph.D program in Pure Mathematics with a concentration in Lie Theory at North Carolina State University at Raleigh.

Acknowledgements

First of all I would like to thank my wife Robin for her help and support throughout this entire process, (and for putting up with having math books along on virtually every family vacation!) I would also like to thank my children, Zachary, Michael, and Eric for giving me the inspiration to do my best, (and for using my rough drafts as coloring books giving me some perspective on the relative importance of this work!) I would also like to thank my parents, Marie and Ronald Gagliardi, who have always been there for me and to whom I can always turn. Of course, directly responsible for my success in this endeavor is my friend and thesis advisor, Dr. Aloysius "Loek" Helminck. I am very fortunate to have such an excellent mentor who has helped me in more ways than I can name. I would also like to acknowledge the following people for their direct and/or indirect support, currently and over time: John Arrucci, Patrick Baumgart, Tom Butland, Jeff Cassens, Nancy Castleman, Brenda Currin, Chris Dometrius, Joseph Dunn, Adam Eisenson, Marc Eisenson, Amassa Fauntleroy, Jennifer Fowler, Veronica Gagliardi, Todd Graff, Anne Hartwich, Carol Hurwitz, Nick Jaroff, Kwangil Koh, Marty Lewinter, Jesus Rodriguez, Beverly and Bill Saffron, Denise Seabrook, James Selgrade, and Ling Wu. Finally, I would like to acknowledge, Dr. Ernest "Stitz" Stitzinger, currently the Associate Head of the Graduate Mathematics Program here at NCSU. He is the heart and soul of our department.

Table of Contents

List of Tables

Chapter 1

Introduction

1.1 Symmetric Spaces

Symmetric spaces are of importance in many fields of science, but their main use is in mathematics and physics. They have been studied for over 100 years. Initially they were only studied over the real numbers, but in the last 15 to 20 years symmetric spaces over other fields have become of importance in other areas of mathematics as well.

Symmetric spaces describe a variety of symmetries in nature. In general one can think of them as "nice" spaces acted on by a group, where the group is a set of symmetries or motions (a Lie group). For an easy example consider the group of translations of the euclidean plane. Within mathematics these symmetric spaces are of importance in the study of many areas including differential geometry, representation theory (harmonic analysis on reductive symmetric spaces and the study of Harish Chandra modules and character sheaves), number theory (the study of Trace formulas, automorphic functions, cohomology are arithmetic subgroups), and also in algebraic geometry, invariant theory, and singularity theory.

In this thesis we are concerned with the fine structure of semisimple symmetric spaces. These symmetric spaces can be defined by an involution on a group. In particular, let G be a semisimple algebraic group defined over an algebraically closed field of characteristic not zero. Let θ be an involution in $\mathrm{Aut}(G)$, i.e. θ^2=id $\in \mathrm{Aut}(G)$. Let K be the fixed point group of θ and let

$$P = \{A\theta(A)^{-1} \mid A \in G\}$$

We refer to P as a *symmetric space* and make the remark that $P \cong G/K$. Similarly as for Lie groups we can study the fine structure of a symmetric space locally. The corresponding local symmetric space is defined as follows: Let $\mathfrak{g}$ be the Lie algebra of G and let $d\theta \in \mathrm{Aut}(\mathfrak{g})$ be the involution induced by θ on $\mathfrak{g}$. By abuse of notation we will also write θ for $d\theta$. For this case we let

$$\mathfrak{k} = \{A \in \mathfrak{g} \mid \theta(A) = A\}$$

$$\mathfrak{p} = \{A \in \mathfrak{g} \mid \theta(A) = -A\}$$

We have that $\mathfrak{k}$ and $\mathfrak{p}$ are the tangent spaces in the identity of K and P above. Furthermore as $\theta \in \mathrm{Aut}(\mathfrak{g})$ is a Lie algebra automorphism, then $\mathfrak{k}$, (the $+1$ eigenspace relative to θ), is a subalgebra of $\mathfrak{g}$. While $\mathfrak{p}$, (the -1 eigenspace), is not a subalgebra of $\mathfrak{g}$ we nevertheless have that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with $\mathfrak{k} \perp \mathfrak{p}$ relative to the killing form. We refer to $\mathfrak{p}$ as a *local symmetric space* of $\mathfrak{g}$ relative to θ.

We note that the definition of symmetric space actually extends to base fields k of char $\neq 2$. However, if the char$(k) \neq 0$ we do not have the same relationship between the group and the Lie algebra. Yet, the fine structure of a semisimple algebraic group G and the corresponding symmetric space is independent of the characteristic of the field provided it is $\neq 2$. So all the results in this thesis derived for the fine structure of local symmetric spaces hold for symmetric spaces defined over any algebraically closed field of char $\neq 2$.

1.2 Problem Overview

In Section 2.6 we present the Classification Theorem for semisimple Lie algebras over algebraically closed fields. This theorem states that the Dynkin diagram for any irreducible root system must be one of A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, or G_2. In [Hel88] A. G. Helminck establishes an analogous result for locally symmetric spaces. Identified therein are twenty-four diagrams, (called *θ-diagrams*), that serve to characterize the fine structure of absolutely irreducible semisimple locally symmetric spaces over $k = \overline{k}$. Implicit in each these diagrams is a root system that comes from a maximal torus $\mathfrak{a}$ contained in the symmetric space $\mathfrak{p}$. We denote this root system by $\Phi(\mathfrak{a})$. In general $\Phi(\mathfrak{a})$ is a *reduced root system*, one for which twice (or half) a root may also be a root. When this occurs, then $\Phi(\mathfrak{a})$ is of type BC_n.

In Section 2.9 we describe $\Phi(\mathfrak{a})$ as a projection (derived from θ) of the root system $\Phi(\mathfrak{t})$ of the related Lie algebra $\mathfrak{g}$, where $\mathfrak{t}$ is a maximal torus (in $\mathfrak{g}$) with $\mathfrak{a} \subseteq \mathfrak{t} \subset \mathfrak{g}$. This projection maps the root lattice of the Lie algebra to the root lattice of the symmetric space. Now, denote the weight lattices of the Lie algebra and symmetric space by $\Lambda_{\mathfrak{t}}$ and $\Lambda_{\mathfrak{a}}$, respectively. Extend π linearly to the weight lattice of the Lie algebra so that $\pi(\Lambda_{\mathfrak{t}})$ makes sense. In this thesis we consider the relationship between $\pi(\Lambda_{\mathfrak{t}})$ and $\Lambda_{\mathfrak{a}}$. In particular, from [Hel88] we have that if $\Phi(\mathfrak{a})$ is not of type BC_n, then $\pi(\Lambda_{\mathfrak{t}}) \subseteq \Lambda_{\mathfrak{a}}$, (Theorem 4.1(1)). In Chapter 4 we establish the converse of this theorem with Theorem 4.2. For the cases where $\Phi(\mathfrak{a})$ is of type BC_n, we have already that $\pi(\Lambda_{\mathfrak{t}}) = \Lambda_{\mathfrak{a}}$ $(= R_{\mathfrak{a}})$, (Theorem 4.1(2)). For these cases we offer a direct proof, and for both cases give explicit formulas for the characters of each in terms of the other.

One of the reasons we are interested in such a formulation is to find descriptions which are algorithmic in nature and can be incorporated as an adjoint to a computer algebra package for computations related to symmetric spaces. Indeed, the formulas we derive in this thesis are highly algorithmic and we plan to develop implementations based on them as a sequel to the work here.

Chapter 2

Preliminaries and Recollections

Our basic references for Lie algebras and Representation theory will be the the books of Humphreys [Hum72] and Fulton and Harris [FH91]. We shall follow their notation and terminology. As a preliminary step we consider the root space decomposition of a finite dimensional semisimple Lie algebra with respect to a maximal toral subalgebra.

2.1 Root Space Decomposition

Let $\mathfrak{g}$ be a nonzero semisimple Lie algebra over an algebraically closed field k, with char(k)=0. Let $\mathfrak{t} \subset \mathfrak{g}$ be a maximal toral subalgebra. Then

$$\mathfrak{g} = \mathfrak{t} \oplus \sum_{\alpha \in \Phi(\mathfrak{t})} \mathfrak{g}_\alpha \tag{2.1}$$

where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [y, x] = \alpha(y)x \quad \forall y \in \mathfrak{t}\}$ and $\Phi(\mathfrak{t}) = \{\alpha \in \mathfrak{t}^* \mid \alpha \neq 0, \mathfrak{g}_\alpha \neq 0\}$

The above decomposition of $\mathfrak{g}$ is called the *root space decomposition* of $\mathfrak{g}$ (with respect to $\mathfrak{t}$). That it exists is due to the fact that $\mathfrak{t}$, being *ad*-semisimple, (i.e. ad($\mathfrak{t}$) is semisimple in $\mathfrak{gl}(\mathfrak{g})$ where $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ with $\mathrm{ad}(h)(x) := [h, x]\ \forall\ x \in \mathfrak{g},\ \forall\ h \in \mathfrak{g}$), induces a common eigenspace decomposition of $\mathfrak{g}$. The elements α of $\Phi(\mathfrak{t})$ are called the *roots* of $\mathfrak{g}$ with respect to $\mathfrak{t}$, and $\{\mathfrak{g}_\alpha \mid \alpha \in \Phi(\mathfrak{t})\}$ the associated *root spaces*.

For each $\alpha \in \Phi(\mathfrak{t})$ we have $\dim(\mathfrak{g}_\alpha)= 1$. We also have that $\mathfrak{g}_0 = \mathfrak{t}$. Note that zero is not usually considered a root nor is $\mathfrak{g}_0$ regarded as a root space.

2.2 Root Systems

While $\mathfrak{t}^*$ is already a vector space, (over k), it is our ultimate aim here to consider the roots of $\mathfrak{g}$ as vectors in a *real* euclidean space E. Surprisingly, the process to accomplish this is rather involved, requiring several steps, and is the subject of this section. However, once done, we can apply our knowledge of the geometry and algebra of $\mathbb{R}^n$ to reveal the symmetries and combinatorial structure of these objects. We begin with the following definition.

Definition 1 (Killing form). Let $\mathfrak{g}$ be any Lie algebra over k. We define a bilinear form κ on $\mathfrak{g}$ by,

$$\kappa : \mathfrak{g} \times \mathfrak{g} \to k$$
$$\kappa(x, y) = \mathrm{Tr}(\mathrm{ad}\, x \cdot \mathrm{ad}\, y)$$

Clearly, κ is a symmetric bilinear form. κ is also associative in the sense that $\kappa([xy], z) = \kappa(x, [yz])$. Furthermore, with regard to its action on $\mathfrak{t}$, we have the following result.

Proposition 1 ([Hum72]). *The restriction of κ to $\mathfrak{t}$ is nondegenerate.*

Proposition 1 provides us with a way to identify, by means of an isomorphism, $\mathfrak{t}^*$ with $\mathfrak{t}$. Indeed, for each element $\alpha \in \mathfrak{t}^*$ there exists a unique element $t_\alpha \in \mathfrak{t}$ such that $\alpha(t) = \kappa(t_\alpha, t)$ for all $t \in \mathfrak{t}$, (see [Hum72] section 8.2.) Let

$$\tau : \mathfrak{t}^* \to \mathfrak{t}$$
$$\tau(\alpha) = t_\alpha$$

We see that τ is an isomorphism. In particular, $\tau(\Phi(\mathfrak{t})) = \{t_\alpha \mid \alpha \in \Phi(\mathfrak{t})\}$. With these remarks we justify transfer of the Killing form to $\mathfrak{t}^*$.

Definition 2. Let α and $\beta \in \mathfrak{t}^*$. We define a bilinear form on $\mathfrak{t}^*$,

$$(\cdot,\cdot) : \mathfrak{t}^* \times \mathfrak{t}^* \to k$$
$$(\alpha,\beta) = \kappa(t_\alpha, t_\beta).$$

Again, $(\cdot,\cdot)$ is a symmetric bilinear form inheriting this property from κ, and is non-degenerate by Proposition 1.

We now proceed to construct the Euclidean Space E of our stated aim. We cite from [Hum72] two helpful lemmas.

Lemma 1 ([Hum72, Proposition 8. 3(a)]). *$\Phi(\mathfrak{t})$ spans $\mathfrak{t}^*$ (over $\mathbb{C}$).*

Lemma 2 ([Hum72, Proposition 8. 4(c)]). *If $\alpha, \beta \in \Phi(\mathfrak{t})$ then $\frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$, and $\beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha \in \Phi(\mathfrak{t})$.*

The numbers $\frac{2(\beta,\alpha)}{(\alpha,\alpha)}$ of Lemma 2 are called the *Cartan integers*.

Now, in light of Lemma 1 let $\Delta = \{\alpha_1, \alpha_2, ..., \alpha_n\} \subset \Phi(\mathfrak{t})$ be a basis for $\mathfrak{t}^*$. Define $E_\mathbb{Q}$ as the $\mathbb{Q}$-span of α. We have the following,

Theorem 2.1 ([Hum72]). *Each α in $\Phi(\mathfrak{t})$ is contained in $E_\mathbb{Q}$. Moreover, the $\mathbb{Q}$-dimension of $E_\mathbb{Q}$ equals the $\mathbb{C}$-dimension of $\mathfrak{t}^*$.*

The second assertion follows immediately from the definition of $E_\mathbb{Q}$ and Proposition 1. For the first part, we refer the reader to [Hum72] section 8.5.

The next proposition shows that the form of Definition 2 transferred now to $E_\mathbb{Q}$ has the properties we desire of an inner product. Indeed, thanks to Lemma 2 we have the following,

Proposition 2 ([Hum72]). *The form $(\cdot,\cdot)$ restricted to $E_\mathbb{Q}$ satisfies,*

(1) $(\cdot,\cdot)\mid_{E_\mathbb{Q}}$ is positive definite and symmetric.

(2) For any $\alpha, \beta \in E_{\mathbb{Q}}$ *then* $(\alpha, \beta) \in \mathbb{Q}$.

To complete our construction we extend the base field of our vector space $E_{\mathbb{Q}}$ to the real numbers. Specifically, let $E = \mathbb{R} \otimes_{\mathbb{R}} E_{\mathbb{Q}}$ and let $\iota : E_{\mathbb{Q}} \to E$ be the inclusion map. Extend the form of Proposition 2 canonically to E and we now have, as desired, our real euclidean inner product space to study the geometric properties of the embedded roots. In particular, we can now define the length of vectors in E and measure angles between them. We will put this to good use soon. First, however, we give the properties of $\Phi(\mathfrak{t})$ that will eventually characterize it as a root system.

Theorem 2.2 ([Hum72, Theorem 8.5]). *Let* E, $\Phi(\mathfrak{t})$, $\mathfrak{g}$, $\mathfrak{t}$, *and* $(\cdot, \cdot)$ *be defined as above. Then,*

(1) $\Phi(\mathfrak{t})$ *spans* E, *and* $0 \notin \mathfrak{t}$.

(2) The only multiples of $\alpha \in \Phi(\mathfrak{t})$ *are* ± 1.

(3) For all $\alpha,\beta \in \Phi(\mathfrak{t})$ *then* $\beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha \in \Phi(\mathfrak{t})$.

(4) For all $\alpha,\beta \in \Phi(\mathfrak{t})$ *then* $\frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$.

2.3 Root Systems in $\mathbb{R}^n$

Now, from a purely geometric perspective we consider a fixed euclidean space G endowed with a positive definite symmetric bilinear form. As usual we define the length of a vector $\alpha \in G$ by the formula $\mid \alpha \mid = \sqrt{(\alpha, \alpha)}$ and the angle between two vectors α and β by $\cos(\theta) = \frac{(\alpha,\beta)}{(\alpha,\alpha)}$. A *reflection* in G is an invertible linear transformation leaving pointwise fixed some hyperplane and sending any vector orthogonal to that hyperplane into its negative. Any nonzero vector α determines a reflection s_α with reflecting hyperplane, $P_\alpha = \{\beta \in G \mid (\beta, \alpha) = 0\}$. Explicitly,

$$s_\alpha(\beta) = \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha \tag{2.2}$$

For convenience we introduce the abbreviation,

$$< \beta, \alpha >= \frac{2(\beta, \alpha)}{(\alpha, \alpha)} \tag{2.3}$$

Thus we may rewrite equation 2.2,

$$s_\alpha(\beta) = \beta - < \beta, \alpha > \alpha \tag{2.4}$$

We now write what we mean by a system of roots in $\mathbb{R}^n$.

Definition 3. A subset Φ of the euclidean space G is called a *root system* in G if the following axioms are satisfied:

(1) Φ is finite, spans G, and $0 \notin \Phi$.

(2) The only multiples of $\alpha \in \Phi$ are ± 1.

(3) If $\alpha \in \Phi$, then s_α leaves Φ invariant.

(4) If $\alpha, \beta \in \Phi$ then $< \beta, \alpha > \in \mathbb{Z}$

Now, because of Definition 3.4 the possible angles between pairs of distinct roots is extremely limited. Indeed, let $\alpha, \beta \in \Phi$ with $\alpha \neq \pm\beta$. In light of Equation 2.4 we have that

$$< \beta, \alpha > \cdot < \alpha, \beta >= 4\cos^2(\theta). \tag{2.5}$$

Note that $< \beta, \alpha >$ and $< \alpha, \beta >$ both have the same sign and $0 \leq \cos^2(\theta) \leq 1$. Without loss of generality let $| \beta | \geq | \alpha |$. The condition that $< \beta, \alpha > \in \mathbb{Z}$ then restricts the possible values for θ, $< \beta, \alpha >$, and $< \alpha, \beta >$ to those listed in Table 2.1 from [Hum72, section 9.4].

Now, in light of Definition 3 and Theorem 2.2 we have the following,

Theorem 2.3 ([Hum72]). $\Phi(\mathbf{t})$ *is a root system in* E.

There are exactly four systems of rank 2. They are denoted by, $A_1 \times A_1$, A_2, B_2 and G_2. These arise when the angles between the roots are, respectively, multiples of $\frac{\pi}{2}$, $\frac{\pi}{3}$, $\frac{\pi}{4}$, and $\frac{\pi}{6}$, (see Table 2.1.)

Table 2.1: Possible angles between roots

$<\alpha,\beta>$	$<\beta,\alpha>$	θ	$\mid\beta\mid^2/\mid\alpha\mid^2$
0	0	$\pi/2$	undetermined
1	1	$\pi/3$	1
-1	-1	$2\pi/3$	1
1	2	$\pi/4$	2
-1	-2	$3\pi/4$	2
1	3	$\pi/6$	3
-1	-3	$5\pi/6$	3

Remark 1. In practice it can quite tedious to apply the inner product of Definition 2 to vectors in E. Fortunately, there is not always a need to do so. Indeed, let Φ be a root system in E, (relative to a given bilinear form). Then any other form that leaves the Cartan integers invariant maintains the structure of the original root system changing, in general, only the scale. For example, let Φ be a root system in E relative to the bilinear form F. Let β be a basis for E and let M be the matrix of F with respect to β. Now let c be a real number. Then the bilinear form obtained from the matrix cM magnifies the scale of the Φ by $\mid c\mid$ but leaves it otherwise unchanged.

Theorem 2.3 is of enormous importance to the study of semisimple Lie algebras. Indeed, we have the fact that *every nonzero finite-dimensional semisimple Lie algebra* $\mathfrak{g}$ *is characterized, up to isomorphism, by its system of roots.* Thus, information we obtain regarding the roots of $\mathfrak{g}$ can be transferred back to facilitate understanding of the structure of $\mathfrak{g}$ itself.

2.4 Bases and Weyl Groups

Definition 4. A subset Δ is called a *base* of Φ if

(1) Δ is a basis of E

(2) Each root $\beta\in\Phi$ can be written as $\beta=\sum k_\alpha\,\alpha(\alpha\in\Delta)$ with integral coefficients

k_α all nonnegative or all nonpositive.

The roots in Δ are called *simple* and the *rank* of Φ (i.e. the dimension of E) is equal to the number of roots in Δ. Definition 4.2 induces a natural partition on the set of roots: Let Φ^+ be the set of roots with coefficients, $k_\alpha > 0$. Define Φ^- similarly. Φ^+ and Φ^- are called, respectively, the *positive* and *negative* roots of Φ relative to Δ. Moreover, $\Phi^+ \cap \Phi^- = \emptyset$ and $\Phi = \Phi^+ \cup \Phi^-$.

Definition 5. Let Φ be a root system in E. The *Weyl group* of Φ, denoted by $W(\Phi)$, is the subgroup of $GL(E)$ generated by the reflections, s_α ($\alpha \in \Phi$).

Actually, it turns out that $W(\Phi)$ can be generated by the elements, s_α with $\alpha \in \Delta$. We also have

Theorem 2.4. *Let Δ be a base of Φ. Then,*

(1) If α is any root, there exists $w \in W(\Phi)$ such that $w(\alpha) \in \Delta$.

(2) If Δ' is another base of Φ, then $w(\Delta') = \Delta$ for some $w \in W(\Phi)$ (so W acts transitively on bases).

2.5 Cartan Matrices and Dynkin Diagrams

Fix an ordering $(\alpha_1, \alpha_2, ..., \alpha_n)$ of the simple roots of Φ. The matrix $(<\alpha_i, \alpha_j>)$ with $1 \leq i, j \leq n$ is called the *Cartan matrix* of Φ. Its entries are the Cartan integers, (cf. Lemma 2). Now since $\Delta = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ is a base, then the Cartan Matrix is nonsingular. This is extremely important in all that follows. We also have that the Cartan Matrix characterizes Φ.

Proposition 3 ([Hum72, Proposition 11.1]). *Let $\Phi' \subset E'$ be another root system, with base $\Delta' = \{\alpha'_1, \alpha'_2, \cdots, \alpha'_n\}$. If $<\alpha_i, \alpha_j> = <\alpha'_i, \alpha'_j>$ for $1 \leq i, j \leq n$, then the bijection $\alpha_i \mapsto \alpha'_i$ extends (uniquely) to an isomorphism $\phi : E \to E'$ mapping Φ onto Φ' and satisfying $<\phi(\alpha), \phi(\beta)> = <\alpha, \beta>$ for all $\alpha, \beta \in \Phi$. Therefore, the Cartan matrix of Φ determines Φ up to isomorphism.*

For the systems of rank 2, the matrices are,

$$A_1 \times A_1 \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad A_2 \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$$B_2 \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix} \quad G_2 \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

The matrix, of course, depends on the ordering of the roots *but not on the choice of* Δ. This is due to the fact that for all $w \in W(\Phi)$ $< w(\alpha), w(\beta) >=< \alpha, \beta >$ for all α, $\beta \in \Phi$ and, Theorem 2.4, that $W(\Phi)$ acts transitively on the set of bases of Φ.

Now, If α and β are distinct roots, then we know from Table 1 that $< \alpha, \beta > \cdot < \beta, \alpha > =$ 0,1,2, or 3. Define the *Coxeter Graph* of Φ to be the multigraph having n vertices corresponding to the simple roots of Δ with $< \alpha_i, \alpha_j > \cdot < \alpha_j, \alpha_i >$ edges from v_i to v_j. Now when a double or triple edge occurs in the Coxeter graph of Φ it means that one of the vertices corresponds to a root that is shorter than the other. If we add an arrow pointing to the shorter root we can now recover the Cartan integers without ambiguity. This directed multigraph is called the *Dynkin diagram.* The Dynkin diagrams for each of the matrices above are all directed multigraphs on two nodes, v_1 and v_2. For $A_1 \times A_1$ the Dynkin diagram consists of two isolated nodes. For A_2 it is P_2, the undirected path on two nodes. The diagrams for B_2 (G_2) have two (resp. three) directed edges from v_1 to v_2.

2.6 Irreducible Root Systems and the Characterization Theorem

Let Φ be a root system in a euclidean space E. Φ is called *irreducible* if it (and hence Δ) cannot be partitioned into two proper orthogonal subsets. Obviously the Dynkin diagram is connected (as a multigraph) if and only if Φ is irreducible. (For the two dimensional root systems of the previous section, we see that A_2, B_2, and G_2 are irreducible, while $A_1 \times A_1$ is not.) Let m be the number of connected components in the Dynkin diagram. We have the following proposition.

Proposition 4 ([Hum72, Proposition 11.3]). *Φ decomposes (uniquely) as the union of irreducible root systems Φ_i (in subspaces E_i of E) such that $E = \bigoplus_{i=1}^m E_i$*

In light of this proposition, (and Proposition 3) to characterize any root system, it suffices to classify the irreducible root systems, or equivalently, the connected Dynkin diagrams. In this regard we have the next remarkable result.

Theorem 2.5 ([Hum72, Theorem 11.4]). *If Φ is an irreducible root system of rank n, its Dynkin diagram is one of the diagrams in Table 2.2.*

The restrictions on n for types A_n, B_n, C_n, and D_n are imposed to avoid duplication. Relative to above labeling of the simple roots we have the corresponding Cartan matrices:

$$A_n : \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdots & 0 \\ \vdots & & & \ddots & & & 0 \\ \vdots & & & & \ddots & & 0 \\ 0 & 0 & \cdots & 0 & -1 & 2 & -1 \\ 0 & 0 & \cdots & & & -1 & 2 \end{pmatrix}$$

$$B_n : \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdots & 0 \\ \vdots & & & \ddots & & & 0 \\ \vdots & & & & \ddots & & 0 \\ 0 & 0 & \cdots & 0 & -1 & 2 & -2 \\ 0 & 0 & \cdots & & & -1 & 2 \end{pmatrix}$$

$$C_n : \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdots & 0 \\ \vdots & & & \ddots & & & 0 \\ \vdots & & & & \ddots & & 0 \\ 0 & 0 & \cdots & 0 & -1 & 2 & -2 \\ 0 & 0 & \cdots & & & -2 & 2 \end{pmatrix}$$

$$D_n : \begin{pmatrix} 2 & -1 & 0 & & \cdot & \cdot & & & 0 \\ -1 & 2 & -1 & & \cdot & \cdot & & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & & \cdot & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & & \cdot & 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & & \cdot & 0 & 0 & -1 & 0 & 2 \end{pmatrix}$$

$$E_6 : \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

$$E_7 : \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

$$E_8 : \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

$$F_4 : \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

$$G_2 : \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

2.7 Example: $\mathfrak{sl}_n(\mathbb{C})$

Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$. Recall that $\mathfrak{g}$ is the semisimple Lie algebra of matrices in $M_n(\mathbb{C})$ with trace zero. The product, $[\cdot, \cdot]$, on $\mathfrak{g}$ is just the commutator. That is, for all $x, y \in \mathfrak{g}$

$$[x, y] := xy - yx \tag{2.6}$$

Let $\mathfrak{t} \subset \mathfrak{g}$ be the diagonal matrices. Now define the functionals (on $\mathfrak{t}$), $e_i, i \in \{1, 2, ...n\}$ by the formula, $e_i(\text{diag}(a_1, a_2, \cdots, a_n)) = a_i$. A straightforward computation reveals that $\Phi(\mathfrak{t}) = \{e_i - e_j, 1 \leq i \neq j \leq n\}$ is the set of roots of $\mathfrak{g}$. Now define $\alpha_i = e_i - e_{i+1}$, with $i \in \{1, 2, ..., n-1\}$. The reader can easily verify that, $\Delta = \{\alpha_1, \alpha_2, ..., \alpha_{n-1}\}$ is a base for $\Phi(\mathfrak{t})$. In particular, we have that $\Phi(\mathfrak{t})^+ = \{e_i - e_j \mid 1 \leq i < j \leq n\}$ and $\Phi(\mathfrak{t})^- = \{e_j - e_i \mid 1 \leq i < j \leq n\}$ are, respectively, the positive and negative roots of $\mathfrak{g}$ (with respect to $\mathfrak{t}$ and relative to Δ.)

Now for the root spaces of $\mathfrak{g}$ we have that,

$$\mathfrak{g}_{e_i - e_j} = \text{span}_{\mathbb{C}}\{e_{i,j}\},\ 1 \leq i \neq j \leq n. \tag{2.7}$$

where the matrices $e_{i,j}$ are the usual standard basis vectors of $M_n(\mathbb{C})$. We record this correspondence symbolically in the form of a matrix:

$$\begin{pmatrix}
0 & \mathfrak{g}_{\alpha_1} & \mathfrak{g}_{\alpha_1+\alpha_2} & \cdot & \cdot & \cdot & \cdot & \mathfrak{g}_{\alpha_1+\cdots+\alpha_{n-1}} \\
\mathfrak{g}_{-\alpha_1} & 0 & \mathfrak{g}_{\alpha_2} & \mathfrak{g}_{\alpha_2+\alpha_3} & \cdot & \cdot & \cdot & \mathfrak{g}_{\alpha_2+\cdots+\alpha_{n-1}} \\
\mathfrak{g}_{-\alpha_1-\alpha_2} & \mathfrak{g}_{-\alpha_2} & 0 & \mathfrak{g}_{\alpha_3} & \mathfrak{g}_{\alpha_3+\alpha_4} & \cdot & \cdot & \mathfrak{g}_{\alpha_3+\cdots+\alpha_{n-1}} \\
\cdot & & & \cdot & & & & \cdot \\
\cdot & & & & \cdot & & & \cdot \\
\cdot & & & & & \cdot & & \cdot \\
\mathfrak{g}_{-\alpha_1-\cdots-\alpha_{n-2}} & \mathfrak{g}_{-\alpha_2-\cdots-\alpha_{n-2}} & \cdot & \cdot & \cdot & \cdot & 0 & \mathfrak{g}_{\alpha_{n-1}} \\
\mathfrak{g}_{-\alpha_1-\cdots-\alpha_{n-1}} & \mathfrak{g}_{-\alpha_2-\cdots-\alpha_{n-1}} & \cdot & \cdot & \cdot & \cdot & \mathfrak{g}_{-\alpha_{n-1}} & 0
\end{pmatrix}$$

Now let E' be the span (over $\mathbb{R}$) of $\{e_1, e_2, \cdots, e_n\}$. In this way we identify E' with $\mathbb{R}^n$. Let E be the subspace of E' defined by the quotient,

$$E = \text{span}_{\mathbb{R}}\{\{e_1, e_2, \cdots, e_n\}/(e_1 + \cdots + e_n)\} \tag{2.8}$$

and identify E with $\mathbb{R}^{n-1}$. We claim that $\Delta = \{e_1 - e_2, \cdots, e_{n-1} - e_n\}$ is a basis for E. Indeed, let a_i be scalars in $\mathbb{R}$. Then $a_1(e_1 - e_2) + a_2(e_2 - e_3) + \cdots + a_{n-1}(e_{n-1} - e_n) = 0$ if and only if $a_i = 0$. Now let v be any vector in E' and let S be the one dimensional subspace (of E') generated by $e_1 + e_2 + \cdots + e_n$. We must show that there exists a vector $e \in \text{span}_{\mathbb{R}}(\Delta)$ such that the cosets $\overline{e} = e + S$ and $\overline{v} = v + S$ are equal. In other words that,

$$v = a_1(e_1 - e_2) + a_2(e_2 - e_3) + \cdots + a_{n-1}(e_{n-1} - e_n) + t(e_1 + e_2 + \cdots + e_n) \tag{2.9}$$

has a solution for $a_1, a_2, \cdots, a_{n-1}$, and t in $\mathbb{R}$. Let $v = (b_1, b_2, \cdots, b_n)$. Then Equation

2.9 is equivalent to,

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 1 \\ -1 & 1 & 0 & \cdots & 0 & 1 \\ 0 & -1 & 1 & \cdots & 0 & 1 \\ 0 & 0 & -1 & & 0 & 1 \\ \vdots & & & & & \vdots \\ & & & & 1 & 1 \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{n-1} \\ t \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ \vdots \\ b_n \end{pmatrix} \tag{2.10}$$

As the matrix of Equation 2.10 is nonsingular, (with its determinant equal to n), a solution is guaranteed thereby establishing the claim.

Now, $\Phi(\mathfrak{t})$ is clearly contained in E. Moreover, we have, by Theorem 2.3, that $\Phi(\mathfrak{t})$ is a root system in E relative to the inner product of Section 2.2. However, as mentioned earlier, while this result is of theoretical significance, in practice the computations are forbidding and finding the elements of the Cartan Matrix can be extremely tedious. Instead, we choose an inner product on E defined by the following: Let the inner product on E' be defined, as usual, by the equation,

$$(e_i, e_j) := \delta_{i,j} \tag{2.11}$$

Now, if we project this down to E we obtain,

$$(e_i - e_{i+1}, e_j - e_{j+1}) := 2\delta_{i,j} - \delta_{i,j+1} - \delta_{i+1,j} \tag{2.12}$$

Clearly, it is easier to compute lengths and angles with this choice of inner product. (Indeed, we may regard E as a hyperplane in E' perpendicular to the vector $(1, 1, \cdots, 1)$). But additionally we have the remarkable fact that the inner product here produces the same set of Cartan integers as the inner product of Section 2.2. In particular, we have the Cartan Matrix A_{n-1}. We also have that the Dynkin diagram for $\Phi(\mathfrak{t})$ is P_{n-1}, the undirected path on $n-1$ nodes.

Figure 2.1: Dynkin Diagram for A_{n-1}

A_{n-1} :

2.8 Representations and Weights

We now turn our attention to representations of Lie algebras. Throughout this section we let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra over an algebraically closed field k of characteristic zero.

Definition 6. Let V be a finite dimensional vector space over k. A *representation* of $\mathfrak{g}$ on V is a homomorphism,

$$\rho : \mathfrak{g} \to \mathfrak{gl}(V).$$

where $\mathfrak{gl}(V)$ is the linear Lie algebra of endomorphisms of V. It is immediate that such a map gives V the structure of a $\mathfrak{g}$-module. That is,

(1) $\rho(ax + by)(v) = a\rho(x)(v) + b\rho(y)(v)$

(2) $\rho(x)(av + bw) = a\rho(x)(v) + b\rho(x)(w)$

(3) $\rho([x, y])(v) = \rho(x)(\rho(y)(v)) - \rho(y)(\rho(x)(v))$

for any $x, y \in \mathfrak{g}$, $v, w \in V$, and $a, b \in k$.

Now, let $\mathfrak{t} \subset \mathfrak{g}$ be a maximal torus. If ρ is a representation of $\mathfrak{g}$ on a vector space V, then $\rho(\mathfrak{t})$ is diagonalizable in $\mathfrak{gl}(V)$, [Hum72, corollary 6. 4]. Accordingly, we have the common eigenspace decomposition $V = \oplus V_\lambda$ where λ runs over $\mathfrak{t}^*$ and

$$V_\lambda = \{v \in V \mid \rho(y)(v) = \lambda(y)(v) \ \forall y \in \mathfrak{t}\}. \tag{2.13}$$

As before, whenever V_λ is nontrivial we call it a *weight space*, and λ a *weight* of $\mathfrak{t}$ on V. (Note that the operation of $\mathfrak{t}$ on $\mathfrak{g}$ in Section 2.1 is just the adjoint representation. Thus the root space decomposition of a finite dimensional semisimple Lie algebra over $k = \overline{k}$ is a special case, and is subsumed by the material here).

Now the question arises: How do we locate the weights for an arbitrary representation V other than to say that they are elements of $\mathfrak{t}^*$? We answer this in due course but first we make the following definition.

Definition 7. Let $\Phi(\mathfrak{t})$ be the roots of $\mathfrak{t}$. Let $R_\mathfrak{t}$ be the integer lattice of points spanned by $\Phi(\mathfrak{t})$. We call $R_\mathfrak{t}$ the *root lattice* of $\mathfrak{t}$.

In general, the weights of a representation are "outside the reach" of the root lattice.

Example 2.1. Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. Then every irreducible $\mathfrak{g}$- module has the following structure:

$$V_{-m} \oplus V_{-m+2} \oplus \cdots \oplus V_{m-2} \oplus V_m \tag{2.14}$$

Now the only roots of $\mathfrak{g}$ are the one dimensional vectors, $\{\pm 2\}$. Thus $R_\mathfrak{t}$ here is just the set of points $\{2n \mid n \in \mathbb{Z}\}$. Clearly, if m is odd then the weights $\{-m, -m + 2, \cdots, m-2, m\}$ are not in $R_\mathfrak{t}$.

Let $\Lambda_\mathfrak{t} = \{\lambda \in \mathfrak{t}^* \mid < \lambda, \alpha > \in \mathbb{Z}\ (\alpha \in \Phi(\mathfrak{t}))\}$. We call $\Lambda_\mathfrak{t}$ the *weight lattice* of $\Phi(\mathfrak{t})$. Observe that this lattice is dependent only on the root system, $\Phi(\mathfrak{t})$. It can be shown that the weights for an arbitrary representation ρ of $\mathfrak{g}$ are contained within $\Lambda_\mathfrak{t}$. This definition for $\Lambda_\mathfrak{t}$, however, is of little practical value. Fortunately, we have the next very useful result.

Proposition 5 ([Hum72]). *Let $\Phi(\mathfrak{t})$ be a root system of rank n and let $\Delta(\mathfrak{t})$ be an ordered base for $\Phi(\mathfrak{t})$. Let $C(\Delta)$ be the Cartan matrix for $\Phi(\mathfrak{t})$. Define a set of vectors, $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$, in $\mathfrak{t}^*$ such that the components of λ_i, with respect to $\Delta(\mathfrak{t})$, are given by the i-th row of $C(\Delta)^{-1}$. In other words,*

$$\lambda_i = \sum_{j=1}^{n} (C(\Delta)^{-1})_{ij} \cdot \alpha_j \qquad i \in \{1, 2, \cdots, n\} \tag{2.15}$$

(These are called the fundamental weights relative to $\Delta(\mathfrak{t})$). Then

$$\Lambda_{\mathfrak{t}} = span_{\mathbb{Z}}\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$$

Example 2.2. Consider the root system $\Phi(\mathfrak{t})$ of type A_7 and let $\Delta(\mathfrak{t}) = \{\alpha_1, \alpha_2, \cdots, \alpha_7\}$ be a base. We see from the example of Section 2.7 that this root system arises from the Lie algebra $\mathfrak{g} = \mathfrak{sl}_8(\mathbb{C})$ with $\mathfrak{t} = \{h \in \mathfrak{g} \mid \mathrm{Tr}(h) = 0\}$. We have again from this example that the Cartan matrix is given by

$$A_7 : \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix} \tag{2.16}$$

so that

$$(A_7)^{-1} : \frac{1}{8}\begin{pmatrix} 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 6 & 12 & 10 & 8 & 6 & 4 & 2 \\ 5 & 10 & 15 & 12 & 9 & 6 & 3 \\ 4 & 8 & 12 & 16 & 12 & 8 & 4 \\ 3 & 6 & 9 & 12 & 15 & 10 & 5 \\ 2 & 4 & 6 & 8 & 10 & 12 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix} \tag{2.17}$$

Let $\alpha = \Delta(\mathfrak{t})$. In accordance with Proposition 5 we have that

$$\begin{aligned}
\lambda_1 &= \frac{1}{8}[7,6,5,4,3,2,1]_\alpha \\
\lambda_2 &= \frac{1}{8}[6,12,10,8,6,4,2]_\alpha \\
\lambda_3 &= \frac{1}{8}[5,10,15,12,9,6,3]_\alpha \\
\lambda_4 &= \frac{1}{8}[4,8,12,16,12,8,4]_\alpha \\
\lambda_5 &= \frac{1}{8}[3,6,9,12,15,10,5]_\alpha \\
\lambda_6 &= \frac{1}{8}[2,4,6,8,10,12,6]_\alpha \\
\lambda_7 &= \frac{1}{8}[1,2,3,4,5,6,7]_\alpha
\end{aligned}$$

with

$$\Lambda_{\mathfrak{t}} = \mathrm{span}\{\lambda_1, \lambda_2, \cdots, \lambda_7\}. \tag{2.18}$$

2.9 Local Symmetric Spaces and θ-diagrams

We now turn our attention to a description of symmetric spaces. To motivate this topic we briefly consider symmetric spaces related to Lie groups. In particular, let G be a linear algebraic group. Let θ be an involution in $\mathrm{Aut}(G)$, i.e. θ^2=id $\in \mathrm{Aut}(G)$. Let K be the fixed point group of θ and let

$$P = \{A\theta(A)^{-1} \mid A \in G\}$$

We refer to P as a symmetric space and make the remark that $P \cong G/K$.

Example 2.3. Let $G = \mathrm{GL}_n(\mathbb{R})$ with $\theta(A) = (A^T)^{-1}$. In this case $K = O_n(\mathbb{R})$ and P is the set of symmetric matrices with positive eigenvalues. This example gives rise to the *polar decomposition*: For any $A \in \mathrm{GL}_n(\mathbb{R})$ we have that $A = O \cdot X$ where $O \in O_n(\mathbb{R})$ and $X \in P$.

To define symmetric spaces on Lie algebras we proceed as follows: Let $\theta \in \mathrm{Aut}(\mathfrak{g})$ be an involution. In this case we let

$$\mathfrak{k} = \{A \in \mathfrak{g} \mid \theta(A) = A\}$$

$$\mathfrak{p} = \{A \in \mathfrak{g} \mid \theta(A) = -A\}$$

Here $\mathfrak{p}$ is called a *local symmetric space* of the Lie algebra $\mathfrak{g}$. Clearly, $\mathfrak{k}$ and $\mathfrak{p}$ are, respectively, the $+1$ and -1 eigenspaces of $\mathfrak{g}$ relative to θ. Using these symmetric spaces we obtain a decomposition of $\mathfrak{g}$ reminiscent of the root space decomposition of Section 2.1. Indeed, let $\mathfrak{a}$ be a toral subalgebra maximal in $\mathfrak{p}$. We have that $\mathfrak{a}$ is ad-semisimple as it inherits this property from $\mathfrak{g}$. Then as before we have that

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\omega \in \Phi(\mathfrak{a})} \mathfrak{g}_\omega \tag{2.19}$$

where $\mathfrak{g}_\omega = \{x \in \mathfrak{g} \mid [t, x] = \omega(t)x \ \forall t \in \mathfrak{a}\}$ and $\Phi(\mathfrak{a}) = \{\omega \in \mathfrak{a}^* \mid \omega \neq 0, \mathfrak{g}_\omega \neq 0\}$. In this setting the eigenspaces, $\mathfrak{g}_\omega$, are multidimensional, in general. We also have that $\Phi(\mathfrak{a})$ is a *reduced root system*, one for which twice (or half) a root may also be a root. When this occurs, then $\Phi(\mathfrak{a})$ is of type BC_n. The analysis and explicit formulation for the elements of $\Phi(\mathfrak{a})$ is carried out in [Fowler03].

Analogous to the situation in 2.6 we present the methodology of A. G. Helminck for characterizing the fine structure of irreducible locally symmetric spaces. This characterization is realized through graphical structures derived from Dynkin diagrams. We carry this out in several steps. First, it is a well known fact that for any operator ϕ on a (finite dimensional) vector space V there exists an (induced) operator ϕ^* on the algebraic dual V^*. In fact, let $\beta = \{\beta_1, \beta_2, \cdots, \beta_n\}$ be an ordered base for V and let $\beta' = \{\beta'_1, \beta'_2, \cdots, \beta'_n\}$ be a base for β' dual to β in the sense that $\beta'_i(\beta_j) = \delta_{ij}$. Let $[M]$ be the matrix of ϕ relative to β. Then $[M]^T$ is the matrix of ϕ^* relative to β'. As $\mathfrak{g}$ is itself a finite dimensional vector space, then for any involution $\theta \in \mathrm{Aut}(\mathfrak{g})$, we similarly construct the induced involution $\theta^* \in \mathrm{Aut}(\mathfrak{g}^*)$. To maintain consistency with the literature, we drop the "*" on θ^*. The underlying space will be clear from the context. Next we make the following definitions.

Definition 8. $\Lambda_0(\theta) = \{\lambda \in \Lambda_{\mathfrak{t}} \mid \theta(\lambda) = \lambda\}$ and $\Phi_0(\theta) = \Lambda_0(\theta) \cap \Phi$

Evidently, $\Lambda_0(\theta)$ and $\Phi_0(\theta)$ are, respectively, the characters and roots fixed by θ. Note that $\Phi_0(\theta)$ is itself a root system. Now let π be the canonical projection,

$$\pi : \Lambda_{\mathfrak{t}} \to \Lambda_{\mathfrak{t}}/\Lambda_0(\theta)$$

so that $\overline{\Lambda_{\mathfrak{t}}} = \pi(\Lambda_{\mathfrak{t}})$ and $\overline{\Phi(\mathfrak{t})} = \pi(\Phi(\mathfrak{t}))$.

The next proposition, due to A. G. Helminck, relates the roots of $\mathfrak{t}$ and $\mathfrak{a}$.

Proposition 6 ([Hel88]). *Let $\mathfrak{t}$ and $\mathfrak{a}$ be as above. Then $\Phi(\mathfrak{a}) = \overline{\Phi(\mathfrak{t})}$. Moreover, let*

$$\pi : \Phi(\mathfrak{t}) \to \Phi(\mathfrak{a})$$
$$\pi(\alpha) = \frac{1}{2}(\alpha - \theta(\alpha))$$

Then π is onto and $\Phi(\mathfrak{a}) = \{\pi(\alpha) \mid \pi(\alpha) \neq 0,\ \alpha \in \Phi(\mathfrak{t})\}$.

We now focus on the choice of base for $\Phi(\mathfrak{t})$. In particular, we develop the notion of a "θ-base", i.e. a base for $\Phi(\mathfrak{t})$ that is, in a specific way, compatible with the action of θ. We require the following

Definition 9. An order $\succ$ on $\Phi(\mathfrak{t})$ is called a *θ-order* if whenever $\chi \in \Lambda_{\mathfrak{t}}$, with $\chi \succ 0,\ \chi \notin \Lambda_0(\theta)$, then $\theta(\chi) \prec 0$.

The base we derive from this order is called a θ-base. The next proposition gives a useful characterization.

Proposition 7 ([Hel88]). *Let $\Delta(\mathfrak{t})$ be a base for $\Phi(\mathfrak{t})$. $\Delta(\mathfrak{t})$ is a θ-base if and only if*

(1) $\pi(\Delta(\mathfrak{t}))$ is a base of $\Phi(\mathfrak{a})$.

(2) $\Delta \cap \Phi_0(\theta)$ is a basis of $\Phi_0(\theta)$.

Note that this proposition implies the existence of a θ-order since one can always choose a base for $\Phi(\mathfrak{a})$ and also one for $\Phi_0(\theta)$ thereby inducing a θ-base for $\Phi(\mathfrak{t})$.

For the next step let $W(\mathfrak{t})$ be the Weyl group of $\Phi(\mathfrak{t})$, (Section 2.4). For a subset $S \subset \Phi(\mathfrak{t})$, denote by $W(S)$ the subgroup of $W(\mathfrak{t})$ generated by $s_\alpha,\ \alpha \in S$. Let $W_0(\theta)$ be the Weyl group of $\Phi_0(\theta)$. Then $W_0(\theta) = W(\Phi_0(\mathfrak{t}))$. Now let $\Delta(\mathfrak{t})$ be a θ-base of $\Phi(\mathfrak{t})$ and let $\Delta_0(\theta) = \Delta(\mathfrak{t}) \cap \Phi_0(\theta)$ be a base of $\Phi_0(\theta)$. Let $w_0(\theta)$ be the longest Weyl group element of $W_0(\theta)$ (with respect to the base $\Delta_0(\theta)$. Then

$$w_0(\theta)[\Phi_0^+(\theta)] = \Phi_0^-(\theta)$$

This equation implies that $w_0(\theta)^2$ leaves $\Delta_0(\theta)$ fixed. And since the Weyl group acts simply transitively, we have that $w_0(\theta)^2$ is the identity on $W_0(\theta)$.

For the next step we define (another) map θ^* on $\Phi(\mathfrak{t})$ by

$$\theta^* = -\operatorname{id} \circ \theta \circ w_0(\theta) \tag{2.20}$$

As each involution on the right hand side of Equation 2.20 commutes we have that

$$\theta^* = \begin{cases} \text{id} \\ \text{Dynkin Diagram automorphism of order 2} \end{cases}$$

In light of this and Equation 2.20 we may recover θ. Indeed,

$$\theta = -\operatorname{id} \circ \theta^* \circ w_0(\theta) \tag{2.21}$$

We are now ready to define and describe the θ-diagram.

Definition 10. Let $\Delta(\mathfrak{t})$ be a θ-base for $\Phi(\mathfrak{t})$ and let D be the corresponding Dynkin Diagram. Color black each $\alpha \in \Delta_0(\theta)$. Denote the action of θ^* on D by arrows. We call these enhanced diagrams *involution diagrams* or *θ-diagrams* for short.

First we make the remark that the existence of a θ-order implies the existence of a θ-diagram. Next, in light of Equation 2.21, we can from the θ-diagram, recover the action of θ. Indeed, $\Delta_0(\theta)$ and θ^* are evident upon inspection of the diagram. Then, construct $W_0(\theta)$ and locate within it $w_0(\theta)$. Computing with θ in this way simplifies enormously the task of finding explicit formulas for the projections of the roots defined in Proposition 6. Again we refer the reader to [Fowler03] for the details.

Remark 2. We can associate a θ-diagram with every semisimple local symmetric space by choosing a maximal toral subalgebra $\mathfrak{a}$ contained in $\mathfrak{p}$ and a maximal toral subalgebra $\mathfrak{t}$ containing $\mathfrak{a}$. This will be called the θ-diagram of the triple $(\mathfrak{g}, \mathfrak{t}, \mathfrak{a})$. It was shown in [Hel88] that the θ-diagram of a semisimple local symmetric space is unique up to isomorphism, i.e. it does not depend on our choice of $\mathfrak{a}$ and $\mathfrak{t}$.

In Section 2.6 we presented the irreducible root systems and their Dynkin diagrams. For symmetric spaces we define an *irreducible θ-diagram* to be one for which the root system (and hence the Dynkin diagram of) $\Phi(\mathfrak{a})$ is irreducible. An *absolutely irreducible θ-diagram* is one for which the root systems (and Dynkin diagrams) for both $\Phi(\mathfrak{t})$ and $\Phi(\mathfrak{a})$ are irreducible. We call a θ-diagram *semi-irreducible* if it is irreducible but not absolutely irreducible. The absolutely irreducible and semi-irreducible θ-diagrams are the building blocks for the irreducible θ-diagrams since every irreducible θ-diagram is either an absolutely irreducible θ-diagram or a semi-irreducible θ-diagram. Similar to the Classification Theorem we have from [Hel88] the following landmark result.

Theorem 2.6 ([Hel88]).

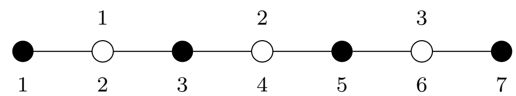
(1) There is a one to one correspondence between semisimple local symmetric spaces and the associated θ-diagrams of the triples $(\mathfrak{g}, \mathfrak{t}, \mathfrak{a})$.

(2) There are, respectively, nine semi-irreducible and twenty-four absolutely irreducible θ-diagrams.

The semi-irreducible θ-diagrams can be constructed in the following way: Let $\mathcal{D}$ be the set of the nine (irreducible) Dynkin diagrams of Section 2.6. Let $D \in \mathcal{D}$ be

Figure 2.2: Subcase A_7^3(II)

A_7^3 :



the Dynkin diagram associated with $\Phi(\mathfrak{t})$. Let θ^* be the diagram automorphism that identifies the corresponding nodes of two copies of D. In this case, then, $\Phi(\mathfrak{a})$ is also of the same type. The twenty-four absolutely irreducible θ-diagrams are given in Tables 2.3 through 2.7.

We proceed with an explanation of the labeling of the θ-diagrams. First observe that the diagram on the left for each case is the θ-diagram. What appears to its right is the Dynkin diagram for the projected reduced root system, $\Phi(\mathfrak{a})$. Let $\Delta(\mathfrak{t}) = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ be a (θ)-base for $\Phi(\mathfrak{t})$ and let $\Delta(\mathfrak{a}) = \{\omega_1, \omega_2, \cdots, \omega_p\}$ be a base for $\Phi(\mathfrak{a})$. Now, from the fact that $\Delta(\mathfrak{t})$ is a θ-base, we have that π maps $\Delta(\mathfrak{t})$ to $\Delta(\mathfrak{a})$. To ascertain the specifics of these projections observe first that labels for the α's do not appear in the θ-diagram at all. These are fixed by convention and purposefully excluded so as to not clutter the θ-diagram, (see Section 2.6.) The numbers that do appear over the nodes of these directed multigraphs are the indices of the ω's in the target. To help avoid any confusion we consider the following

Example 2.4. We consider the case, AII. The θ-diagram for this case is a (modified) path on $2n+1$ nodes. From left to right, these nodes correspond to the simple roots, α_i with $i \in \{1, 2, \cdots, 2n+1\}$. Thus, α_1, is represented by the leftmost node and so on. We see from this diagram that every odd node is colored black. Thus, $\pi(\alpha_k) = 0$ for $k \in \{1, 3, \cdots, 2n+1\}$. On the other hand, for $k = \{2, 4, \cdots, 2n\}$ we have that $\pi(\alpha_k) = \omega_{\frac{k}{2}}$ since these nodes are explicitly affixed with the labels $\frac{k}{2}$. If we apply this to the subcase A_7^3(II), $(n = 3)$, we have that $\pi(\alpha_1) = \pi(\alpha_3) = \pi(\alpha_5) = \pi(\alpha_7) = 0$. And for the nontrivial elements we have that $\pi(\alpha_2) = \omega_1$, $\pi(\alpha_4) = \omega_2$, and $\pi(\alpha_6) = \omega_3$, (see Figure 2.2).

At this point we are now able to define the structures we will need for the main result of this thesis. First, from the linearity of π and Proposition 6 we have that $\overline{R_{\mathfrak{t}}} = R_{\mathfrak{a}}$. We may, however, extend the domain of π to include the weight lattice of the Lie algebra giving meaning to the expression $\pi(\Lambda_{\mathfrak{t}})$. In particular, recall from the last section (Equation 2.15) that

$$\lambda_i = \sum_{j=1}^{n} (C(\Delta)^{-1})_{ij} \cdot \alpha_j \qquad i \in \{1, 2, \cdots, n\}$$

where $C(\Delta)^{-1}$ is the Cartan matrix of $\Phi(\mathfrak{t})$ relative to the ordered base $\Delta(\mathfrak{t})$. We define vectors γ_i with $i \in \{1, 2, \cdots, n\}$ by applying π to both sides of this equation. So that

$$\gamma_i = \pi(\lambda_i) = \sum_{j=1}^{n} (C(\Delta)^{-1})_{ij} \cdot \pi(\alpha_j) \tag{2.22}$$

with

$$\pi(\Lambda_{\mathfrak{t}}) = \text{span}_{\mathbb{Z}}\{\gamma_1, \gamma_2, \cdots, \gamma_n\} \tag{2.23}$$

Next let $\Delta(\mathfrak{a}) = \{\omega_1, \omega_2, \cdots, \omega_p\}$ be a base for $\Phi(\mathfrak{a})$. Let $C(\Delta(\mathfrak{a}))$ be the Cartan matrix of $\Phi(\mathfrak{a})$ relative to $\Delta(\mathfrak{a})$. Using Proposition 5 we define

$$\mu_i = \sum_{j=1}^{p} (C(\Delta(\mathfrak{a}))^{-1})_{ij} \cdot \omega_j \qquad i \in \{1, 2, \cdots, p\} \tag{2.24}$$

We refer to the integer span of the μ's as the *weight lattice of the symmetric space* and denote it by $\Lambda_{\mathfrak{a}}$.

Evidently, $\pi(\Lambda_{\mathfrak{t}})$ is in the $\mathbb{Q}$-span of $\Delta(\mathfrak{a})$. In fact we can say more. Later we will see that $\pi(\Lambda_{\mathfrak{t}}) \subseteq \Lambda_{\mathfrak{a}}$, (see Theorem 4.1(1)). In the next example we compute the γ's and μ's for the special case A_7^3 of AII.

Example 2.5. From Example 2.2 we have that the fundamental weights for this case are given by

$$\begin{aligned}
\lambda_1 &= \frac{1}{8}[7,6,5,4,3,2,1]_\alpha \\
\lambda_2 &= \frac{1}{8}[6,12,10,8,6,4,2]_\alpha \\
\lambda_3 &= \frac{1}{8}[5,10,15,12,9,6,3]_\alpha \\
\lambda_4 &= \frac{1}{8}[4,8,12,16,12,8,4]_\alpha \\
\lambda_5 &= \frac{1}{8}[3,6,9,12,15,10,5]_\alpha \\
\lambda_6 &= \frac{1}{8}[2,4,6,8,10,12,6]_\alpha \\
\lambda_7 &= \frac{1}{8}[1,2,3,4,5,6,7]_\alpha
\end{aligned}$$

Applying π to these weights and using the information of Example 2.4 we obtain

$$\begin{aligned}
\gamma_1 &= \frac{1}{4}[3,2,1]_\omega \\
\gamma_2 &= \frac{1}{4}[6,4,2]_\omega \\
\gamma_3 &= \frac{1}{4}[5,6,3]_\omega \\
\gamma_4 &= \frac{1}{4}[4,8,4]_\omega \\
\gamma_5 &= \frac{1}{4}[3,6,5]_\omega \\
\gamma_6 &= \frac{1}{4}[2,4,6]_\omega \\
\gamma_7 &= \frac{1}{4}[1,2,3]_\omega
\end{aligned}$$

To compute the weight lattice of the symmetric space observe that for this case $\Phi(\mathfrak{a})$

is A_3. Again by invoking Proposition 5 we have that

$$\mu_1 = \frac{1}{4}[3,2,1]_\omega$$
$$\mu_2 = \frac{1}{4}[2,4,2]_\omega$$
$$\mu_3 = \frac{1}{4}[1,2,3]_\omega$$

Now, observe that

$$\gamma_1 = \mu_1$$
$$\gamma_2 = 2\mu_1$$
$$\gamma_3 = \mu_1 + \mu_2$$
$$\gamma_4 = 2\mu_2$$
$$\gamma_5 = \mu_2 + \mu_3$$
$$\gamma_6 = 2\mu_3$$
$$\gamma_7 = \mu_3$$

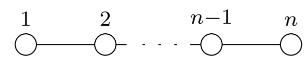
We also have that

$$\mu_1 = \gamma_1$$
$$\mu_2 = -\gamma_1 + \gamma_3$$
$$\mu_3 = \gamma_1 - \gamma_3 + \gamma_5$$

Thus for this (sub)case we have that $\pi(\Lambda_{\mathfrak{t}}) = \Lambda_{\mathfrak{a}}$. That this is always the case is part of the main result of this thesis.

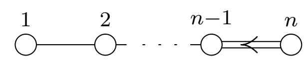
Table 2.2: Dynkin Diagrams of Irreducible Root Systems

A_n $(n \geq 1)$:

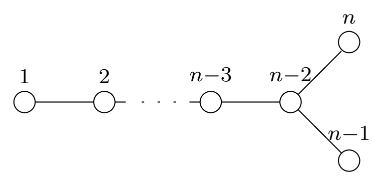


B_n $(n \geq 2)$:

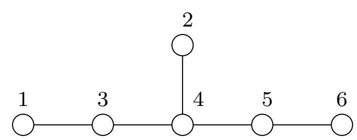
C_n $(n \geq 3)$:



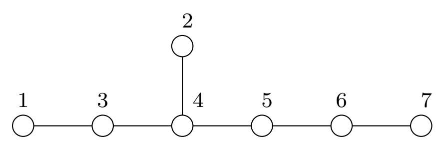
D_n $(n \geq 4)$:



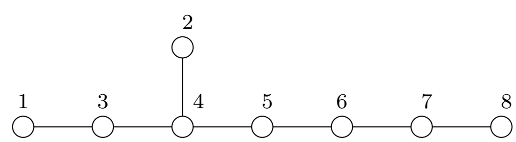
E_6 :



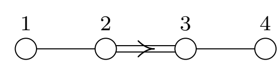
E_7 :



E_8 :



F_4 :



G_2 :

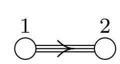


Table 2.3: θ-diagrams

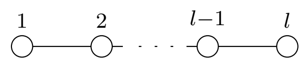
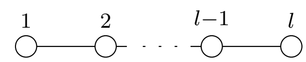
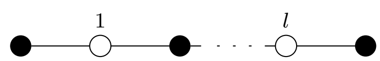
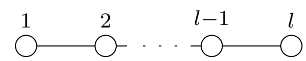
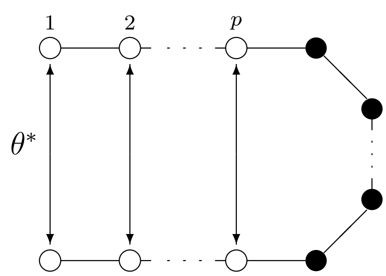
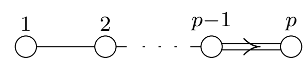
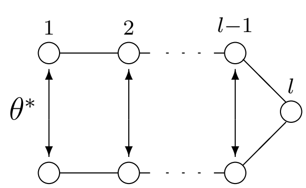
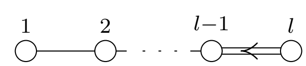
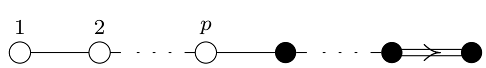
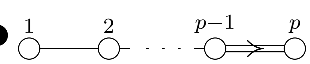
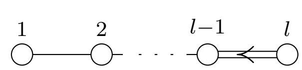
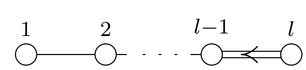
Type θ	θ-Diagram	$\Phi(\mathfrak{a})$
AI		
AII		
AIII$_a$ (AIV ($p=1$)) ($1 \le 2p \le l$)		
AIII$_b$ ($l \ge 2$)		
BI (BII ($p=1$)) ($l \ge 2,\ 1 \le p \le l$)		
CI		

Table 2.4: θ-diagrams (continued)

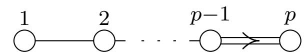
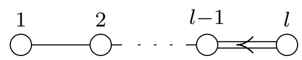
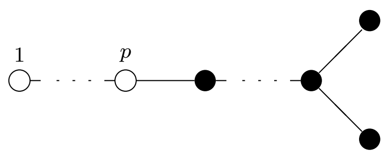
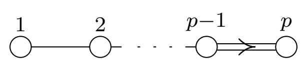
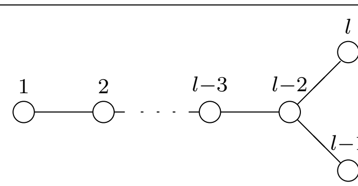
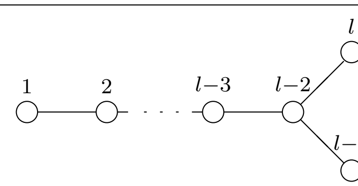
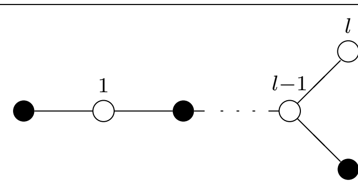
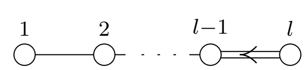
Type θ	θ-Diagram	$\Phi(\mathfrak{a})$
$C\text{II}_a$ $(l \geq 3)$ $(1 \leq p \leq \frac{1}{2}(l-1))$	 I	
$C\text{II}_b$ $(l \geq 2)$		
$\Phi(\mathfrak{a})$:		
$D\text{I}_a$ $D\text{II}$ $(p = 1)$ $l \geq 4$ $(1 \leq p \leq l-1)$		
$\Phi(\mathfrak{a})$:		
$D\text{I}_b$ $(l \geq 4)$		
$D\text{III}_a$ $(l \geq 2)$		

Table 2.5: θ-diagrams (continued)

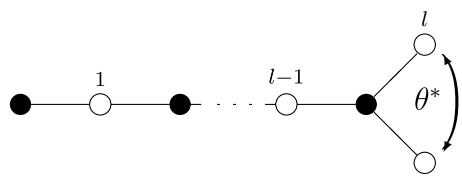
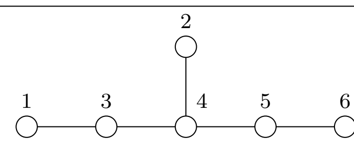
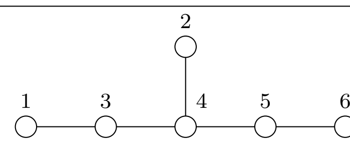
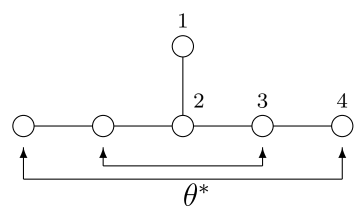
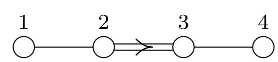
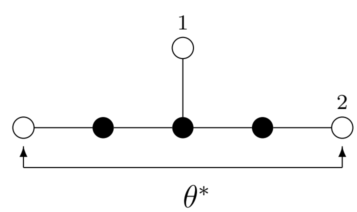
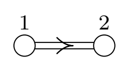
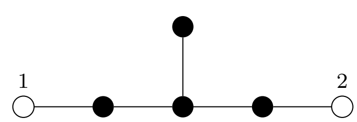
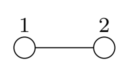
Type θ	θ-Diagram	$\Phi(\mathfrak{a})$
$D\mathrm{III}_b$ ($l \geq 2$)		
$E\mathrm{I}$		
$E\mathrm{II}$		
$E\mathrm{III}$		
$E\mathrm{IV}$		

Table 2.6: θ-diagrams (continued)

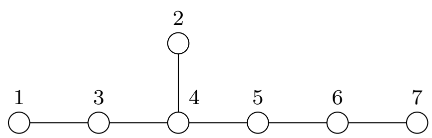
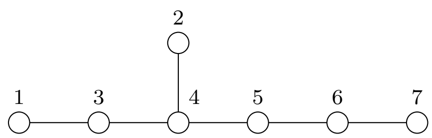
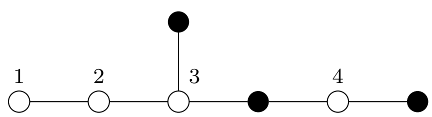
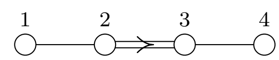
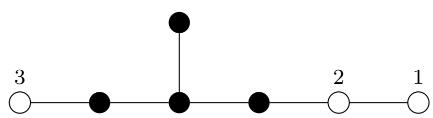
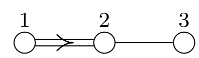
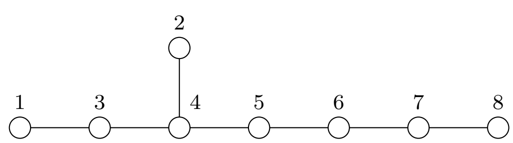
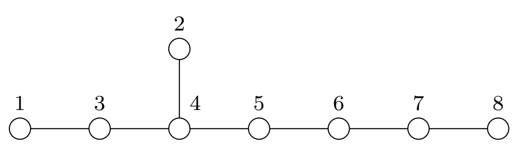
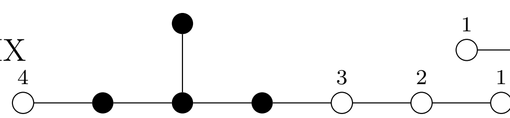
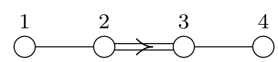
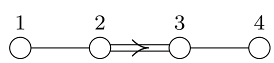
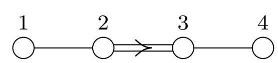
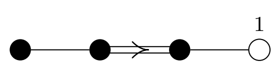
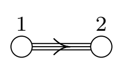
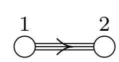
Type θ	θ-Diagram	$\Phi(\mathfrak{a})$
EV		
$\Phi(\mathfrak{a})$:		
EVI		
EVII		
EVIII		
$\Phi(\mathfrak{a})$:		

Table 2.7: θ-diagrams (continued)

Type θ	θ-Diagram	$\Phi(\mathfrak{a})$
EIX		
FI		
FII		
G		

Chapter 3

Inverses of Cartan Matrices

As was mentioned in Section 2.8 expressions for the fundamental weights of an irreducible representation of a Lie Algebra, $\mathfrak{g}$, can be found by inverting the Cartan matrix of $\mathfrak{g}$. In this section we provide explicit formulas for the inverses of each of the Cartan matrices of the Classification Theorem, (Section 2.6).

3.1 Case 1: A_n

Lemma 3. *Consider the Cartan matrix A_n of Section 2.6. Then the inverse is given by,*

$$(A_n^{-1})_{i,j} = \begin{cases} i(1-\frac{j}{n+1}) & i \leq j \\ j(1-\frac{i}{n+1}) & i > j \end{cases} \tag{3.1}$$

Proof. As A_n is invertible, (Section 2.8), it suffices to show that the matrix given in equation 3.1 is a right inverse of A_n. Specifically we need to show that

$$\sum_{k=1}^{n} (A_n)_{l,k} \cdot (A_n^{-1})_{k,j} = \delta_{l,j} \tag{3.2}$$

for all $l, j \in \{1, 2, \cdots, n\}$. Let $l \in \{1, 2, \cdots, n\}$ denote the row of A_n. Observe that the rows of A_n have three obvious partitions: We have the top row with $l = 1$; the middle rows where $2 \leq l < n - 1$; and finally, the last row with $l = n$.

Figure 3.1: Partitions of the rows of A_7

2	-1	0	0	0	0	0
-1	2	-1	0	0	0	0
0	-1	2	-1	0	0	0
0	0	-1	2	-1	0	0
0	0	0	-1	2	-1	0
0	0	0	0	-1	2	-1
0	0	0	0	0	-1	2

Consistent with this partition we structure our proof. In particular, if $l = 1$ then Equation 3.2 is upheld as

$$2(1 - \frac{1}{n+1}) - (1 - \frac{2}{n+1}) = 1 \qquad j = 1$$
$$2(1 - \frac{j}{n+1}) - 2(1 - \frac{j}{n+1}) = 0 \qquad 2 \leq j \leq n$$

For $2 \leq l \leq n-1$ we have,

$$-j(1 - \frac{l-1}{n+1}) + 2j(1 - \frac{l}{n+1}) - j(1 - \frac{l+1}{n+1}) = 0 \qquad j < l$$
$$-1(l-1)(1 - \frac{j}{n+1}) + 2l(1 - \frac{j}{n+1}) - j(1 - \frac{l+1}{n+1}) = 1 \qquad j = l$$
$$-(l-1)(1 - \frac{j}{n+1}) + 2l(1 - \frac{j}{n+1}) - (l+1)(1 - \frac{j}{n+1}) = 0 \qquad j > l$$

Finally for $l = n$ the product of Equation 3.2 evaluates to

$$-j(1 - \frac{n-1}{n+1}) + 2j(1 - \frac{n}{n+1}) = 0 \qquad j < n$$
$$-(n-1)(1 - \frac{n}{n+1}) + 2n(1 - \frac{n}{n+1}) = 1 \qquad j = n$$

In each of these cases we see that Equation 3.2 is upheld thus completing the proof. □

The inverses for the A_n cases have a particularly nice form. After factoring out $\frac{1}{n+1}$ from each element of the matrix we have that each of the columns, starting from the right and up to the diagonal, have successive multiples of the integers $n - j + 1$.

For example for $n = 7$ we have that

$$(A_7)^{-1} : \frac{1}{8}\begin{pmatrix} 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 6 & 12 & 10 & 8 & 6 & 4 & 2 \\ 5 & 10 & 15 & 12 & 9 & 6 & 3 \\ 4 & 8 & 12 & 16 & 12 & 8 & 4 \\ 3 & 6 & 9 & 12 & 15 & 10 & 5 \\ 2 & 4 & 6 & 8 & 10 & 12 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \end{pmatrix} \tag{3.3}$$

3.2 Case 2: B_n

Lemma 4. *Consider the Cartan matrix B_n of Section 2.6. Then the inverse, B_n^{-1}, is the matrix given by,*

$$(B_n^{-1})_{i,j} = \begin{cases} i & i < j \leq n \\ j & j \leq i < n \\ \frac{j}{2} & i = n \end{cases} \tag{3.4}$$

Proof. As before we show that the above matrix is a right inverse of its counterpart B_n, i.e. that

$$\sum_{k=1}^{n}(B_n)_{l,k} \cdot (B_n^{-1})_{k,j} = \delta_{l,j}. \tag{3.5}$$

for every $l, j \in \{1, \cdot, n\}$. Again we consider several separate cases depending on the value of l. For $l = 1$ Equation 3.5 is clearly upheld for any $j \in \{1, \cdots, n\}$. If $2 \leq l \leq n-2$ the product on the left hand side of 3.5 evaluates to

$$\begin{aligned} -j + 2j - j &= 0 & j &< l \\ -(l-1) + 2l - l &= 1 & j &= l \\ -(l-1) + 2l - (l+1) &= 0 & j &> l \end{aligned}$$

Now, if $l = n - 1$ then

$$\begin{aligned} -j + 2j - 2\frac{j}{2} &= 0 & j < n-1 \\ -(n-2) + 2(n-1) - 2\frac{n-1}{2} &= 1 & j = n-1 \\ -(n-2) + 2(n-1) - 2\frac{n}{2} &= 0 & j = n \end{aligned}$$

Finally, for $l = n$ then

$$\begin{aligned} -j + 2\frac{j}{2} &= 0 & j < n-1 \\ -(n-1) + 2\frac{n-1}{2} &= 0 & j = n-1 \\ -(n-1) + 2\frac{n}{2} &= 1 & j = n \end{aligned}$$

Again in each instance Equation 3.5 is upheld. □

For an example, let $n = 6$. We then have that

$$(B_7)^{-1} : \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 & 5 \\ \frac{1}{2} & 1 & \frac{3}{2} & 2 & \frac{5}{2} & 3 \end{pmatrix} \tag{3.6}$$

3.3 Case 3: C_n

Recall from Section 2.6 that $C_n = (B_n)^t$. It follows that $C_n^{-1} = (B_n^{-1})^t$. For completeness we record a formula in the lemma below.

Lemma 5. *Let C_n be the Cartan matrix of Section 2.6. Then the inverse, C_n^{-1}, is the matrix given by,*

$$(C_n^{-1})_{i,j} = \begin{cases} j & j < i \leq n \\ i & i \leq j < n \\ \frac{i}{2} & j = n \end{cases} \tag{3.7}$$

Here for an example we again let $n = 6$. Notice that $C_7^{-1} = (B_7^{-1})^t$.

$$(C_7)^{-1} : \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \frac{1}{2} \\ 1 & 2 & 2 & 2 & 2 & 1 \\ 1 & 2 & 3 & 3 & 3 & \frac{3}{2} \\ 1 & 2 & 3 & 4 & 4 & 2 \\ 1 & 2 & 3 & 4 & 5 & \frac{5}{2} \\ 1 & 2 & 3 & 4 & 5 & 3 \end{pmatrix} \tag{3.8}$$

3.4 Case 4: D_n

Lemma 6. *Consider the Cartan matrix D_n of Section 2.6. Then the inverse of this matrix is given by*

$$(D_n^{-1})_{i,j} = \begin{cases} i & i \leq j \leq n-2 \\ j & j < i \leq n-2 \\ \frac{j}{2} & i \in \{n-1, n\}, j \leq n-2 \\ \frac{i}{2} & j \in \{n-1, n\}, i \leq n-2 \\ \frac{n}{4} & i = n-1, j = n-1 \\ \frac{n-2}{4} & i = n-1, j = n \\ \frac{n-2}{4} & i = n, j = n-1 \\ \frac{n}{4} & i = n, j = n \end{cases} \tag{3.9}$$

Proof. Again we show that the above matrix is a right inverse of its counterpart D_n, i.e. that

$$\sum_{k=1}^{n} (D_n)_{l,k} \cdot (D_n^{-1})_{k,j} = \delta_{l,j} \tag{3.10}$$

for all $l, j \in \{1, \cdots, n\}$. Once more we consider several cases. First, if $l = 1$ then Equation 3.10 is clearly upheld for any $j \in \{1, \cdots, n\}$. Now if $2 \leq l \leq n-3$ then the product of Equation 3.10 becomes

$$\begin{aligned} -j + 2j - j &= 0 && j < l \\ -(l-1) + 2l - l &= 1 && j = l \\ -(l-1) + 2l - (l+1) &= 0 && j > l \end{aligned}$$

Next if $l = n - 2$ then

$$\begin{aligned} -j + 2j - \frac{j}{2} - \frac{j}{2} &= 0 && j \le n-4 \\ -(n-3) + 2(n-3) - \frac{n-3}{2} - \frac{n-3}{2} &= 0 && j = n-3 \\ -(n-3) + 2(n-2) - \frac{n-2}{2} - \frac{n-2}{2} &= 1 && j = n-2 \\ -\frac{n-3}{2} + 2\frac{n-2}{2} - \frac{n}{4} - \frac{n-2}{4} &= 0 && j = n-1 \\ -\frac{n-3}{2} + 2\frac{n-2}{2} - \frac{n-2}{4} - \frac{n}{4} &= 0 && j = n \end{aligned}$$

For if $l = n - 1$ we have that

$$\begin{aligned} -j + 2\frac{j}{2} &= 0 && j \le n-3 \\ -(n-2) + 2\frac{n-2}{2} &= 0 && j = n-2 \\ -\frac{n-2}{2} + 2\frac{n}{4} &= 1 && j = n-1 \\ -\frac{n-2}{2} + 2\frac{n-2}{4} &= 0 && j = n \end{aligned}$$

Finally, if $l = n$ the product of equation 3.10 evaluates to

$$\begin{aligned} -j + 2\frac{j}{2} &= 0 && j \le n-3 \\ -(n-2) + 2\frac{n-2}{2} &= 0 && j = n-2 \\ -\frac{n-2}{2} + 2\frac{n-2}{4} &= 0 && j = n-1 \\ -\frac{n-2}{2} + 2\frac{n}{4} &= 1 && j = n \end{aligned}$$

Once again Equation 3.10 is upheld in every case thus completing the proof. □

Remark 3. Observe from the above formulas that $(A_n)^{-1}$ and $(D_n)^{-1}$ are symmetric matrices. This is expected of course as their respective counterparts A_n and D_n are themselves symmetric.

We present D_7^{-1} as an example.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \frac{1}{2} & \frac{1}{2} \\ 1 & 2 & 2 & 2 & 2 & 1 & 1 \\ 1 & 2 & 3 & 3 & 3 & \frac{3}{2} & \frac{3}{2} \\ 1 & 2 & 3 & 4 & 4 & 2 & 2 \\ 1 & 2 & 3 & 4 & 5 & \frac{5}{2} & \frac{5}{2} \\ \frac{1}{2} & 1 & \frac{3}{2} & 2 & \frac{5}{2} & \frac{7}{4} & \frac{5}{4} \\ \frac{1}{2} & 1 & \frac{3}{2} & 2 & \frac{5}{2} & \frac{5}{4} & \frac{7}{4} \end{pmatrix} \tag{3.11}$$

3.5 The Remaining Cases: E_6, E_7, E_8, F_4, and G_2

Recall that the matrices of this section are all of fixed dimension and we simply compute the inverse for each case. First for the matrices E_n with $n = 6$, 7, and 8, we have that the inverses are respectively,

$$(E_6)^{-1} = \begin{pmatrix} \frac{4}{3} & 1 & \frac{5}{3} & 2 & \frac{4}{3} & \frac{2}{3} \\ 1 & 2 & 2 & 3 & 2 & 1 \\ \frac{5}{3} & 2 & \frac{10}{3} & 4 & \frac{8}{3} & \frac{4}{3} \\ 2 & 3 & 4 & 6 & 4 & 2 \\ \frac{4}{3} & 2 & \frac{8}{3} & 4 & \frac{10}{3} & \frac{5}{3} \\ \frac{2}{3} & 1 & \frac{4}{3} & 2 & \frac{5}{3} & \frac{4}{3} \end{pmatrix} \tag{3.12}$$

$$(E_7)^{-1} = \begin{pmatrix} 2 & 2 & 3 & 4 & 3 & 2 & 1 \\ 2 & \frac{7}{2} & 4 & 6 & \frac{9}{2} & 3 & \frac{3}{2} \\ 3 & 4 & 6 & 8 & 6 & 4 & 2 \\ 4 & 6 & 8 & 12 & 9 & 6 & 3 \\ 2 & \frac{9}{2} & 6 & 9 & \frac{15}{2} & 5 & \frac{5}{2} \\ 2 & 3 & 4 & 6 & 5 & 4 & 2 \\ 1 & \frac{3}{2} & 2 & 3 & \frac{5}{2} & 2 & \frac{3}{2} \end{pmatrix} \tag{3.13}$$

$$(E_8)^{-1} = \begin{pmatrix} 4 & 5 & 7 & 10 & 8 & 6 & 4 & 2 \\ 5 & 8 & 10 & 15 & 12 & 9 & 6 & 3 \\ 7 & 10 & 14 & 20 & 16 & 12 & 8 & 4 \\ 10 & 15 & 20 & 30 & 24 & 18 & 12 & 6 \\ 8 & 12 & 16 & 24 & 20 & 15 & 10 & 5 \\ 6 & 9 & 12 & 18 & 15 & 12 & 8 & 4 \\ 4 & 6 & 8 & 12 & 10 & 8 & 6 & 4 \\ 2 & 3 & 4 & 6 & 5 & 4 & 3 & 2 \end{pmatrix} \tag{3.14}$$

For the remaining matrices we have that

$$(F_4)^{-1} = \begin{pmatrix} 2 & 3 & 4 & 2 \\ 3 & 6 & 8 & 4 \\ 2 & 4 & 6 & 3 \\ 1 & 2 & 3 & 2 \end{pmatrix} \tag{3.15}$$

and

$$(G_2)^{-1} = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}. \tag{3.16}$$

Chapter 4

Relations between Characters of Lie Algebras and Symmetric Spaces

Let $\Lambda_{\mathfrak{t}}$, $\Lambda_{\mathfrak{a}}$, and $R_{\mathfrak{a}}$ be the lattices of Section 2.9. On their inter-relationship we have the following theorem due to A. G. Helminck.

Theorem 4.1 ([Hel88]). *Let $\Phi(\mathfrak{t})$ and its projection $\Phi(\mathfrak{a})$ be irreducible. If*

(1) $\Phi(\mathfrak{a})$ is not of type BC_n then $R_{\mathfrak{a}} = \pi(R_{\mathfrak{t}}) \subseteq \pi(\Lambda_{\mathfrak{t}}) \subseteq \Lambda_{\mathfrak{a}}$.

(2) $\Phi(\mathfrak{a})$ is of type BC_n then $\pi(\Lambda_{\mathfrak{t}}) = R_{\mathfrak{a}} = \Lambda_{\mathfrak{a}}$.

The main result of this thesis is the converse of Theorem 4.1.(1).

Theorem 4.2. *Let $\Phi(\mathfrak{t})$ and its projection $\Phi(\mathfrak{a})$ be irreducible. Then $\pi(\Lambda_{\mathfrak{t}}) = \Lambda_{\mathfrak{a}}$.*

We not only seek to establish this result but also to provide explicit formulas for the characters of each in terms of the other. In addition, for the cases where $\Phi(\mathfrak{a})$ is of type BC_n, we offer a direct proof of Theorem 4.1.(2) again with formulas for the characters.

Remark 4. For root systems of type E_8, F_4, or G_2 the weight lattice and the root lattice of $\Phi(\mathfrak{t})$ are the same, i.e. $\Lambda_{\mathfrak{t}} = R_{\mathfrak{t}}$. For either one of these we have as a direct consequence of Theorem 4.2, (and Proposition 6), that the same result is true for $\Phi(\mathfrak{a})$, i.e. that $\Lambda_{\mathfrak{a}} = R_{\mathfrak{a}}$, which of course is, in general, a root system of a different type.

Our proof will be comprised of twenty-four parts corresponding to each of the absolutely irreducible theta diagrams of Section 2.9. There are, however, two basic proof strategies depending on whether or not $\Phi(\mathfrak{a})$ is of type BC_n. We provide an outline for each type.

Type 1. $\Phi(\mathfrak{a})$ is not of type BC_n

Step 1 Use Proposition 5 to compute the fundamental dominant weights, $\lambda_i,\ i \in \{1, \cdots, n\}$ with respect to the base $\Delta(\mathfrak{t}) = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$. Recall that the integer span of these vectors is the lattice $\Lambda_\mathfrak{t}$, (see Section 2.8, Equation 2.15.)

Step 2 Use the implicitly defined projection map of the θ-diagram to compute $\gamma_i = \pi(\lambda_i),\ i \in \{1, 2, \cdots, n\}$ with respect to the base $\Delta(\mathfrak{a}) = \{\omega_1, \omega_2, \cdots, \omega_p\}$ where $p =\mid \Delta(\mathfrak{a}) \mid$. The integer span of the γ's is the lattice $\pi(\Lambda_\mathfrak{t})$, (see Section 2.9, Equation 2.22.)

Step 3 Again use Proposition 5 to compute the fundamental dominant weights $\mu_i,\ i \in \{1, 2, \cdots, p\}$ with respect to the base $\Delta(\mathfrak{a}) = \{\omega_1, \omega_2, \cdots, \omega_p\}$. Recall that the integer span of the μ's is the lattice $\Lambda_\mathfrak{a}$, (see Section 2.9, Equation 2.24.)

Step 4 Show explicitly that $\pi(\Lambda_\mathfrak{t}) \subseteq \Lambda_\mathfrak{a}$ by finding integers b_i such that $\gamma_i = \sum_1^p b_i\mu_i$ $\forall j = \{1, 2, \cdots, n\}$. That this is possible is guaranteed by Theorem 4.1(1).

Step 5 Use the result of Step 4 to prove the theorem for the (particular) symmetric space; i.e. show that $\pi(\Lambda_\mathfrak{t}) \supseteq \Lambda_\mathfrak{a}$ explicitly by finding integers a_i such that $\mu_i = \sum_1^p a_i\gamma_i$ $\forall j = \{1, 2, \cdots, p\}$.

Type 2. $\Phi(\mathfrak{a})$ is of type BC_n

Step 1 As before use Proposition 5 to compute the fundamental dominant weights, $\lambda_i,\ i \in \{1, \cdots, n\}$ with respect to the base $\Delta(\mathfrak{t}) = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$.

Step 2 Again use θ-diagram to compute $\gamma_i = \pi(\lambda_i),\ i \in \{1, 2, \cdots, n\}$ with respect to the base $\Delta(\mathfrak{a}) = \{\omega_1, \omega_2, \cdots, \omega_p\}$. Show that the coefficients for the γ's relative to $\Phi(\mathfrak{a})$ are integers. This gives immediately an explicit formula verifying that $\pi(\Lambda_\mathfrak{t}) \subseteq R_\mathfrak{a} = \Lambda_\mathfrak{a}$ as guaranteed by Theorem 4.1(2).

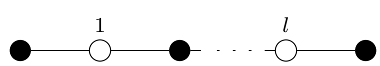
Step 3 Use the result of Step 2 to verify again Theorem 4.1(2), (this time in the other direction), with an explicit formulation to show that $\pi(\Lambda_{\mathfrak{t}}) \supseteq R_{\mathfrak{a}} = \Lambda_{\mathfrak{a}}$; i.e. find integers a_j such that $\omega_i = \sum_1^n a_j \gamma_j \forall i = \{1, 2, \cdots, p\}$.

We begin our proof with the following remark.

Remark 5. Recall that an involution θ of $\mathfrak{g}$ is called θ-split if $\mathfrak{a} = \mathfrak{t}$. From [Hel88] it follows that for each type of simple Lie algebra there exists a unique isomorphy class of θ-split involutions. In these cases $\Phi(\mathfrak{a}) = \Phi(\mathfrak{t})$. They are as follows: AI, BI ($p = n$), CI, DI$_b$, EI, EV, EVIII, G, and FI. For each of these cases we have that π is identity so that $\pi(\Phi(\mathfrak{t})) = \Phi(\mathfrak{a})$ and $\pi(\Lambda_{\mathfrak{t}}) = \Lambda_{\mathfrak{a}}$.

We partition the remainder of this proof into two main sections starting with the (nontrivial) cases for which $\Phi(\mathfrak{a})$ is not of type BC_n. At this point, the reader may wish to review the particular characteristics of the θ-diagrams of Section 2.9. The action of π on the elements of $\Phi(\mathfrak{t})$ for each case will be of special importance.

4.1 Case AII



We require the following lemma.

Lemma 7. *Let A_n^{-1} be the inverse Cartan matrix of Lemma 3. Let $k \in \{1, 2, \cdots, n-1\}$ and let s be a row vector whose components are the sums of the entries of the k-th and $(k+1)$-th rows of this matrix, i.e. $s_j = (A_n^{-1})_{k,j} + (A_n^{-1})_{k+1,j}$. Then*

$$s_j = \begin{cases} j(1 - \frac{k}{n+1}) + j(1 - \frac{k+1}{n+1}) & 1 \leq j \leq k-1 \\ k(1 - \frac{k}{n+1}) + k(1 - \frac{k+1}{n+1}) & j = k \\ k(1 - \frac{j}{n+1}) + (k+1)(1 - \frac{j}{n+1}) & k+1 \leq j \leq n \end{cases} \tag{4.1}$$

The proof of this lemma is a direct consequence of Lemma 3 and is omitted. We are now ready to state explicitly the formulation for the containment of $\pi(\Lambda_{\mathfrak{t}})$ in $\Lambda_{\mathfrak{a}}$ as guaranteed by Theorem 4.1.(1).

Proposition 8. *Let γ and μ be as above (relative to the case AII). We then have that*

$$\gamma_k = \begin{cases} \mu_1 & k = 1 \\ 2\mu_{\frac{k}{2}} & k = \textit{even} \\ \mu_{\frac{k-1}{2}} + \mu_{\frac{k+1}{2}} & k = \textit{odd, and } \neq 2n+1 \\ \mu_n & k = 2n+1 \end{cases} \tag{4.2}$$

Proof. Let $\Delta(\mathfrak{t}) = \{\alpha_1, \alpha_2, \cdots, \alpha_{2n+1}\}$ be a base for $\Phi(\mathfrak{t})$ and $\Delta(\mathfrak{a}) = \{\omega_1, \omega_2, \cdots, \omega_n\}$ be a base for $\Phi(\mathfrak{a})$. In accordance with our aforementioned strategy, we start by computing coefficients for the weights, λ_i, with respect to $\Delta(\mathfrak{t})$. For this case we have that

$$\lambda_i = \sum_{j=1}^{2n+1} (A_{2n+1}^{-1})_{ij} \cdot \alpha_j \qquad i \in \{1, 2, \cdots, 2n+1\} \tag{4.3}$$

Next, to compute the γ_i's, observe from the θ-diagram for AII that

$$\pi(\alpha_j) = \begin{cases} 0 & j = \text{odd} \\ \omega_{\frac{j}{2}} & j = \text{even} \end{cases} \qquad j \in \{1, 2, \cdots, 2n+1\} \tag{4.4}$$

Thus, with respect to $\Delta(\mathfrak{a})$, we have that

$$\gamma_i = \pi(\lambda_i) = \sum_{j=1}^{n} (A_{2n+1}^{-1})_{i,2j} \cdot \omega_j \qquad i \in \{1, 2, \cdots, 2n+1\} \tag{4.5}$$

For convenience, we define $[\Gamma]_\omega$ to be the $(2n+1) \times n$ matrix whose rows are the coefficients of γ_i relative to $\Delta(\mathfrak{a})$.

$$([\Gamma]_\omega)_{i,j} = ([\gamma_i]_\omega)_j \tag{4.6}$$

From Equation 4.5 we have that

$$\Gamma_{ij} = \begin{cases} i(1 - \frac{2j}{2(n+1)}) & i \leq 2j \\ 2j(1 - \frac{i}{2(n+1)}) & i > 2j \end{cases} \tag{4.7}$$

with $1 \le i \le 2n+1$ and $1 \le j \le n$. The next step is to compute the coefficients for the weights, μ_i, with $i \in \{1, 2, \cdots, n\}$. Again we refer to the θ-diagram for AII. For this case we have that $\Phi(\mathfrak{a})$ is of type A_n. Thus,

$$\mu_i = \sum_{j=1}^{n} (A_n^{-1})_{ij} \cdot \omega_j \qquad 1 \le i, j \le n \tag{4.8}$$

As before let $[M]_\omega$ be the $n \times n$ matrix whose rows are the coefficients of the μ_i's relative to $\Delta(\mathfrak{a})$. Here we have that

$$M_{ij} = (A_n^{-1})_{i,j} = \begin{cases} i(1 - \frac{j}{n+1}) & i \le j \\ j(1 - \frac{i}{n+1}) & i > j \end{cases} \tag{4.9}$$

with $1 \le i, j \le n$. We claim that for each $k \in \{1, 2, \cdots, 2n+1\}$, the γ_k can be written as an integer linear combination of the $\mu_i' s$. Indeed, for $k = 1$ or $2n+1$ the result is immediate. Let $k \le 2n$ be even. From Equations 4.7 and 4.9 we have that

$$\begin{aligned} \Gamma_{kj} &= \begin{cases} k(1 - \frac{j}{n+1}) & k \le 2j \\ 2j(1 - \frac{k}{2(n+1)} & k > 2j \end{cases} \\ &= 2M_{\frac{k}{2},j}. \end{aligned}$$

For the third case of the proposition, let $k \in \{1, 2, \cdots, n-1\}$. We make the equivalent claim that

$$\gamma_{2k+1} = \mu_k + \mu_{k+1} \tag{4.10}$$

To establish this claim, we must show that for each $k \in \{1, 2, \cdots, n-1\}$

$$\Gamma_{2k+1,j} = M_{k,j} + M_{k+1,j} \qquad 1 \le j \le n. \tag{4.11}$$

Fix $k \in \{1, 2, \cdots, n-1\}$. Obviously, we invoke Lemma 7 to show that in each case, Equation 4.11 is upheld. In accordance with the structure of this lemma we consider the following three cases: First, if $j \leq k-1$ then $2j < 2k+1$ so that

$$\begin{aligned} M_{k,j} + M_{k+1,j} &= j[(1 - \frac{k}{n+1}) + (1 - \frac{k+1}{n+1})] \\ &= 2j[1 - \frac{2k+1}{2(n+1)}] \\ &= \Gamma_{2k+1,j}. \end{aligned}$$

Next, if $j = k$, then (again) $2j < 2k+1$, and

$$\begin{aligned} M_{k,j} + M_{k+1,j} &= k[(1 - \frac{k}{n+1}) + (1 - \frac{k+1}{n+1})] \\ &= 2j[1 - \frac{2k+1}{2(n+1)}] \\ &= \Gamma_{2k+1,j}. \end{aligned}$$

Finally, if $j \geq k+1$ then $2j \geq 2k+1$, and

$$\begin{aligned} M_{k,j} + M_{k+1,j} &= k(1 - \frac{j}{n+1}) + (k+1)(1 - \frac{j}{n+1}) \\ &= (2k+1)(1 - \frac{j}{n+1}) \\ &= \Gamma_{2k+1,j}. \end{aligned}$$

□

The main result for case AII follows.

Theorem 4.3. *Let γ, μ, $\pi(\Lambda_{\mathfrak{t}})$, and $\Lambda_{\mathfrak{a}}$, be as above (relative to the case AII). We then have that*

(1)

$$\mu_k = \sum_{j=1}^{k} (-1)^{k+j} \gamma_{2j-1} \qquad k = \{1, 2, \cdots n\} \tag{4.12}$$

(2) $\pi(\Lambda_{\mathfrak{t}}) \supseteq \Lambda_{\mathfrak{a}}$

Proof. For Part 1 we use induction on k. From Proposition 8 we immediately have the result for $k = 1$. Assume that Equation 4.12 holds for $k \in \{1, 2, \cdots, n-1\}$. Again by Proposition 8 we have that

$$\mu_{k+1} + \mu_k = \gamma_{2k+1} \qquad k \in \{1, 2, n-1\}$$

Thus, by the inductive hypothesis, we obtain

$$\mu_{k+1} = \gamma_{2k+1} - \sum_{j=1}^{k} (-1)^{j+k} \gamma_{2j-1} \qquad k \in \{1, 2, n-1\}$$

or

$$\mu_{k+1} = \sum_{j=1}^{k+1} (-1)^{j+k+1} \gamma_{2j-1} \qquad k \in \{1, 2, \cdots, n-1\}$$

This establishes the result for any $k \in \{1, 2, \cdots, n\}$. Part 2 follows immediately. □

Example 4.1. We consider the special case, $A_7^3(\text{II})$. Let $\Delta(\mathfrak{t}) = \{\alpha_1, \alpha_2, \cdots, \alpha_7\}$ and $\Delta(\mathfrak{a}) = \{\omega_1, \omega_2, \omega_3\}$. Again, in keeping with our strategy, we proceed with the usual steps.

Step 1 Compute the weights, $\lambda_i,\ i \in \{1, 2, \cdots, 7\}$, with respect to $\Delta(\mathfrak{t})$ for the lattice $\Lambda_{\mathfrak{t}}$.

$$\lambda_1 = \frac{1}{8}[7, 6, 5, 4, 3, 2, 1]_\alpha$$
$$\lambda_2 = \frac{1}{8}[6, 12, 10, 8, 6, 4, 2]_\alpha$$
$$\lambda_3 = \frac{1}{8}[5, 10, 15, 12, 9, 6, 3]_\alpha$$
$$\lambda_4 = \frac{1}{8}[4, 8, 12, 16, 12, 8, 4]_\alpha$$
$$\lambda_5 = \frac{1}{8}[3, 6, 9, 12, 15, 10, 5]_\alpha$$
$$\lambda_6 = \frac{1}{8}[2, 4, 6, 8, 10, 12, 6]_\alpha$$
$$\lambda_7 = \frac{1}{8}[1, 2, 3, 4, 5, 6, 7]_\alpha$$

Step 2 Now use the θ-diagram for this case to compute the projections of the above weights, $\gamma_i = \pi(\lambda_i),\ i \in \{1, 2, \cdots, 7\}$, with respect to $\Delta(\mathfrak{a})$.

$$\gamma_1 = \frac{1}{4}[3, 2, 1]_\omega$$
$$\gamma_2 = \frac{1}{4}[6, 4, 2]_\omega$$
$$\gamma_3 = \frac{1}{4}[5, 6, 3]_\omega$$
$$\gamma_4 = \frac{1}{4}[4, 8, 4]_\omega$$
$$\gamma_5 = \frac{1}{4}[3, 6, 5]_\omega$$
$$\gamma_6 = \frac{1}{4}[2, 4, 6]_\omega$$
$$\gamma_7 = \frac{1}{4}[1, 2, 3]_\omega$$

Step 3 Compute the fundamental dominant weights, $\mu_i,\ i \in \{1, 2, 3\}$ with respect to $\Delta(\mathfrak{a})$ for the lattice $\Lambda_\mathfrak{a}$.

$$\mu_1 = \frac{1}{4}[3, 2, 1]_\omega$$
$$\mu_2 = \frac{1}{4}[2, 4, 2]_\omega$$
$$\mu_3 = \frac{1}{4}[1, 2, 3]_\omega$$

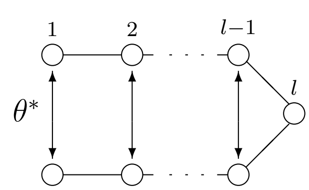
<u>Step 4</u> Show explicitly that $\pi(\Lambda_{\mathfrak{t}}) \subseteq \Lambda_{\mathfrak{a}}$ as guaranteed by Theorem 4.1(1).

$$\begin{aligned}
\gamma_1 &= \mu_1 \\
\gamma_2 &= 2\mu_1 \\
\gamma_3 &= \mu_1 + \mu_2 \\
\gamma_4 &= 2\mu_2 \\
\gamma_5 &= \mu_2 + \mu_3 \\
\gamma_6 &= 2\mu_3 \\
\gamma_7 &= \mu_3
\end{aligned}$$

<u>Step 5</u> Show explicitly that $\pi(\Lambda_{\mathfrak{t}}) \supseteq \Lambda_{\mathfrak{a}}$ thereby verifying Theorem 4.2 for this particular case.

$$\begin{aligned}
\mu_1 &= \gamma_1 \\
\mu_2 &= -\gamma_1 + \gamma_3 \\
\mu_3 &= \gamma_1 - \gamma_3 + \gamma_5
\end{aligned}$$

4.2 Case *AIII$_b$*



As before we begin this section with a proposition stating that $\pi(\Lambda_{\mathfrak{t}})$ is contained $\Lambda_{\mathfrak{a}}$.

Proposition 9. *Let γ and μ be as above (relative to the case AIII$_b$). We then have* *that*

$$\gamma_k = \begin{cases} \mu_k & k \leq n \\ \mu_{2n-k} & n < k \leq 2n-1 \end{cases} \tag{4.13}$$

Proof. Let $\Delta(\mathfrak{t}) = \{\alpha_1, \alpha_2, \cdots, \alpha_{2n-1}\}$ be a base for $\Phi(\mathfrak{t})$ and $\Delta(\mathfrak{a}) = \{\omega_1, \omega_2, \cdots, \omega_n\}$ be a base for $\Phi(\mathfrak{a})$. In this case for the fundamental dominant weights of the lattice $\Lambda_{\mathfrak{t}}$ we have

$$\lambda_i = \sum_{j=1}^{2n-1} (A_{2n-1}^{-1})_{ij} \cdot \alpha_j \qquad i \in \{1, 2, \cdots, 2n-1\} \tag{4.14}$$

Next, the θ-diagram for case $A\mathrm{III}_b$ from Section 2.9 implies that

$$\pi(\alpha_j) = \begin{cases} \omega_j & 1 \leq j \leq n \\ \omega_{2n-j} & n+1 \leq j \leq 2n-1 \end{cases} \tag{4.15}$$

Applying this map to the weights of Equation 4.14 we obtain

$$\gamma_i = \sum_{j=1}^{n-1} \{(A_{2n-1}^{-1})_{ij} + (A_{2n-1}^{-1})_{i,2n-j}\} \cdot \omega_j + (A_{2n-1}^{-1})_{in} \cdot \omega_n \qquad i \in \{1, 2, \cdots, 2n-1\} \tag{4.16}$$

As was done in the previous section we let $[\Gamma]_\omega$ be the matrix of coefficients of the γ's relative to the base $\Delta_{\mathfrak{a}}$. We claim that in this case,

$$\Gamma_{ij} = \begin{cases} j & j < i \leq n \\ i & i \leq j < n \\ \frac{i}{2} & i \leq j < n \\ j & j < 2n-i, \ i \geq n+1 \\ 2n-i & 2n-i \leq j < n, \ (i \geq n+1) \\ \frac{2n-i}{2} & j = n, \ i \geq n+1 \end{cases} \tag{4.17}$$

To establish this claim let, $s_{ij} = (A_{2n-1}^{-1})_{ij} + (A_{2n-1}^{-1})_{i,2n-j}$. We consider several cases. First if $i < n$ then

$$s_{ij} = \begin{cases} j(1 - \frac{i}{2n}) + i(1 - \frac{2n-j}{2n}) = j & 1 \leq j \leq i-1 \\ i(1 - \frac{j}{2n}) + i(1 - \frac{2n-j}{2n}) = i & i \leq j \leq n-1 \end{cases}$$

Next, if $i = n$ then

$$s_{ij} = j(1 - \frac{n}{2n}) + n(1 - \frac{2n-j}{2n}) = j.$$

If $i \geq n+1$ then

$$s_{ij} = \begin{cases} j(1 - \frac{i}{2n}) + i(1 - \frac{2n-j}{2n}) = j & 1 \leq j \leq 2n - i \\ j(1 - \frac{i}{2n}) + (2n-j)(1 - \frac{i}{2n}) = 2n - i & 2n - i + 1 \leq j \leq n - 1. \end{cases}$$

Finally, we have that

$$(A_{2n-1}^{-1})_{in} = \begin{cases} \frac{i}{2} & 1 \leq i \leq n \\ \frac{2n-i}{2} & n+1 \leq i \leq 2n-1 \end{cases}$$

thereby finishing the proof of the claim.

Now, for the fundamental weights, μ_i, we return again to the θ-diagram for $A\text{III}_b$. We see that $\Phi(\mathfrak{a})$ is of type C_n. Thus,

$$\mu_i = \sum_{j=1}^{n} (C_n^{-1})_{ij} \cdot \omega_j \qquad 1 \leq i \leq n. \tag{4.18}$$

As before we let $[M]_\omega$ be the matrix of coefficients for the μ's relative to $\Delta(\mathfrak{a})$. Obviously,

$$M_{ij} = (C_n^{-1})_{ij} \qquad 1 \leq i, j \leq n. \tag{4.19}$$

We are now ready to establish Proposition 9. Observe from Equations 4.17 and 4.19 that

$$\Gamma_{kj} = M_{kj} \qquad 1 \leq k, j \leq n.$$

So that for $k \in \{1, 2, 3, \cdots, n\}$ then $\gamma_k = \mu_k$. For $k \in \{n+1, n+2, \cdots, 2n-1\}$, we have that

$$\Gamma_{k,j} = \Gamma_{2n-k,j} \qquad 1 \leq j \leq n.$$

Thus $\gamma_k = \gamma_{2n-k} = \mu_{2n-k}$. $\square$

We state the main result for this section whose proof is now immediate.

Theorem 4.4. *Let γ, μ, $\pi(\Lambda_{\mathfrak{t}})$, and $\Lambda_{\mathfrak{a}}$, be as above (relative to the case AIII$_b$). We then have that*

(1) $\mu_k = \gamma_k \qquad 1 \leq k \leq n$

(2) $\pi(\Lambda_{\mathfrak{t}}) \supseteq \Lambda_{\mathfrak{a}}$.

Example 4.2. We consider the special case, $A_7^4(\mathrm{III}_b)$. Let $\Delta(\mathfrak{t}) = \{\alpha_1, \alpha_2, \cdots, \alpha_7\}$ and $\Delta(\mathfrak{a}) = \{\omega_1, \omega_2, \omega_3, \omega_4\}$. Again, in keeping with our strategy, we proceed as usual.
Step 1 Compute the weights, $\lambda_i,\ i \in \{1, 2, \cdots, 7\}$, with respect to $\Delta(\mathfrak{t})$ for the lattice $\Lambda_{\mathfrak{t}}$.

$$\begin{aligned}
\lambda_1 &= \frac{1}{8}[7,6,5,4,3,2,1]_\alpha \\
\lambda_2 &= \frac{1}{8}[6,12,10,8,6,4,2]_\alpha \\
\lambda_3 &= \frac{1}{8}[5,10,15,12,9,6,3]_\alpha \\
\lambda_4 &= \frac{1}{8}[4,8,12,16,12,8,4]_\alpha \\
\lambda_5 &= \frac{1}{8}[3,6,9,12,15,10,5]_\alpha \\
\lambda_6 &= \frac{1}{8}[2,4,6,8,10,12,6]_\alpha \\
\lambda_7 &= \frac{1}{8}[1,2,3,4,5,6,7]_\alpha
\end{aligned}$$

Step 2 Now use the θ-diagram for this case to compute the projections of the above weights, $\gamma_i = \pi(\lambda_i),\ i \in \{1, 2, \cdots, 7\}$, with respect to $\Delta(\mathfrak{a})$.

$$\gamma_1 = [1, 1, 1, \frac{1}{2}]_\omega$$
$$\gamma_2 = [1, 2, 2, 1]_\omega$$
$$\gamma_3 = [1, 2, 3, \frac{3}{2}]_\omega$$
$$\gamma_4 = [1, 2, 3, 2]_\omega$$
$$\gamma_5 = [1, 2, 3, \frac{3}{2}]_\omega$$
$$\gamma_6 = [1, 2, 2, 1]_\omega$$
$$\gamma_7 = [1, 1, 1, \frac{1}{2}]_\omega$$

Step 3 Compute the fundamental dominant weights, $\mu_i,\ i \in \{1, 2, 3\}$ with respect to $\Delta(\mathfrak{a})$ for the lattice $\Lambda_{\mathfrak{a}}$.

$$\mu_1 = [1, 1, 1, \frac{1}{2}]_\omega$$
$$\mu_2 = [1, 2, 2, 1]_\omega$$
$$\mu_3 = [1, 2, 3, \frac{3}{2}]\omega$$
$$\mu_4 = [1, 2, 3, 2]_\omega$$

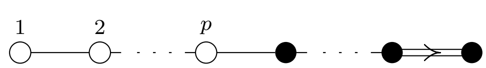
Step 4 Show explicitly that $\pi(\Lambda_{\mathfrak{t}}) \subseteq \Lambda_{\mathfrak{a}}$ as guaranteed by Theorem 4.1.

$$\gamma_1 = \mu_1$$
$$\gamma_2 = \mu_2$$
$$\gamma_3 = \mu_3$$
$$\gamma_4 = \mu_4$$
$$\gamma_5 = \mu_3$$
$$\gamma_6 = \mu_2$$
$$\gamma_7 = \mu_1$$

Step 5 Show explicitly that $\pi(\Lambda_{\mathfrak{t}}) \supseteq \Lambda_{\mathfrak{a}}$ thereby verifying Theorem 4.2 for this particular case.

$$\mu_1 = \gamma_1$$
$$\mu_2 = \gamma_2$$
$$\mu_3 = \gamma_3$$
$$\mu_4 = \gamma_4$$

4.3 Case BI ($p < n$)



In this section we assume that $p < n$. Again we begin with a proposition giving formulas for the (guaranteed) containment of $\pi(\Lambda_{\mathfrak{t}})$ in $\Lambda_{\mathfrak{a}}$.

Proposition 10. *Let γ and μ be as above (relative to the case BI). We then have that*

$$\gamma_k = \begin{cases} \mu_k & k \in \{1, \cdots, p-1\} \\ 2\mu_p & k \in \{p, \cdots, n-1\} \\ \mu_p & k = n \end{cases} \tag{4.20}$$

Proof. Recall from the θ-diagram for this case, (Section 2.9), that $n \geq p+2$ where $n =\mid \Delta(\mathfrak{t}) \mid$ and $p =\mid \Delta(\mathfrak{a}) \mid$. Let $\Delta(\mathfrak{t}) = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ be a base for $\Phi(\mathfrak{t})$ and let $\Delta(\mathfrak{a}) = \{\omega_1, \omega_2, \cdots, \omega_p\}$ be a base for $\Phi(\mathfrak{a})$. Here the fundamental weights for the lattice $\Lambda_\mathfrak{t}$ are given by

$$\lambda_i = \sum_{j=1}^{n} (B_n^{-1})_{ij} \cdot \alpha_j \qquad i \in \{1, 2, \cdots, n\} \tag{4.21}$$

Again we refer to the θ-diagram to obtain

$$\pi(\alpha_j) = \begin{cases} \omega_j & 1 \leq j \leq p \\ 0 & p+1 \leq j \leq n \end{cases} \tag{4.22}$$

Applying this map to the weights of Equation 4.21 we obtain

$$\gamma_i = \sum_{j=1}^{p} (B_n^{-1})_{ij} \cdot \omega_j \qquad i \in \{1, 2, \cdots, n\} \tag{4.23}$$

Define $[\Gamma]_\omega$ as before. In this case Γ is the matrix obtained from the first p columns of B_n^{-1}. Thus Γ is he $n \times p$ matrix given by

$$\Gamma_{ij} = \begin{cases} i & i < j \ (j \leq p) \\ j & j \leq i < n \ (j \leq p) \\ \frac{j}{2} & i = n \ (j \leq p) \end{cases} \tag{4.24}$$

Next we compute the fundamental weights μ_i, with $i \in \{1, \cdots, p\}$. As $\Phi(\mathfrak{a})$ is of type B_p we have that,

$$\mu_i = \sum_{j=1}^{p} (B_p^{-1})_{ij} \cdot \omega_j \qquad 1 \leq i \leq p. \tag{4.25}$$

As before, we let $[M]_\omega$ be the matrix of coefficients for the μ's relative to $\Delta(\mathfrak{a})$. Obviously, in this case

$$M = B_p^{-1} \tag{4.26}$$

To prove the proposition observe that for $1 \leq k \leq p-1$ then

$$\Gamma_{kj} = M_{kj} \qquad 1 \leq j \leq p$$

So that $\gamma_k = \mu_k$. For $p \leq k \leq n-1$ then

$$\Gamma_{kj} = 2M_{pj} \qquad 1 \leq j \leq p$$

and $\gamma_k = 2\mu_k$. Finally for $k = n$ then

$$\Gamma_{nj} = (B_p^{-1})_{pj} \qquad 1 \leq j \leq p$$

and $\gamma_n = \mu_p$ thus completing the proof. □

The main result for this section is (again) an immediate consequence of Proposition 10.

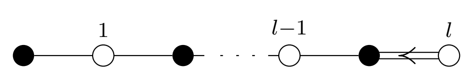
Theorem 4.5. *Let γ, μ, $\pi(\Lambda_\mathfrak{t})$, and $\Lambda_\mathfrak{a}$, be as above (relative to the case BI). We then have that*

(1)

$$\mu_k = \begin{cases} \gamma_k & 1 \leq k \leq p-1 \\ \gamma_n & k = p \end{cases} \tag{4.27}$$

(2) $\pi(\Lambda_\mathfrak{t}) \supseteq \Lambda_\mathfrak{a}$.

4.4 Case $C\mathrm{II}_b$



We require the following

Lemma 8. *Let C_n^{-1} be the inverse Cartan matrix of Lemma 5. Let $k \in \{1, 2, \cdots, n-1\}$ and let s be a row vector whose components are the sums of the entries of the k-th and $(k+1)$-th rows of this matrix, i.e. $s_j = (C_n^{-1})_{k,j} + (C_n^{-1})_{k+1,j}$. Then*

$$s_j = \begin{cases} 2j & j \leq k < n \\ 2k+1 & k < j < n \\ \frac{2k+1}{2} & j = n \end{cases} \tag{4.28}$$

Similar to Lemma 7 the proof of this lemma is a direct consequence of Lemma 5 and is omitted. Proceeding as before we have the following

Proposition 11. *Let γ and μ be as above (relative to the case CII_b). We then have that*

$$\gamma_k = \begin{cases} \mu_1 & k = 1 \\ \mu_{\frac{k-1}{2}} + \mu_{\frac{k+1}{2}} & k \in \{3, 5, \cdots, 2n-1\} \\ 2\mu_{\frac{k}{2}} & k \in \{2, 4, \cdots, 2n\} \end{cases} \tag{4.29}$$

Proof. For this case the θ-diagram specifies that $\mid \Delta(\mathfrak{t}) \mid= 2n$ and $\mid \Delta(\mathfrak{a}) \mid= n$. Let $\Delta(\mathfrak{t}) = \{\alpha_1, \alpha_2, \cdots, \alpha_{2n}\}$ be a base for $\Phi(\mathfrak{t})$ and let $\Delta(\mathfrak{a}) = \{\omega_1, \omega_2, \cdots, \omega_n\}$ be a base for $\Phi(\mathfrak{a})$. As $\Phi(\mathfrak{t})$ here is of type C_{2n}, then the fundamental weights for the lattice $\Lambda_\mathfrak{t}$ are given by

$$\lambda_i = \sum_{j=1}^{2n} (C_{2n}^{-1})_{ij} \cdot \alpha_j \qquad i \in \{1, 2, \cdots, 2n\} \tag{4.30}$$

Proceeding as before we have that

$$\pi(\alpha_j) = \begin{cases} 0 & j \in \{1, 3, \cdots, 2n-1\} \\ \omega_{\frac{j}{2}} & j \in \{2, 4, \cdots, 2n\} \end{cases} \tag{4.31}$$

Thus

$$\gamma_i = \sum_{j=1}^{n} (C_{2n}^{-1})_{i,2j} \cdot \omega_j \qquad i \in \{1, 2, \cdots, n\} \tag{4.32}$$

with

$$\Gamma_{ij} = \begin{cases} 2j & 2j < i \leq 2n \\ i & i \leq 2j < 2n \\ \frac{i}{2} & 2j = 2n \end{cases} \tag{4.33}$$

As $\Phi(\mathfrak{a})$ is of type C_n we have that,

$$\mu_i = \sum_{j=1}^{n} (C_n^{-1})_{ij} \cdot \omega_j \qquad 1 \leq i \leq n. \tag{4.34}$$

With

$$M = C_n^{-1} \tag{4.35}$$

Now for $k = 1$ the proposition is immediate. Let $k \in \{2, 4, \cdots, 2n\}$. Then

$$\begin{aligned} \Gamma_{kj} &= \begin{cases} 2j & 2j < k \leq 2n \\ k & k \leq 2j < 2n \\ \frac{k}{2} & 2j = 2n \end{cases} \\ &= \begin{cases} 2j & j < \frac{k}{2} \leq n \\ 2(\frac{k}{2}) & \frac{k}{2} \leq j < n \\ 2(\frac{k}{4}) & j = n \end{cases} \\ &= 2M_{\frac{k}{2},j}. \end{aligned}$$

Finally, to complete the proof of the proposition we make the equivalent claim that for $k = \{1, 2, \cdots, n-1\}$ then

$$\gamma_{2k+1} = \mu_k + \mu_{k+1}.$$

Indeed,

$$\Gamma_{2k+1,j} = \begin{cases} 2j & 2j < 2k+1 < 2n \\ 2k+1 & 2k+1 < 2j < 2n \\ \frac{2k+1}{2} & 2j = 2n \end{cases} \tag{4.36}$$

Now

$$2j < 2k + 1 \Leftrightarrow j \leq k$$
$$2k + 1 \leq 2j \Leftrightarrow k < j$$
$$2j = 2n \Leftrightarrow j = n.$$

Therefore, Lemma 8 implies that

$$\Gamma_{2k+1,j} = M_{kj} + M_{k+1,j}$$

proving the claim. □

We now give the main result for this section.

Theorem 4.6. *Let γ, μ, $\pi(\Lambda_{\mathfrak{t}})$, and $\Lambda_{\mathfrak{a}}$, be as above (relative to the case CIIb). We then have that*

(1)

$$\mu_k = \sum_{j+1}^{k} (-1)^{k+j} \gamma_{2j-1} \qquad k \in \{1, 2, \cdots n\} \tag{4.37}$$

(2) $\pi(\Lambda_{\mathfrak{t}}) \supseteq \Lambda_{\mathfrak{a}}$.

The proof is this theorem is analogous to that of Theorem 4.12 and is omitted.

Example 4.3. We consider the special case, $C_8^4(\mathrm{II}_b)$. Let $\Delta(\mathfrak{t}) = \{\alpha_1, \alpha_2, \cdots, \alpha_7, \alpha_8\}$ and $\Delta(\mathfrak{a}) = \{\omega_1, \omega_2, \omega_3, \omega_4\}$.

Step 1 Compute the weights, λ_i, $i \in \{1, 2, \cdots, 8\}$, with respect to $\Delta(\mathfrak{t})$ for the lattice

$\Lambda_{\mathfrak{t}}$.

$$\lambda_1 = [1,1,1,1,1,1,1,\frac{1}{2}]_\alpha$$
$$\lambda_2 = [1,2,2,2,2,2,2,1]_\alpha$$
$$\lambda_3 = [1,2,3,3,3,3,3,\frac{3}{2}]_\alpha$$
$$\lambda_4 = [1,2,3,4,4,4,4,2]_\alpha$$
$$\lambda_5 = [1,2,3,4,5,5,5,\frac{5}{2}]_\alpha$$
$$\lambda_6 = [1,2,3,4,5,6,6,3]_\alpha$$
$$\lambda_7 = [1,2,3,4,5,6,7,\frac{7}{2}]_\alpha$$
$$\lambda_8 = [1,2,3,4,5,6,7,4]_\alpha$$

Step 2 Now use the θ-diagram for this case to compute the projections of the above weights, $\gamma_i = \pi(\lambda_i),\ i \in \{1,2,\cdots,8\}$, with respect to $\Delta(\mathfrak{a})$.

$$\gamma_1 = [1,1,1,\frac{1}{2}]_\omega$$
$$\gamma_2 = [2,2,2,1]_\omega$$
$$\gamma_3 = [2,3,3,\frac{3}{2}]_\omega$$
$$\gamma_4 = [2,4,4,2]_\omega$$
$$\gamma_5 = [2,4,5,\frac{5}{2}]_\omega$$
$$\gamma_6 = [2,4,6,3]_\omega$$
$$\gamma_7 = [2,4,6,\frac{7}{2}]_\omega$$
$$\gamma_8 = [2,4,6,4]_\omega$$

Step 3 Compute the fundamental dominant weights, $\mu_i,\ i \in \{1,2,3\}$ with respect to

$\Delta(\mathfrak{a})$ for the lattice $\Lambda_{\mathfrak{a}}$.

$$\mu_1 = [1, 1, 1, \frac{1}{2}]_\omega$$
$$\mu_2 = [1, 2, 2, 1]_\omega$$
$$\mu_3 = [1, 2, 3, \frac{3}{2}]$$
$$\mu_4 = [1, 2, 3, 2]_\omega$$

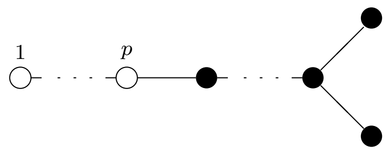
<u>Step 4</u> Show explicitly that $\pi(\Lambda_{\mathfrak{t}}) \subseteq \Lambda_{\mathfrak{a}}$ as guaranteed by Theorem 4.1(1).

$$\gamma_1 = \mu_1$$
$$\gamma_2 = 2\mu_1$$
$$\gamma_3 = \mu_1 + \mu_2$$
$$\gamma_4 = 2\mu_2$$
$$\gamma_5 = \mu_2 + \mu_3$$
$$\gamma_6 = 2\mu_3$$
$$\gamma_7 = \mu_3 + \mu_4$$
$$\gamma_8 = 2\mu_4$$

<u>Step 5</u> Show explicitly that $\pi(\Lambda_{\mathfrak{t}}) \supseteq \Lambda_{\mathfrak{a}}$ thereby verifying Theorem 4.2 for this particular case.

$$\mu_1 = \gamma_1$$
$$\mu_2 = -\gamma_1 + \gamma_3$$
$$\mu_3 = \gamma_1 - \gamma_3 + \gamma_5$$
$$\mu_4 = -\gamma_1 + \gamma_3 - \gamma_5 + \gamma_7$$

4.5 Case $D\mathrm{I}_a$



Again we begin with a proposition,

Proposition 12. *Let γ and μ be as above (relative to the case DI_a). We then have that*

$$\gamma_k = \begin{cases} \mu_k & k \in \{1, \cdots, p-1\} \\ 2\mu_p & k \in \{p, \cdots, n-2\} \\ \mu_p & k \in \{n-1, n\} \end{cases} \tag{4.38}$$

Proof. For this case we have that $|\ \Delta(\mathfrak{t})\ | = n$ and $|\ \Delta(\mathfrak{a})\ | = p$ with $1 \leq p \leq n-2$. Let $\Delta(\mathfrak{t}) = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ be a base for $\Phi(\mathfrak{t})$ and let $\Delta(\mathfrak{a}) = \{\omega_1, \omega_2, \cdots, \omega_p\}$ be a base for $\Phi(\mathfrak{a})$. As $\Phi(\mathfrak{t})$ here is of type D_n, then the fundamental weights for the lattice $\Lambda_{\mathfrak{t}}$ are given by

$$\lambda_i = \sum_{j=1}^{n} (D_n^{-1})_{ij} \cdot \alpha_j \qquad i \in \{1, 2, \cdots, n\} \tag{4.39}$$

Proceeding as before we have that

$$\pi(\alpha_j) = \begin{cases} \omega_j & 1 \leq j \leq p \\ 0 & p+1 \leq j \leq n \end{cases} \tag{4.40}$$

Thus

$$\gamma_i = \sum_{j=1}^{p} (D_n^{-1})_{i,j} \cdot \omega_j \qquad i \in \{1, 2, \cdots, n\} \tag{4.41}$$

Γ in this case is the $n \times p$ matrix given by

$$\Gamma_{ij} = \begin{cases} i & i < j \leq p \\ j & j \leq i \leq n-2 \\ \frac{j}{2} & i \in \{n-1, n\} \end{cases} \tag{4.42}$$

Now as $\Phi(\mathfrak{a})$ is of type B_p we have that,

$$\mu_i = \sum_{j=1}^{p} (B_p^{-1})_{ij} \cdot \omega_j \qquad 1 \leq i \leq p. \tag{4.43}$$

Thus we have that M is the $p \times p$ matrix given by

$$M = B_p^{-1} = \begin{cases} i & i < j \leq p \\ j & j \leq i < p \\ \frac{j}{2} & i = p \end{cases} \tag{4.44}$$

To prove the proposition first observe that for $k \in \{1, \cdots, p-1\}$ then

$$\Gamma_{kj} = M_{kj}.$$

For $k \in \{p, \cdots, n-2\}$ we have that

$$\Gamma_{kj} = 2M_{pj}.$$

Finally,

$$\Gamma_{n-1,j} = \Gamma_{nj} = M_{pj}.$$

□

We now give the main result for this section whose proof, once again, is an immediate consequence of the preceding proposition.

Theorem 4.7. *Let γ, μ, $\pi(\Lambda_{\mathfrak{t}})$, and $\Lambda_{\mathfrak{a}}$, be as above (relative to the case DI_a). We then have that*

(1)

$$\mu_k = \begin{cases} \gamma_k & 1 \leq k \leq p-1 \\ \gamma_n & k = p \end{cases} \tag{4.45}$$

(2) $\pi(\Lambda_{\mathfrak{t}}) \supseteq \Lambda_{\mathfrak{a}}$.

Example 4.4. We consider the special case, $D_7^4(\mathrm{I}_a)$. Let $\Delta(\mathfrak{t}) = \{\alpha_1, \alpha_2, \cdots, \alpha_7\}$ and $\Delta(\mathfrak{a}) = \{\omega_1, \omega_2, \omega_3, \omega_4\}$.

Step 1 Compute the weights, $\lambda_i,\ i \in \{1, 2, \cdots, 7\}$, with respect to $\Delta(\mathfrak{t})$ for the lattice $\Lambda_{\mathfrak{t}}$.

$$\begin{aligned}
\lambda_1 &= [1, 1, 1, 1, 1, \frac{1}{2}, \frac{1}{2}]_\alpha \\
\lambda_2 &= [1, 2, 2, 2, 2, 1, 1]_\alpha \\
\lambda_3 &= [1, 2, 3, 3, 3, \frac{3}{2}, \frac{3}{2}]_\alpha \\
\lambda_4 &= [1, 2, 3, 4, 4, 2, 2]_\alpha \\
\lambda_5 &= [1, 2, 3, 4, 5, \frac{5}{2}, \frac{5}{2}]_\alpha \\
\lambda_6 &= [\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \frac{7}{4}, \frac{5}{4}]_\alpha \\
\lambda_7 &= [\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \frac{5}{4}, \frac{7}{4}]_\alpha
\end{aligned}$$

Step 2 Now use the θ-diagram for this case to compute the projections of the above weights, $\gamma_i = \pi(\lambda_i),\ i \in \{1, 2, \cdots, 7\}$, with respect to $\Delta(\mathfrak{a})$.

$$\begin{aligned}
\gamma_1 &= [1, 1, 1, 1]_\omega \\
\gamma_2 &= [1, 2, 2, 2]_\omega \\
\gamma_3 &= [1, 2, 3, 3]_\omega \\
\gamma_4 &= [1, 2, 3, 4]_\omega \\
\gamma_5 &= [1, 2, 3, 4]_\omega \\
\gamma_6 &= [\frac{1}{2}, 1, \frac{3}{2}, 2]_\omega \\
\gamma_7 &= [\frac{1}{2}, 1, \frac{3}{2}, 2]_\omega
\end{aligned}$$

Step 3 Compute the fundamental dominant weights, $\mu_i,\ i \in \{1, 2, 3, 4\}$ with respect to $\Delta(\mathfrak{a})$ for the lattice $\Lambda_{\mathfrak{a}}$.

$$\mu_1 = [1, 1, 1, 1]_\omega$$
$$\mu_2 = [1, 2, 2, 2]_\omega$$
$$\mu_3 = [1, 2, 3, 3]_\omega$$
$$\mu_4 = [\frac{1}{2}, 1, \frac{3}{2}, 2]_\omega$$

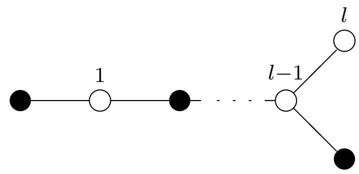
<u>Step 4</u> Show explicitly that $\pi(\Lambda_{\mathfrak{t}}) \subseteq \Lambda_{\mathfrak{a}}$ as guaranteed by Theorem 4.1(1).

$$\gamma_1 = \mu_1$$
$$\gamma_2 = \mu_2$$
$$\gamma_3 = \mu_3$$
$$\gamma_4 = 2\mu_4$$
$$\gamma_5 = 2\mu_4$$
$$\gamma_6 = \mu_4$$
$$\gamma_7 = \mu_4$$

<u>Step 5</u> Show explicitly that $\pi(\Lambda_{\mathfrak{t}}) \supseteq \Lambda_{\mathfrak{a}}$ thereby verifying Theorem 4.2 for this particular case.

$$\mu_1 = \gamma_1$$
$$\mu_2 = \gamma_2$$
$$\mu_3 = \gamma_3$$
$$\mu_4 = \gamma_7$$

4.6 Case $D\mathrm{III}_a$



In this case we have the following

Proposition 13. *Let γ and μ be as above (relative to the case DIII$_a$). We then have that*

$$\gamma_k = \begin{cases} \mu_1 & k = 1 \\ 2\mu_{\frac{k}{2}} & k \in \{2, 4, \cdots, 2n-2\} \\ \mu_{\frac{k-1}{2}} + \mu_{\frac{k+1}{2}} & k \in \{3, 5, \cdots, 2n-3\} \\ \mu_{n-1} & k = 2n-1 \\ \mu_n & k = n \end{cases} \tag{4.46}$$

Proof. Here we have that $\mid \Delta(\mathfrak{t}) \mid = 2n$ and $\mid \Delta(\mathfrak{a}) \mid = n$. Let $\Delta(\mathfrak{t}) = \{\alpha_1, \alpha_2, \cdots, \alpha_{2n}\}$ be a base for $\Phi(\mathfrak{t})$ and let $\Delta(\mathfrak{a}) = \{\omega_1, \omega_2, \cdots, \omega_n\}$ be a base for $\Phi(\mathfrak{a})$. As $\Phi(\mathfrak{t})$ in this case is of type D_{2n}, then the fundamental weights for the lattice $\Lambda_{\mathfrak{t}}$ are given by

$$\lambda_i = \sum_{j=1}^{2n} (D_{2n}^{-1})_{ij} \cdot \alpha_j \qquad i \in \{1, 2, \cdots, 2n\} \tag{4.47}$$

Proceeding as before we have that

$$\pi(\alpha_j) = \begin{cases} 0 & k \in \{1, 3, \cdots, 2n-1\} \\ \omega_{\frac{j}{2}} & k \in \{2, 4, \cdots, 2n\} \end{cases} \tag{4.48}$$

Thus

$$\gamma_i = \sum_{j=1}^{n} (D_{2n}^{-1})_{i,2j} \cdot \omega_j \qquad i \in \{1, 2, \cdots, 2n\} \tag{4.49}$$

Γ in this case is the $2n \times n$ matrix given by

$$\Gamma_{ij} = \begin{cases} i & i < 2j \leq 2n-2 \\ 2j & 2j \leq i \leq 2n-2 \\ j & i \in \{2n-1, 2n\}, 2j \leq 2n-2 \\ \frac{i}{2} & 2j \in \{2n-1, 2n\}, i \leq 2n-2 \\ \frac{2n-2}{4} & i = 2n-1, 2j = 2n \\ \frac{2n}{4} & i = 2n, 2j = 2n \end{cases} \tag{4.50}$$

Now as $\Phi(\mathfrak{a})$ is of type C_n we have that,

$$\mu_i = \sum_{j=1}^{n} (C_n^{-1})_{ij} \cdot \omega_j \qquad 1 \leq i \leq n. \tag{4.51}$$

Thus we have that M is the $n \times n$ matrix given by

$$M = C_n^{-1} = \begin{cases} j & j < i \leq n \\ i & i \leq j < n \\ \frac{i}{2} & j = n \end{cases} \tag{4.52}$$

To prove the proposition we first observe that $\Gamma_{1,j} = M_{1,j}$, $\Gamma_{2n-1,j} = M_{n-1,j}$, and $\Gamma_{2n,j} = M_{n,j}$. Now let $k \in \{1, 2, \cdots, n-1\}$. Then

$$\begin{aligned} \Gamma_{2k,j} &= \begin{cases} 2k & 2k < 2j \leq 2n-2 \\ 2j & 2j \leq 2k \leq 2n-2 \\ k & 2j = 2n, 2k \leq 2n-2 \end{cases} \\ &= \begin{cases} 2j & j < k \leq n-1 \\ 2k & k \leq j \leq n-1 \\ k & j = n \end{cases} \\ &= 2M_{k,j}. \end{aligned}$$

For the next step we make the equivalent claim that for $k \in \{1, 2, \cdots, n-2\}$ then

$$\gamma_{2k+1} = \mu_k + \mu_k + 1$$

Indeed, for these values of k, we have by Lemma 8 that

$$\Gamma_{2k+1,j} = \begin{cases} 2k+1 & 2k+1 < 2j \leq 2n-2 \\ 2j & 2j \leq 2k+1 \leq 2n-2 \\ \frac{2k+1}{2} & 2k+1 \leq 2n-2, 2j = 2n \end{cases}$$
$$= M_{k,j} + M_{k+1,j}$$

thus completing the proof. □

We now give the main result for this section.

Theorem 4.8. *Let γ, μ, $\pi(\Lambda_{\mathfrak{t}})$, and $\Lambda_{\mathfrak{a}}$, be as above (relative to the case DIII$_a$). We then have that*

(1)

$$\mu_k = \begin{cases} \sum_{j+1}^{k} (-1)^{k+j} \gamma_{2j-1} & k \in \{1, 2, \cdots n-1\} \\ \gamma_{2n} & k = n \end{cases} \tag{4.53}$$

(2) $\pi(\Lambda_{\mathfrak{t}}) \supseteq \Lambda_{\mathfrak{a}}$.

For the proof use induction on n and Proposition 13

Example 4.5. We consider the special case, $D_8^4(\text{III}_a)$. Let $\Delta(\mathfrak{t}) = \{\alpha_1, \alpha_2, \cdots, \alpha_8\}$ and $\Delta(\mathfrak{a}) = \{\omega_1, \omega_2, \omega_3, \omega_4\}$.

Step 1 Compute the weights, $\lambda_i,\ i \in \{1, 2, \cdots, 8\}$, with respect to $\Delta(\mathfrak{t})$ for the lattice

$\Lambda_{\mathfrak{t}}$.

$$\lambda_1 = [1, 1, 1, 1, 1, 1, \frac{1}{2}, \frac{1}{2}]_\alpha$$
$$\lambda_2 = [1, 2, 2, 2, 2, 2, 1, 1]_\alpha$$
$$\lambda_3 = [1, 2, 3, 3, 3, 3, \frac{3}{2}, \frac{3}{2}]_\alpha$$
$$\lambda_4 = [1, 2, 3, 4, 4, 4, 2, 2]_\alpha$$
$$\lambda_5 = [1, 2, 3, 3, 3, 3, \frac{3}{2}, \frac{3}{2}]_\alpha$$
$$\lambda_6 = [1, 2, 3, 4, 5, 5, \frac{5}{2}, \frac{5}{2}]_\alpha$$
$$\lambda_7 = [\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{8}{4}, \frac{6}{4}]_\alpha$$
$$\lambda_8 = [\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{6}{4}, \frac{8}{4}]_\alpha$$

<u>Step 2</u> Now use the θ-diagram for this case to compute the projections of the above weights, $\gamma_i = \pi(\lambda_i),\ i \in \{1, 2, \cdots, 8\}$, with respect to $\Delta(\mathfrak{a})$.

$$\gamma_1 = [1, 1, 1, \frac{1}{2}]_\omega$$
$$\gamma_2 = [1, 2, 2, 1]_\omega$$
$$\gamma_3 = [2, 3, 3, \frac{3}{2}]_\omega$$
$$\gamma_4 = [2, 4, 4, 2]_\omega$$
$$\gamma_5 = [2, 4, 5, \frac{5}{2}]_\omega$$
$$\gamma_6 = [2, 4, 6, 3]_\omega$$
$$\gamma_7 = [1, 2, 3, \frac{6}{4}]_\omega$$
$$\gamma_8 = [1, 2, 3, \frac{8}{4}]_\omega$$

<u>Step 3</u> Compute the fundamental dominant weights, $\mu_i,\ i \in \{1, 2, 3, 4\}$ with respect

to $\Delta(\mathfrak{a})$ for the lattice $\Lambda_{\mathfrak{a}}$.

$$\mu_1 = [1,1,1,\frac{1}{2}]_\omega$$
$$\mu_2 = [1,2,2,1]_\omega$$
$$\mu_3 = [1,2,3,\frac{3}{2}]_\omega$$
$$\mu_4 = [1,2,3,2]_\omega$$

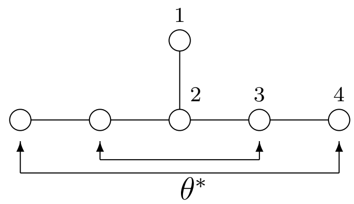
Step 4 Show explicitly that $\pi(\Lambda_{\mathfrak{t}}) \subseteq \Lambda_{\mathfrak{a}}$ as guaranteed by Theorem 4.1(1).

$$\gamma_1 = \mu_1$$
$$\gamma_2 = 2\mu_1$$
$$\gamma_3 = \mu_1 + \mu_2$$
$$\gamma_4 = 2\mu_2$$
$$\gamma_5 = \mu_2 + \mu_3$$
$$\gamma_6 = 2\mu_3$$
$$\gamma_7 = \mu_3$$
$$\gamma_8 = \mu_4$$

Step 5 Show explicitly that $\pi(\Lambda_{\mathfrak{t}}) \supseteq \Lambda_{\mathfrak{a}}$ thereby verifying Theorem 4.2 for this particular case.

$$\mu_1 = \gamma_1$$
$$\mu_2 = -\gamma_1 + \gamma_3$$
$$\mu_3 = \gamma_1 - \gamma_3 + \gamma_5$$
$$\mu_4 = \gamma_8$$

4.7 Case EII



Being of fixed dimension, the treatment for this case, (and the next five to follow), is similar to that of the previous examples. Indeed, we have that $\Delta(\mathfrak{t}) = \{\alpha_1, \alpha_2, \cdots, \alpha_6\}$ and $\Delta(\mathfrak{a}) = \{\omega_1, \omega_2, \omega_3, \omega_4\}$.

Step 1 Compute the weights, $\lambda_i,\ i \in \{1, 2, \cdots, 6\}$, with respect to $\Delta(\mathfrak{t})$ for the lattice $\Lambda_{\mathfrak{t}}$.

$$\lambda_1 = [\frac{4}{3}, 1, \frac{5}{3}, 2, \frac{4}{3}, \frac{2}{3}]_\alpha$$
$$\lambda_2 = [1, 2, 2, 3, 2, 1]_\alpha$$
$$\lambda_3 = [\frac{5}{3}, 2, \frac{10}{3}, 4, \frac{8}{3}, \frac{4}{3}]_\alpha$$
$$\lambda_4 = [2, 3, 4, 6, 4, 2]_\alpha$$
$$\lambda_5 = [\frac{4}{3}, 2, \frac{8}{3}, 4, \frac{10}{3}, \frac{5}{3}]_\alpha$$
$$\lambda_6 = [\frac{2}{3}, 1, \frac{4}{3}, 2, \frac{5}{3}, \frac{4}{3}]_\alpha$$

Step 2 Now from the θ-diagram for this case we have that

$$\pi(\alpha_1) = \pi(\alpha_6) = \omega_4$$
$$\pi(\alpha_2) = \omega_1$$
$$\pi(\alpha_3) = \pi(\alpha_5) = \omega_3$$
$$\pi(\alpha_4) = \omega_2$$

We therefore have that

$$\gamma_1 = [1,2,3,2]_\omega$$
$$\gamma_2 = [2,3,4,2]_\omega$$
$$\gamma_3 = [2,4,6,3]_\omega$$
$$\gamma_4 = [3,6,8,4]_\omega$$
$$\gamma_5 = [2,4,6,3]_\omega$$
$$\gamma_6 = [1,2,3,2]_\omega$$

Step 3 Compute the fundamental dominant weights, $\mu_i,\ i \in \{1,2,3,4\}$ with respect to $\Delta(\mathfrak{a})$ for the lattice $\Lambda_\mathfrak{a}$.

$$\mu_1 = [2,3,4,2]_\omega$$
$$\mu_2 = [3,6,8,4]_\omega$$
$$\mu_3 = [2,4,6,3]_\omega$$
$$\mu_4 = [1,2,3,2]_\omega$$

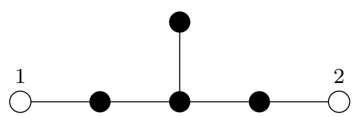
Step 4 Show explicitly that $\pi(\Lambda_\mathfrak{t}) \subseteq \Lambda_\mathfrak{a}$ as guaranteed by Theorem 4.1(1).

$$\gamma_1 = \gamma_6 = \mu_4$$
$$\gamma_2 = \mu_1$$
$$\gamma_3 = \gamma_5 = \mu_3$$
$$\gamma_4 = \mu_2$$

Step 5 Show explicitly that $\pi(\Lambda_{\mathfrak{t}}) \supseteq \Lambda_{\mathfrak{a}}$ thereby proving Theorem 4.2 for this particular case.

$$\mu_1 = \gamma_2$$
$$\mu_2 = \gamma_4$$
$$\mu_3 = \gamma_5$$
$$\mu_4 = \gamma_6$$

4.8 Case *E*IV



For the situation here we have that $\Delta(\mathfrak{t}) = \{\alpha_1, \alpha_2, \cdots, \alpha_6\}$ and $\Delta(\mathfrak{a}) = \{\omega_1, \omega_2\}$. We proceed, as before, with the following steps:

Step 1 Compute the weights, $\lambda_i,\ i \in \{1, 2, \cdots, 6\}$, with respect to $\Delta(\mathfrak{t})$ for the lattice $\Lambda_{\mathfrak{t}}$.

$$\lambda_1 = [\frac{4}{3}, 1, \frac{5}{3}, 2, \frac{4}{3}, \frac{2}{3}]_\alpha$$
$$\lambda_2 = [1, 2, 2, 3, 2, 1]_\alpha$$
$$\lambda_3 = [\frac{5}{3}, 2, \frac{10}{3}, 4, \frac{8}{3}, \frac{4}{3}]_\alpha$$
$$\lambda_4 = [2, 3, 4, 6, 4, 2]_\alpha$$
$$\lambda_5 = [\frac{4}{3}, 2, \frac{8}{3}, 4, \frac{10}{3}, \frac{5}{3}]_\alpha$$
$$\lambda_6 = [\frac{2}{3}, 1, \frac{4}{3}, 2, \frac{5}{3}, \frac{4}{3}]_\alpha$$

Step 2 Now from the θ-diagram for this case we have that

$$\pi(\alpha_1) = \omega_1$$
$$\pi(\alpha_2) = \pi(\alpha_3) = \pi(\alpha_4) = \pi(\alpha_5) = 0$$
$$\pi(\alpha_6) = \omega_2$$

We therefore have that

$$\gamma_1 = [\frac{4}{3}, \frac{2}{3}]_\omega$$
$$\gamma_2 = [1, 1]_\omega$$
$$\gamma_3 = [\frac{5}{3}, \frac{4}{3}]_\omega$$
$$\gamma_4 = [2, 2]_\omega$$
$$\gamma_5 = [\frac{4}{3}, \frac{5}{3}]_\omega$$
$$\gamma_6 = [\frac{2}{3}, \frac{4}{3}]_\omega$$

Step 3 Compute the fundamental dominant weights, $\mu_i,\ i \in \{1, 2\}$ with respect to $\Delta(\mathfrak{a})$ for the lattice $\Lambda_{\mathfrak{a}}$.

$$\mu_1 = [\frac{2}{3}, \frac{1}{3}]_\omega$$
$$\mu_2 = [\frac{1}{3}, \frac{2}{3}]_\omega$$

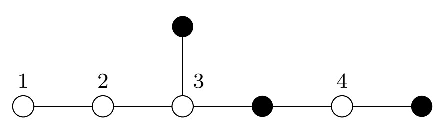
Step 4 Show explicitly that $\pi(\Lambda_{\mathfrak{t}}) \subseteq \Lambda_{\mathfrak{a}}$ as guaranteed by Theorem 4.1(1).

$$\begin{aligned}
\gamma_1 &= 2\mu_1 \\
\gamma_2 &= \mu_1 + \mu_2 \\
\gamma_3 &= 2\mu_1 + \mu_2 \\
\gamma_4 &= 2(\mu_1 + \mu_2) \\
\gamma_5 &= \mu_1 + 2\mu_2 \\
\gamma_6 &= 2\mu_2
\end{aligned}$$

Step 5 Show explicitly that $\pi(\Lambda_{\mathfrak{t}}) \supseteq \Lambda_{\mathfrak{a}}$ thereby proving Theorem 4.2 for this particular case.

$$\begin{aligned}
\mu_1 &= \gamma_3 - \gamma_2 \\
\mu_2 &= \gamma_5 - \gamma_2
\end{aligned}$$

4.9 Case *E*VI



In this case we have that $\Delta(\mathfrak{t}) = \{\alpha_1, \alpha_2, \cdots, \alpha_7\}$ and $\Delta(\mathfrak{a}) = \{\omega_1, \omega_2, \omega_3, \omega_4\}$.
We proceed, as before, with the following steps:
Step 1 Compute the weights, $\lambda_i,\ i \in \{1, 2, \cdots, 7\}$, with respect to $\Delta(\mathfrak{t})$ for the lattice

$\Lambda_{\mathfrak{t}}$.

$$\lambda_1 = [2, 2, 3, 4, 3, 2, 1]_\alpha$$
$$\lambda_2 = [2, \frac{7}{2}, 4, 6, \frac{9}{2}, 3, \frac{3}{2}]_\alpha$$
$$\lambda_3 = [3, 4, 6, 8, 6, 4, 2]_\alpha$$
$$\lambda_4 = [4, 6, 8, 12, 9, 6, 3]_\alpha$$
$$\lambda_5 = [3, \frac{9}{2}, 6, 9, \frac{15}{2}, 5, \frac{5}{2}]_\alpha$$
$$\lambda_6 = [2, 3, 4, 6, 5, 4, 2]_\alpha$$
$$\lambda_7 = [1, \frac{3}{2}, 2, 3, \frac{5}{2}, 2, \frac{3}{2}]_\alpha$$

Step 2 Now from the θ-diagram for this case we have that

$$\pi(\alpha_1) = \omega_1$$
$$\pi(\alpha_2) = 0$$
$$\pi(\alpha_3) = \omega_2$$
$$\pi(\alpha_4) = \omega_3$$
$$\pi(\alpha_5) = 0$$
$$\pi(\alpha_6) = \omega_4$$
$$\pi(\alpha_7) = 0$$

We therefore have that

$$\gamma_1 = [2,3,4,2]_\omega$$
$$\gamma_2 = [2,4,6,3]_\omega$$
$$\gamma_3 = [3,6,8,4]_\omega$$
$$\gamma_4 = [4,8,12,6]_\omega$$
$$\gamma_5 = [3,6,9,5]_\omega$$
$$\gamma_6 = [2,4,6,4]_\omega$$
$$\gamma_7 = [1,2,3,2]_\omega$$

Step 3 Compute the fundamental dominant weights, $\mu_i,\ i \in \{1,2,3,4\}$ with respect to $\Delta(\mathfrak{a})$ for the lattice $\Lambda_{\mathfrak{a}}$.

$$\mu_1 = [2,3,4,2]_\omega$$
$$\mu_2 = [3,6,8,4]_\omega$$
$$\mu_3 = [2,4,6,3]_\omega$$
$$\mu_4 = [1,2,3,2]_\omega$$

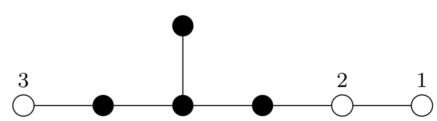
Step 4 Show explicitly that $\pi(\Lambda_{\mathfrak{t}}) \subseteq \Lambda_{\mathfrak{a}}$ as guaranteed by Theorem 4.1(1).

$$\gamma_1 = \mu_1$$
$$\gamma_2 = \mu_3$$
$$\gamma_3 = \mu_2$$
$$\gamma_4 = 2\mu_3$$
$$\gamma_5 = \mu_3 + \mu_4$$
$$\gamma_6 = 2\mu_4$$
$$\gamma_7 = \mu_4$$

Step 5 Show explicitly that $\pi(\Lambda_{\mathfrak{t}}) \supseteq \Lambda_{\mathfrak{a}}$ thereby proving Theorem 4.2 for this particular case.

$$\mu_1 = \gamma_1$$
$$\mu_2 = \gamma_3$$
$$\mu_3 = \gamma_2$$
$$\mu_4 = \gamma_7$$

4.10 Case *E*VII



In this case we have that $\Delta(\mathfrak{t}) = \{\alpha_1, \alpha_2, \cdots, \alpha_7\}$ and $\Delta(\mathfrak{a}) = \{\omega_1, \omega_2, \omega_3\}$.

We proceed, as before, with the following steps:

Step 1 Compute the weights, $\lambda_i,\ i \in \{1, 2, \cdots, 7\}$, with respect to $\Delta(\mathfrak{t})$ for the lattice $\Lambda_{\mathfrak{t}}$.

$$\lambda_1 = [2, 2, 3, 4, 3, 2, 1]_\alpha$$
$$\lambda_2 = [2, \frac{7}{2}, 4, 6, \frac{9}{2}, 3, \frac{3}{2}]_\alpha$$
$$\lambda_3 = [3, 4, 6, 8, 6, 4, 2]_\alpha$$
$$\lambda_4 = [4, 6, 8, 12, 9, 6, 3]_\alpha$$
$$\lambda_5 = [3, \frac{9}{2}, 6, 9, \frac{15}{2}, 5, \frac{5}{2}]_\alpha$$
$$\lambda_6 = [2, 3, 4, 6, 5, 4, 2]_\alpha$$
$$\lambda_7 = [1, \frac{3}{2}, 2, 3, \frac{5}{2}, 2, \frac{3}{2}]_\alpha$$

Step 2 Now from the θ-diagram for this case we have that

$$\pi(\alpha_1) = \omega_3$$
$$\pi(\alpha_2) = \pi(\alpha_3) = \pi(\alpha_4) = \pi(\alpha_5) = 0$$
$$\pi(\alpha_6) = \omega_2$$
$$\pi(\alpha_7) = \omega_3$$

We therefore have that

$$\gamma_1 = [1, 2, 2]_\omega$$
$$\gamma_2 = [\frac{3}{2}, 3, 2]_\omega$$
$$\gamma_3 = [2, 4, 3]_\omega$$
$$\gamma_4 = [3, 6, 4]_\omega$$
$$\gamma_5 = [\frac{5}{2}, 5, 3]_\omega$$
$$\gamma_6 = [2, 4, 2]_\omega$$
$$\gamma_7 = [\frac{3}{2}, 2, 1]_\omega$$

Step 3 Compute the fundamental dominant weights, $\mu_i,\ i \in \{1, 2, 3\}$ with respect to $\Delta(\mathfrak{a})$ for the lattice $\Lambda_{\mathfrak{a}}$.

$$\mu_1 = [\frac{3}{2}, 2, 1]_\omega$$
$$\mu_2 = [1, 2, 1]_\omega$$
$$\mu_3 = [\frac{1}{2}, 1, 1]_\omega$$

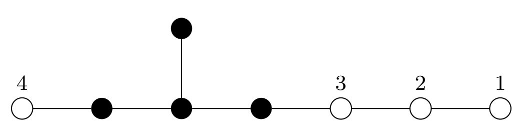
Step 4 Show explicitly that $\pi(\Lambda_{\mathfrak{t}}) \subseteq \Lambda_{\mathfrak{a}}$ as guaranteed by Theorem 4.1(1).

$$\gamma_1 = 2\mu_3$$
$$\gamma_2 = \mu_2 + \mu_3$$
$$\gamma_3 = \mu_2 + 2\mu_3$$
$$\gamma_4 = 2(\mu_2 + \mu_3)$$
$$\gamma_5 = 2\mu_2 + \mu_3$$
$$\gamma_6 = 2\mu_2$$
$$\gamma_7 = \mu_1$$

Step 5 Show explicitly that $\pi(\Lambda_{\mathfrak{t}}) \supseteq \Lambda_{\mathfrak{a}}$ thereby proving Theorem 4.2 for this particular case.

$$\mu_1 = \gamma_7$$
$$\mu_2 = \gamma_3 - \gamma_1$$
$$\mu_3 = \gamma_3 - \gamma_2$$

4.11 Case *E*IX



In this case we have that $\Delta(\mathfrak{t}) = \{\alpha_1, \alpha_2, \cdots, \alpha_8\}$ and $\Delta(\mathfrak{a}) = \{\omega_1, \omega_2, \omega_3, \omega_4\}$.

We proceed, as before, with the following steps:

Step 1 Compute the weights, $\lambda_i,\ i \in \{1, 2, \cdots, 8\}$, with respect to $\Delta(\mathfrak{t})$ for the lattice

$\Lambda_{\mathfrak{t}}$.

$$\lambda_1 = [4, 5, 7, 10, 8, 6, 4, 2]_\alpha$$
$$\lambda_2 = [5, 8, 10, 15, 12, 9, 6, 3]_\alpha$$
$$\lambda_3 = [7, 10, 14, 20, 16, 12, 8, 4]_\alpha$$
$$\lambda_4 = [10, 15, 20, 30, 24, 18, 12, 6]_\alpha$$
$$\lambda_5 = [8, 12, 16, 24, 20, 15, 10, 5]_\alpha$$
$$\lambda_6 = [6, 9, 12, 18, 15, 12, 8, 4]_\alpha$$
$$\lambda_7 = [4, 6, 8, 12, 10, 8, 6, 3]_\alpha$$
$$\lambda_8 = [2, 3, 4, 6, 5, 4, 3, 2]_\alpha$$

Step 2 Now from the θ-diagram for this case we have that

$$\pi(\alpha_1) = \omega_4$$
$$\pi(\alpha_2) = \pi(\alpha_3) = \pi(\alpha_4) = \pi(\alpha_5) = 0$$
$$\pi(\alpha_6) = \omega_3$$
$$\pi(\alpha_7) = \omega_2$$
$$\pi(\alpha_8) = \omega_1$$

We therefore have that

$$\gamma_1 = [2, 4, 6, 4]_\omega$$
$$\gamma_2 = [3, 6, 9, 5]_\omega$$
$$\gamma_3 = [4, 8, 12, 7]_\omega$$
$$\gamma_4 = [6, 12, 18, 10]_\omega$$
$$\gamma_5 = [5, 10, 15, 8]_\omega$$
$$\gamma_6 = [4, 8, 12, 6]_\omega$$
$$\gamma_7 = [3, 6, 8, 4]_\omega$$
$$\gamma_8 = [2, 3, 4, 2]_\omega$$

Step 3 Compute the fundamental dominant weights, $\mu_i,\ i \in \{1, 2, 3, 4\}$ with respect to $\Delta(\mathfrak{a})$ for the lattice $\Lambda_{\mathfrak{a}}$.

$$\mu_1 = [2, 3, 4, 2]_\omega$$
$$\mu_2 = [3, 6, 8, 4]_\omega$$
$$\mu_3 = [2, 4, 6, 3]_\omega$$
$$\mu_4 = [1, 2, 3, 2]_\omega$$

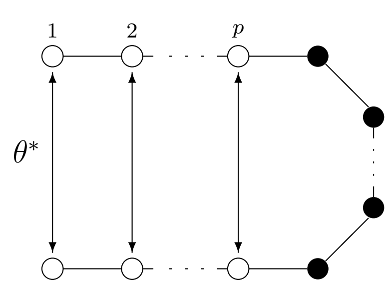
Step 4 Show explicitly that $\pi(\Lambda_{\mathfrak{t}}) \subseteq \Lambda_{\mathfrak{a}}$ as guaranteed by Theorem 4.1(1).

$$\gamma_1 = 2\mu_4$$
$$\gamma_2 = \mu_3 + \mu_4$$
$$\gamma_3 = \mu_3 + 2\mu_4$$
$$\gamma_4 = 2(\mu_3 + \mu_4)$$
$$\gamma_5 = 2\mu_3 + \mu_4$$
$$\gamma_6 = 2\mu_3$$
$$\gamma_7 = \mu_2$$
$$\gamma_8 = \mu_1$$

Step 5 Show explicitly that $\pi(\Lambda_{\mathfrak{t}}) \supseteq \Lambda_{\mathfrak{a}}$ thereby proving Theorem 4.2 for this particular case.

$$\mu_1 = \gamma_8$$
$$\mu_2 = \gamma_7$$
$$\mu_2 = \gamma_5 - \gamma_2$$
$$\mu_3 = \gamma_3 - \gamma_2$$

4.12 Case $AIII_a$



We alert the reader to the fact that this is the first case such that $\Phi(\mathfrak{a})$ is of type BC_n. We therefore apply the "type 2" method of proof outlined at the beginning of this chapter. We have the following

Proposition 14. *Let γ and ω be as above (relative to the case $AIII_a$). We have that for each $k \in \{1, 2, \cdots, n\}$ then*

$$\gamma_k = \sum_{j=1}^{p} \Gamma_{k,j}\omega_j \tag{4.54}$$

where $\Gamma_{i,j}$ is the $n \times p$ matrix (with $2p \leq n$) given by

$$\Gamma_{i,j} = \begin{cases} j & j < i \leq p \\ i & i \leq j \leq p \\ j & p+1 \leq i \leq n-p \\ n-i+1 & n-i \leq j, n-p+1 \leq i \leq n \\ j & j < n-i, n-p+1 \leq i \leq n \end{cases}$$

Proof. For this case we have that $\mid \Delta(\mathfrak{t}) \mid = n$ and $\mid \Delta(\mathfrak{a}) \mid = p$ with $2p \leq n$. Let $\Delta(\mathfrak{t}) = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ be a base for $\Phi(\mathfrak{t})$ and let $\Delta(\mathfrak{a}) = \{\omega_1, \omega_2, \cdots, \omega_p\}$ be a base for $\Phi(\mathfrak{a})$. As $\Phi(\mathfrak{t})$ here is of type A_n, then the fundamental weights for the lattice $\Lambda_{\mathfrak{t}}$ are given by

$$\lambda_i = \sum_{j=1}^{n} (A_n^{-1})_{ij} \cdot \alpha_j \qquad i \in \{1, 2, \cdots, n\} \tag{4.55}$$

Proceeding as before we have from the θ-diagram for this case that

$$\pi(\alpha_j) = \begin{cases} \omega_j & 1 \leq j \leq p \\ 0 & p+1 \leq j \leq n-p \\ \omega_{n-j+1} & n-p+1 \leq j \leq n \end{cases} \tag{4.56}$$

Thus

$$\gamma_i = \sum_{j=1}^{p} \{(A_n^{-1})_{i,j} + (A_n)^{-1}_{i,n+1-j}\} \cdot \omega_j \qquad i \in \{1, 2, \cdots, n\} \tag{4.57}$$

To prove the proposition it suffices to show that

$$\Gamma_{i,j} = (A_n)^{-1}_{i,j} + (A_n)^{-1}_{i,n+1-j} \tag{4.58}$$

for each $j \in \{1, 2, \cdots, p\}$ and $i \in \{1, 2, \cdots, n\}$. Let $s_{i,j} = (A_n)^{-1}_{i,j} + (A_n)^{-1}_{i,n+1-j}$. We consider several cases for the values of i and j corresponding to those for Γ_{ij} in Proposition 14. First, let $j < i (\leq p \leq \frac{n}{2})$. Then $i < n+1-j$. Thus by Lemma 3,

$$s_{i,j} = j(1 - \frac{i}{n+1}) + i(1 - \frac{n+1-j}{n+1}) = j.$$

Next, let $i \leq j (\leq p \leq \frac{n}{2})$. Then $i \leq n+1-j$ so that

$$s_{i,j} = i(1 - \frac{j}{n+1}) + i(1 - \frac{n+1-j}{n+1}) = i$$

again by Lemma 3. If $p+1 \leq i \leq n-p$ then as $1 \leq j \leq p$ we have that $i > j$ and $i < n+1-j$. Thus

$$s_{i,j} = j(1 - \frac{i}{n+1}) + i(1 - \frac{n+1-j}{n+1}) = j.$$

Now, if $n - i \leq j \leq p$ and $n-p+1 \leq i \leq n$ then clearly $i > max\{j, n+1-j\}$. Therefore,

$$s_{i,j} = j(1 - \frac{i}{n+1}) + (n+1-j)(1 - \frac{i}{n+1}) = n+1-j.$$

Finally, let $j < n-i$ and $n-p+1 \leq i \leq n$. Clearly $j < i$. Also, $j < n-i$ implies that $i \leq n+1-j$. Then

$$s_{i,j} = j(1 - \frac{i}{n+1}) + i(1 - \frac{n+1-j}{n+1}) = j.$$

□

Remark 6. Observe that Γ (above) is an integer matrix, verifying, in this case, that $\pi(\Lambda_{\mathfrak{t}}) \subseteq R_{\mathfrak{a}}$.

We now give the main result for this section.

Theorem 4.9. *Let γ, ω, $\pi(\Lambda_{\mathfrak{t}})$, and $R_{\mathfrak{a}}$, be as above (relative to the case $AIII_a$). We then have that*

(1)

$$\omega_k = \begin{cases} 2\gamma_1 - \gamma_2 & k = 1 \\ -\gamma_{k-1} + 2\gamma_k - \gamma_{k+1} & 2 \le k \le p-1 \\ \gamma_p - \gamma_{p-1} & k = p \end{cases} \tag{4.59}$$

(2) $\pi(\Lambda_{\mathfrak{t}}) \supseteq R_{\mathfrak{a}}$.

Proof. Let $G_{i,j}$ be the $p \times p$ matrix defined by

$$G_{i,j} = \Gamma_{i,j} \qquad 1 \le i, j \le p$$

Thus

$$G_{i,j} = \begin{cases} j & j < i \le p \\ i & i \le j \le p \end{cases}$$

Observe that for each $i \in \{1, 2, \cdots, p\}$

$$\gamma_i = \sum_j^p G_{i,j} \cdot \omega_j.$$

In other words, $G_{i,j}$ gives the coefficients for the first p γ_i's, relative to the base $\Delta(\mathfrak{a})$. Now for $k = 1$ or p the result is obvious. Let $k = \{2, 3, \cdots, p-1\}$. We have that

$$-G_{k-1,j} + 2G_{k,j} - G_{k+1,j} = \delta_{kj}$$

for each $j \in \{1, 2, \cdots, p\}$. This implies that

$$-\gamma_{k-1} + 2\gamma_k - \gamma_{k+1} = \omega_k.$$

$\square$

Example 4.6. We consider the special case, $A_7^3(\mathrm{III}_a)$. Let $\Delta(\mathfrak{t}) = \{\alpha_1, \alpha_2, \cdots, \alpha_7\}$ and $\Delta(\mathfrak{a}) = \{\omega_1, \omega_2, \omega_3\}$.

Step 1 Compute the weights, $\lambda_i,\ i \in \{1, 2, \cdots, 7\}$, with respect to $\Delta(\mathfrak{t})$ for the lattice $\Lambda_{\mathfrak{t}}$.

$$\begin{aligned}
\lambda_1 &= \frac{1}{8}[7, 6, 5, 4, 3, 2, 1]_\alpha \\
\lambda_2 &= \frac{1}{8}[6, 12, 10, 8, 6, 4, 2]_\alpha \\
\lambda_3 &= \frac{1}{8}[5, 10, 15, 12, 9, 6, 3]_\alpha \\
\lambda_4 &= \frac{1}{8}[4, 8, 12, 16, 12, 8, 4]_\alpha \\
\lambda_5 &= \frac{1}{8}[3, 6, 9, 12, 15, 10, 5]_\alpha \\
\lambda_6 &= \frac{1}{8}[2, 4, 6, 8, 10, 12, 6]_\alpha \\
\lambda_7 &= \frac{1}{8}[1, 2, 3, 4, 5, 6, 7]_\alpha
\end{aligned}$$

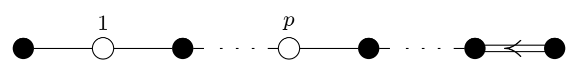
Step 2 Using the θ-diagram we compute the projections $\gamma_i = \pi(\lambda_i),\ i \in \{1, 2, \cdots, 7\}$, with respect to the base $\Delta(\mathfrak{a})$. This gives an explicit formulation for the relationship between the lattices $\pi(\Lambda_{\mathfrak{t}})$ and $R_{\mathfrak{a}}$, verifying, for this example, the fact that $\pi(\Lambda_{\mathfrak{t}}) \subseteq R_{\mathfrak{a}}$.

$$\begin{aligned}
\gamma_1 &= [1, 1, 1]_\omega \\
\gamma_2 &= [1, 2, 2]_\omega \\
\gamma_3 &= [1, 2, 3]_\omega \\
\gamma_4 &= [1, 2, 3]_\omega \\
\gamma_5 &= [1, 2, 3]_\omega \\
\gamma_6 &= [1, 2, 2]_\omega \\
\gamma_7 &= [1, 1, 1]_\omega
\end{aligned}$$

<u>Step 3</u> Show explicitly that $\pi(\Lambda_{\mathfrak{t}}) \supseteq R_{\mathfrak{a}}$ thereby verifying Theorem 4.2 for this example.

$$\omega_1 = 2\gamma_1 - \gamma_2$$
$$\omega_2 = -\gamma_1 + 2\gamma_2 - \gamma_3$$
$$\omega_1 = \gamma_3 - \gamma_2$$

4.13 Case $C\mathrm{II}_a$



As in the previous case, we have here also that $\Phi(\mathfrak{a})$ is of type BC_n. We again apply the "type 2" method of proof outlined above and begin with the following

Proposition 15. *Let γ and ω be as above (relative to the case CII_a). We have that for each $k \in \{1, 2, \cdots, n\}$, then*

$$\gamma_k = \sum_{j=1}^{p} \Gamma_{k,j}\omega_j \tag{4.60}$$

where $\Gamma_{i,j}$ is the $n \times p$ matrix (with $2p + 2 \leq n$) given by

$$\Gamma_{i,j} = \begin{cases} 2j & 2j < i \leq n \\ i & i \leq 2j \leq n \end{cases}$$

Proof. For this case we have that $\mid \Delta(\mathfrak{t}) \mid= n$ and $\mid \Delta(\mathfrak{a}) \mid= p$ with $2p + 2 \leq n$. Let $\Delta(\mathfrak{t}) = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ be a base for $\Phi(\mathfrak{t})$ and let $\Delta(\mathfrak{a}) = \{\omega_1, \omega_2, \cdots, \omega_p\}$ be a base for $\Phi(\mathfrak{a})$. As $\Phi(\mathfrak{t})$ here is of type C_n, then the fundamental weights for the lattice $\Lambda_{\mathfrak{t}}$ are given by

$$\lambda_i = \sum_{j=1}^{n} (C_n^{-1})_{ij} \cdot \alpha_j \qquad i \in \{1, 2, \cdots, n\} \tag{4.61}$$

Proceeding as before we have from the θ-diagram that

$$\pi(\alpha_j) = \begin{cases} 0 & j \in \{1,3,5,\cdots,2p-1\} \\ \omega_{\frac{j}{2}} & j \in \{2,4,6,\cdots,2p\} \\ 0 & 2p+1 \leq j \leq n \end{cases} \tag{4.62}$$

Thus

$$\gamma_i = \sum_{j=1}^{p} (C_n^{-1})_{i,2j} \cdot \omega_j \qquad i \in \{1,2,\cdots,n\} \tag{4.63}$$

The proposition follows at once from this equation and Lemma 5. □

Remark 7. As Γ is an integer matrix, then thanks to this proposition, we again have that $\pi(\Lambda_{\mathfrak{t}}) \subseteq R_{\mathfrak{a}}$.

We now give the main result for this section.

Theorem 4.10. *Let γ, ω, $\pi(\Lambda_{\mathfrak{t}})$, and $R_{\mathfrak{a}}$, be as above (relative to the case CII_a). We then have that*

(1)

$$\omega_k = \begin{cases} -\gamma_{2k-1} + 2\gamma_{2k} - \gamma_{2k+1} & 1 \leq k \leq p-1 \\ \gamma_{2p-1} - \gamma_{2(p-1)} & k = p \end{cases} \tag{4.64}$$

(2) $\pi(\Lambda_{\mathfrak{t}}) \supseteq R_{\mathfrak{a}}$.

Proof. The second case follows at once from the definition of Γ in the proposition above. To establish the first case fix $k \in \{1,2,\cdots,p-1\}$. We have that

$$\begin{aligned} -\Gamma_{2k-1,j} + 2\Gamma_{2k,j} - \Gamma_{2k+1,j} &= \begin{cases} -2j + 2(2j) - 2j = 0 & 2 \leq 2j \leq 2k-2 \\ -(2k-1) + 2(2k) - 2k = 1 & 2j = 2k \\ -(2k-1) - 2(2k) - (2k+1) = 0 & 2k+2 \leq 2j \end{cases} \\ &= \delta_{kj}. \end{aligned}$$

As Γ gives the coefficients for the γ_i's relative to $\Delta(\mathfrak{a})$ the result follows. □

Example 4.7. We consider the special case, $C_{12}^{5}(\mathrm{II}_a)$. Let $\Delta(\mathfrak{t}) = \{\alpha_1, \alpha_2, \cdots, \alpha_{12}\}$ and $\Delta(\mathfrak{a}) = \{\omega_1, \cdots, \omega_5\}$.

Step 1 Compute the weights, $\lambda_i,\ i \in \{1, 2, \cdots, 12\}$, with respect to $\Delta(\mathfrak{t})$ for the lattice $\Lambda_{\mathfrak{t}}$.

$$\begin{aligned}
\lambda_1 &= [1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, \frac{1}{2}]_\alpha \\
\lambda_2 &= [1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 1]_\alpha \\
\lambda_3 &= [1, 2, 3, 3, 3, 3, 3, 3, 3, 3, 3, \frac{3}{2}]_\alpha \\
\lambda_4 &= [1, 2, 3, 4, 4, 4, 4, 4, 4, 4, 4, 2]_\alpha \\
\lambda_5 &= [1, 2, 3, 4, 5, 5, 5, 5, 5, 5, 5, \frac{5}{2}]_\alpha \\
\lambda_6 &= [1, 2, 3, 4, 5, 6, 6, 6, 6, 6, 6, 3]_\alpha \\
\lambda_7 &= [1, 2, 3, 4, 5, 6, 7, 7, 7, 7, 7, \frac{7}{2}]_\alpha \\
\lambda_8 &= [1, 2, 3, 4, 5, 6, 7, 8, 8, 8, 8, 4]_\alpha \\
\lambda_9 &= [1, 2, 3, 4, 5, 6, 7, 8, 9, 9, 9, \frac{9}{2}]_\alpha \\
\lambda_{10} &= [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 10, 5]_\alpha \\
\lambda_{11} &= [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, \frac{11}{2}]_\alpha \\
\lambda_{12} &= [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 6]_\alpha
\end{aligned}$$

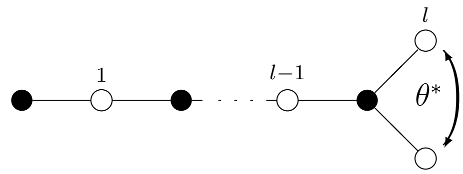
Step 2 Using the θ-diagram we compute the projections $\gamma_i = \pi(\lambda_i),\ i \in \{1, 2, \cdots, 12\}$, with respect to the base $\Delta(\mathfrak{a})$. In this way we verify that $\pi(\Lambda_{\mathfrak{t}}) \subseteq R_{\mathfrak{a}}$.

$$\begin{aligned}
\gamma_1 &= [1,1,1,1,1]_\omega \\
\gamma_2 &= [2,2,2,2,2]_\omega \\
\gamma_3 &= [2,3,3,3,3]_\omega \\
\gamma_4 &= [2,4,4,4,4]_\omega \\
\gamma_5 &= [2,4,5,5,5]_\omega \\
\gamma_6 &= [2,4,6,6,6]_\omega \\
\gamma_7 &= [2,4,6,7,7]_\omega \\
\gamma_8 &= [2,4,6,8,8]_\omega \\
\gamma_9 &= [2,4,6,8,9]_\omega \\
\gamma_{10} &= [2,4,6,8,10]_\omega \\
\gamma_{11} &= [2,4,6,8,10]_\omega \\
\gamma_{12} &= [2,4,6,8,10]_\omega
\end{aligned}$$

Step 3 Show explicitly that $\pi(\Lambda_{\mathfrak{t}}) \supseteq R_{\mathfrak{a}}$ thereby verifying Theorem 4.2 for this example.

$$\begin{aligned}
\omega_1 &= -\gamma_1 + 2\gamma_2 - \gamma_3 \\
\omega_2 &= -\gamma_3 + 2\gamma_4 - \gamma_5 \\
\omega_3 &= -\gamma_5 + 2\gamma_6 - \gamma_7 \\
\omega_4 &= -\gamma_7 + 2\gamma_8 - \gamma_9 \\
\omega_5 &= \gamma_9 - \gamma_8
\end{aligned}$$

4.14 Case $D\mathrm{III}_b$



The formulation for the containment of $\pi(\Lambda_{\mathfrak{t}})$ in the root lattice $R_{\mathfrak{a}}$ for $D\mathrm{III}_b$ is given herewith.

Proposition 16. *Let γ and ω be as above (relative to the case DIII$_b$). We have that for each $k \in \{1, 2, \cdots, 2n+1\}$, then*

$$\gamma_k = \sum_{j=1}^{n} \Gamma_{k,j}\omega_j \tag{4.65}$$

where $\Gamma_{i,j}$ is the $(2n+1) \times n$ matrix given by

$$\Gamma_{i,j} = \begin{cases} i & i < 2j \\ 2j & 2j \leq i \leq 2n-1 \\ j & i \in \{2n, 2n+1\} \end{cases}$$

Proof. For this case we have that $\mid \Delta(\mathfrak{t}) \mid = 2n+1$ and $\mid \Delta(\mathfrak{a}) \mid = n$. Fix a base $\Delta(\mathfrak{t}) = \{\alpha_1, \alpha_2, \cdots, \alpha_{2n+1}\}$ for $\Phi(\mathfrak{t})$ and let $\Delta(\mathfrak{a}) = \{\omega_1, \omega_2, \cdots, \omega_n\}$ be a base for $\Phi(\mathfrak{a})$. As $\Phi(\mathfrak{t})$ here is of type D_{2n+1}, then the fundamental weights for the lattice $\Lambda_{\mathfrak{t}}$ are given by

$$\lambda_i = \sum_{j=1}^{2n+1} (D_{2n+1}^{-1})_{ij} \cdot \alpha_j \qquad i \in \{1, 2, \cdots, 2n+1\} \tag{4.66}$$

Proceeding as before we have from the θ-diagram that

$$\pi(\alpha_j) = \begin{cases} 0 & j \in \{1, 3, 5, \cdots, 2n-1\} \\ \omega_{\frac{j}{2}} & j \in \{2, 4, 6, \cdots, 2n-2\} \\ \omega_n & j \in \{2n, 2n+1\} \end{cases} \tag{4.67}$$

Thus

$$\gamma_i = \sum_{j=1}^{n-1} (D_{2n+1}^{-1})_{i,2j} \cdot \omega_j + \{(D_{2n+1}^{-1})_{i,2n} + (D_{2n+1}^{-1})_{i,2n+1}\} \cdot \omega_n \qquad i \in \{1, 2, \cdots, 2n+1\} \tag{4.68}$$

The proposition follows at once from this equation and Lemma 6. □

Remark 8. As Γ is an integer matrix, then thanks to this proposition, we again have that $\pi(\Lambda_{\mathfrak{t}}) \subseteq R_{\mathfrak{a}}$.

We now give the main result for this section.

Theorem 4.11. *Let γ, ω, $\pi(\Lambda_{\mathfrak{t}})$, and $R_{\mathfrak{a}}$, be as above (relative to the case DIII$_b$). We then have that*

(1)

$$\omega_k = \begin{cases} -\gamma_{2k-1} + 2\gamma_{2k} - \gamma_{2k+1} & 1 \le k \le n-1 \\ \gamma_{2n-1} - \gamma_{2(n-1)} & k = n \end{cases} \tag{4.69}$$

(2) $\pi(\Lambda_{\mathfrak{t}}) \supseteq R_{\mathfrak{a}}$.

Proof. The second case follows at once from the definition of Γ in the proposition above. To establish the first case fix $k \in \{1, 2, \cdots, n-1\}$. We have that

$$\begin{aligned} -\Gamma_{2k-1,j} + 2\Gamma_{2k,j} - \Gamma_{2k+1,j} &= \begin{cases} -2j + 2(2j) - 2j = 0 & 2 \le 2j \le 2k-2 \\ -(2k-1) + 2(2k) - 2k = 1 & 2j = 2k \\ -(2k-1) - 2(2k) - (2k+1) = 0 & 2k+2 \le 2j \end{cases} \\ &= \delta_{kj}. \end{aligned}$$

As Γ gives the coefficients for the γ_i's relative to $\Delta(\mathfrak{a})$ the result follows. □

Example 4.8. We consider the special case, $D_9^4(\mathrm{III}_b)$. Let $\Delta(\mathfrak{t}) = \{\alpha_1, \alpha_2, \cdots, \alpha_9\}$ and $\Delta(\mathfrak{a}) = \{\omega_1, \cdots, \omega_4\}$.

Step 1 Compute the weights, $\lambda_i,\ i \in \{1, 2, \cdots, 9\}$, with respect to $\Delta(\mathfrak{t})$ for the lattice $\Lambda_{\mathfrak{t}}$.

$$\lambda_1 = [1, 1, 1, 1, 1, 1, 1, \frac{1}{2}, \frac{1}{2}]_\alpha$$
$$\lambda_2 = [1, 2, 2, 2, 2, 2, 2, 1, 1]_\alpha$$
$$\lambda_3 = [1, 2, 3, 3, 3, 3, 3, \frac{3}{2}, \frac{3}{2}]_\alpha$$
$$\lambda_4 = [1, 2, 3, 4, 4, 4, 4, 2, 2]_\alpha$$
$$\lambda_5 = [1, 2, 3, 4, 5, 5, 5, \frac{5}{2}, \frac{5}{2}]_\alpha$$
$$\lambda_6 = [1, 2, 3, 4, 5, 6, 6, 3, 3]_\alpha$$
$$\lambda_7 = [1, 2, 3, 4, 5, 6, 7, \frac{7}{2}, \frac{7}{2}]_\alpha$$
$$\lambda_8 = [\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{9}{4}, \frac{7}{4}]_\alpha$$
$$\lambda_9 = [\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{4}, \frac{9}{4}]_\alpha$$

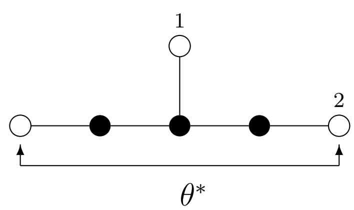
Step 2 Using the θ-diagram we compute the projections $\gamma_i = \pi(\lambda_i),\ i \in \{1, 2, \cdots, 9\}$, with respect to the base $\Delta(\mathfrak{a})$. In this way we verify that $\pi(\Lambda_{\mathfrak{t}}) \subseteq R_{\mathfrak{a}}$.

$$\gamma_1 = [1, 1, 1, 1]_\omega$$
$$\gamma_2 = [2, 2, 2, 2]_\omega$$
$$\gamma_3 = [2, 3, 3, 3]_\omega$$
$$\gamma_4 = [2, 4, 4, 4]_\omega$$
$$\gamma_5 = [2, 4, 5, 5]_\omega$$
$$\gamma_6 = [2, 4, 6, 6]_\omega$$
$$\gamma_7 = [2, 4, 6, 7]_\omega$$
$$\gamma_8 = [1, 2, 3, 4]_\omega$$
$$\gamma_9 = [1, 2, 3, 4]_\omega$$

Step 3 Show explicitly that $\pi(\Lambda_{\mathfrak{t}}) \supseteq R_{\mathfrak{a}}$ thereby verifying Theorem 4.2 for this example.

$$\omega_1 = -\gamma_1 + 2\gamma_2 - \gamma_3$$
$$\omega_2 = -\gamma_3 + 2\gamma_4 - \gamma_5$$
$$\omega_3 = -\gamma_5 + 2\gamma_6 - \gamma_7$$
$$\omega_4 = \gamma_7 - \gamma_6$$

4.15 Case *E*III



Being of fixed dimension, the treatment for this and the next case is similar to that of the previous examples. Indeed, we have that $\Delta(\mathfrak{t}) = \{\alpha_1, \alpha_2, \cdots, \alpha_6\}$ and $\Delta(\mathfrak{a}) = \{\omega_1, \omega_2\}$.

Step 1 Compute the weights, $\lambda_i,\ i \in \{1, 2, \cdots, 6\}$, with respect to $\Delta(\mathfrak{t})$ for the lattice

$\Lambda_{\mathfrak{t}}$.

$$\lambda_1 = [\frac{4}{3}, 1, \frac{5}{3}, 2, \frac{4}{3}, \frac{2}{3}]_\alpha$$
$$\lambda_2 = [1, 2, 2, 3, 2, 1]_\alpha$$
$$\lambda_3 = [\frac{5}{3}, 2, \frac{10}{3}, 4, \frac{8}{3}, \frac{4}{3}]_\alpha$$
$$\lambda_4 = [2, 3, 4, 6, 4, 2]_\alpha$$
$$\lambda_5 = [\frac{4}{3}, 2, \frac{8}{3}, 4, \frac{10}{3}, \frac{5}{3}]_\alpha$$
$$\lambda_6 = [\frac{2}{3}, 1, \frac{4}{3}, 2, \frac{5}{3}, \frac{4}{3}]_\alpha$$

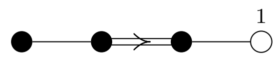
Step 2 Using the θ-diagram we compute the projections $\gamma_i = \pi(\lambda_i),\ i \in \{1, 2, \cdots, 6\}$, with respect to the base $\Delta(\mathfrak{a})$. This gives an explicit formulation for the relationship between the lattices $\pi(\Lambda_{\mathfrak{t}})$ and $R_{\mathfrak{a}}$, verifying, for this case, the fact that $\pi(\Lambda_{\mathfrak{t}}) \subseteq R_{\mathfrak{a}}$.

$$\gamma_1 = [2, 1]_\omega$$
$$\gamma_2 = [2, 2]_\omega$$
$$\gamma_3 = [3, 2]_\omega$$
$$\gamma_4 = [4, 3]_\omega$$
$$\gamma_5 = [3, 2]_\omega$$
$$\gamma_6 = [2, 1]_\omega$$

Step 3 Show explicitly that $\pi(\Lambda_{\mathfrak{t}}) \supseteq R_{\mathfrak{a}}$ thereby verifying Theorem 4.2 for this example.

$$\omega_1 = \gamma_3 - \gamma_2$$
$$\omega_2 = \gamma_2 - \gamma_1$$

4.16 Case ***F*II**



For the case FII we have that $\Delta(\mathfrak{t}) = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ and $\Delta(\mathfrak{a}) = \{\omega_1\}$.

Step 1 Compute the weights, $\lambda_i,\ i \in \{1, 2, 3, 4\}$, with respect to $\Delta(\mathfrak{t})$ for the lattice $\Lambda_{\mathfrak{t}}$.

$$\lambda_1 = [2, 3, 4, 2]_\alpha$$
$$\lambda_2 = [3, 6, 8, 4]_\alpha$$
$$\lambda_3 = [2, 4, 6, 3]_\alpha$$
$$\lambda_4 = [1, 2, 3, 2]_\alpha$$

Step 2 Using the θ-diagram we compute the projections $\gamma_i = \pi(\lambda_i),\ i \in \{1, 2, 3, 4\}$, with respect to the base $\Delta(\mathfrak{a})$. This gives an explicit formulation for the relationship between the lattices $\pi(\Lambda_{\mathfrak{t}})$ and $R_{\mathfrak{a}}$, verifying, for this case, the fact that $\pi(\Lambda_{\mathfrak{t}}) \subseteq R_{\mathfrak{a}}$.

$$\gamma_1 = [2]_\omega$$
$$\gamma_2 = [4]_\omega$$
$$\gamma_3 = [3]_\omega$$
$$\gamma_4 = [2]_\omega$$

Step 3 Show explicitly that $\pi(\Lambda_{\mathfrak{t}}) \supseteq R_{\mathfrak{a}}$ thereby verifying Theorem 4.2 for this example.

$$\omega_1 = \gamma_3 - \gamma_1$$

List of References

[Bor91] A. Borel. *Linear Algebraic Groups*, volume 126 of *Graduate texts in mathematics*.

[FH91] W. Fulton and J. Harris. *Representation Theory*, volume 129 of *Graduate Texts in Mathematics*. Springer Verlag, New York, 1991.

[Fowler03] J. Fowler. *Algorithms for computations in local symmetric spaces*. PhD thesis, North Carolina State University, March 2003.

[Hel88] A. G. Helminck. Algebraic groups with a commuting pair of involutions and semisimple symmetric spaces. *Adv. in Math.*, 71:21–91, 1988.

[Hel00] A. G. Helminck. On the classification of k-involutions I. *Adv. in Math.*, 153(1):1–117, 2000.

[Hum72] J. E. Humphreys. *Linear algebraic groups*, volume 21 of *Graduate Texts in Mathematics*. Springer Verlag, New York, 1975.

[HW93] A. G. Helminck and S. P. Wang. On rationality properties of involutions of reductive groups. *Adv. in Math.*, 99:26–96, 1993.

[HW02] Aloysius G. Helminck and Ling Wu. Classification of involutions of $\mathrm{sl}(2, k)$. *Comm. Algebra*, 30(1):193–203, 2002.

[Spr81] T. A. Springer. *Linear algebraic groups*, volume 9 of *Progr. Math.* Birkhäuser, Boston/Basel/Stuttgart, 1981.