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# ADVANCED APPLICATIONS OF BOUNDARY-INTEGRAL EQUATION METHODS

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## SUMMARY

Numerical analysis has become the basic tool for both design and research problems in solid mechanics. The need for accuracy and detail, plus the availability of the high speed computer has led to the development of many new modeling methods ranging from general purpose structural analysis finite element programs to special purpose research programs. The boundary-integral equation (BIE) method is based on classical mathematical techniques but is finding new life as a basic stress analysis tool for engineering applications.

The BIE method is based on the numerical solution of a set of integral constraint equations which couple boundary tractions (stresses) to boundary displacements. Thus the dimensionality of the problem is reduced by one; only boundary geometry and data are discretized. Stresses at any set of selected interior points are computed following the boundary solution without any further numerical approximations. Thus, the BIE method has inherently greater resolution capability for stress gradients than does the finite element method.

Conversely, the BIE method is not efficient for problems involving significant inhomogeneity such as in multi-thin-layered materials, or in elastoplasticity. Some progress in applying the BIE method to the latter problem has been made but much more work remains. Further, the BIE method is only optional for problems with significant stress risers, and only when boundary stresses are most important. Interior stress calculations are expensive, per point, and can drive the solution costs up rapidly.

The current report summarizes some of the advanced elastic applications of fracture mechanics and three-dimensional stress analysis, while referencing some of the much broader developmental effort. Future emphasis is needed to exploit the BIE method in conjunction with other techniques such as the finite element method through the creation of hybrid stress analysis methods.

# 1. Overview of the Method

## 1.1 Introduction

Numerical analysis has become the basic tool for both design and research problems in solid mechanics. The need for accuracy and detail, plus the availability of the high speed computer has led to the development of many new modeling methods ranging from general purpose structural analysis programs to specialized research programs. The boundary-integral equation (BIE) method is one such program; it has been used for research problems but is now being implemented world-wide as a basic stress analysis tool for engineering application [Ref. 1].

The BIE method is one of a class of boundary solution methods. Most boundary methods in solid mechanics are derived from governing integral equations such as in classical potential theory [2] and complex variable methods [3], or on a variational basis such as collocation methods [4]. Boundary methods have the advantage of modeling the boundary data, rather than the interior data, with the result that problem size is reduced, and the interior solution may be obtained with greater resolution, with respect to finite element models.

The BIE method is derived from the application of classical potential theory used to provide solutions for both Laplace's and Navier's equations. Potential theory refers to the use of certain real variable integral identities that are Newtonian gravitational potentials for different mass (or charge) distributions [5].

Originally developed for solving Laplace's equations, potential methods were first applied to elasticity problems by Betti [6] to obtain relations for the internal dilation and rotation fields in terms of known boundary data. Somigliana [7] obtained similar integral identities for the internal displacement field. A number of extensions of these methods were investigated by Cerruti and Lauricella, whose major works are summarized by Love [8]. Fredholm [9] made use of similar integral identities to obtain conditions for the existence and uniqueness of solutions to the Dirichlet (displacement) problem; this work is the origin of the Fredholm alternative theorems in integral equation theory. Extension of the Fredholm proofs to the Neumann (traction) problem of elasticity, to elastodynamics, elastic inclusions, and anisotropic elasticity was accomplished by a number of investigators, including Mikhlin [10] and Kupradze [11].

## 1.2 Review of the Basic Equation

Boundary-integral equation (BIE) methods have been applied to elastic stress analysis of two and three dimensinal bodies (References [12, 13]) and show distinct advantages over other numerical methods such as the finite element methods. The advantages, as described further in Reference [14] include reduced problem size and increased solution accuracy, particularly for problems with high stress gradients such as elastic fracture mechanics problems.

The analytical basis of the BIE method is the replacement of the governing partial differential equations by an integral identity for the elastic displacements.

$$U_i(p) = \int_S T_{ij}(p,Q)u_j(Q)ds + \int_S U_{ij}(p,Q)t_j(Q)ds \tag{1}$$

In (1), $u_i(p)$ is the displacement vector at an interior point $p(x)$; $t_i(Q)$, $u_i(Q)$ are the boundary values of traction and displacement. The kernel functions $T_{ij}(p,Q)$, $U_{ij}(p,Q)$ are the tractions and displacements in the $x_i$ directions at $Q(x)$ due to orthogonal unit loads in the $x_j$ directions at $p(x)$.

Equation (1) contains the totality of boundary data; in general, a well-posed elasticity problem consists of specifying values of $t_j$ on part of the boundary, $S_t$, and $u_j$ on the remainder of the boundary $S_u$. Allowing $p(x) \rightarrow P(x)$, a boundary point, Eq. (1) becomes a set of integral constraint equations relating boundary displacements to boundary tractions; following References [12, 13] the BIE is obtained

$$U_i(p)/2 + \int_S T_{ij}(p,Q)u_j(Q)ds = \int_S U_{ij}(p,Q)\, t_j(Q)ds \tag{2}$$

In order to solve (2) for the unknown boundary data we approproximate the integral equation by a set of algebraic relations. The details of the numerical modeling of the BIE are discussed in the next section.

Interior stresses are obtained directly from the differentiation of Eq. (1) and application of Hooke's law (Ref. [13]). The interior stresses are obtained at only those interior points for which the solution is required; further, the numerical approximations are only in terms of the modeled boundary data.

## 2. Advanced Structural Modeling

### 2.1 Numerical Reduction of the BIE

The practical application of the BIE method requires the analysis of geometrically complex parts, frequently subject to some combination of gravitational, rotational, and thermoelastic loads in addition to surface mechanical loads. This section discusses some recent developments in the BIE method bearing on these types of applications.

A major problem in BIE modeling of complex parts is to establish a technique for the description of part geometry as well as of surface displacement and traction. One approach is to choose some suitable variation (most often piecewise linear) for these quantities in the global coordinate system (Ref. [14]). A second approach (Ref. [15]) is the use of isoparametric shape functions for the representation of both geometry and boundary data. The basis of the isoparametric method is the mapping of a planar curve or a surface patch to a standard interval or square by means of a fixed set of shape functions. For example, the three-dimensional analysis of Ref. [15] models geometry using quadratic shape functions. The three Cartesian coordinates are separately mapped, each varying quadratically along the edges of the square. Boundary displacements and tractions can also be expressed using isoparametric shape functions either of the same order as the geometric representation or of higher or lower order.

Figure 1 (taken from Ref. [15]) illustrates the ease with which complex geometries can be represented. The BIE model of the 15° sector of a pipe flange uses only 51 elements and 177 nodes despite the presence of several curved surfaces, including a doubly curved element. Figure 2 (also from Ref. [15]) shows the meridional stress on the outer surface for linear, quadratic and cubic representations of the boundary solution. There is a rather marked improvement in the results for quadratic variation but substantially less, at much increased cost, in the use of cubic variation.

The isoparametric approach requires the use of numerical integration to construct the equation system and will always be more costly, for the same model, than a technique (Ref. [14]) using exact integration. The advantage of the method lies in its ability to model much more efficiently the complex geometries encountered in practice.

### 2.2 Use of Substructuring and Hybridization

A different approach to the geometrical modeling problem is found in substructuring and hybridization. Substructuring refers to the BIE modeling of a part in two or more separate subregions which are then joined by enforcing appropriate continuity conditions on the common elements which form the interfaces between subregions (Ref. [15]).

Substructuring can effect substantial reductions in computing time, for both construction and solution of the equation system. The pipe flange problem of Figure 1 required 30% more time when run using only two subregions than when run using the four subregions shown. In addition to computing time reductions, substructuring can be used to allow analysis of multimaterial parts and to improve numerical conditioning by reducing overall aspect ratios in long, thin parts. The major disadvantage of substructiring is the approximation of displacement and traction imposed on the interface.

Another attractive technique for extending the usefulness of the BIE method is hybridization, that is, the joining of regions modeled using integral equations to regions modeled using other methods. Perhaps the most obvious type of hybridization is the merging of integral equation and finite element analyses. The combined analysis can offer many advantages with perhaps the most important being simultaneous access to the general structural modeling capability of the finite element method and the stress concentration and fracture mechanics capabilities of the BIE method. A two-dimensional merged program has been constructed (Ref. [16] from the CHILES finite element code (Ref. [1] and the

BIE/CRX code as described in the next section. The main technical problem in the hybridization process is the development of an algorithm for joining the finite element and BIE regions which allows consistent load transfer across the interface. A general formulation has been developed (Ref. [16]) which displays the effect of different region joining strategies.

Incorporation of gravity and rotational loading in BIE analysis has been discussed in Refs. [18] and [19]. These same references also discuss a technique for reducing the volume integral required for thermoelastic analysis to a surface integral. The method's general usefulness is limited by its explicit dependence on the steady state heat equation. This causes rather large sensitivity to the accuracy of both boundary temperatures and heat fluxes, which are normally numerically determined. A more promising approach, especially for problems involving transient thermal loads, would be based on construction of a special numerical volume integration rule to allow the efficient calculation of the thermal load vector. It is also necessary, particularly in transient thermoelastic problems, to develop effective techniques for dealing with inhomogeneous material properties. One possible approach is discussed in Ref. [20], others are being explored.

## 3. Fracture Mechanics Applications

### 3.1 Singularity Analysis

The BIE method of elastic stress analysis can be used directly, with certain limitations, to model cracked structures. A limitation in the basic formulation of the elasticity problem precludes direct modeling of two, co-planer crack surfaces, as discussed in detail in Ref. [18]. Essentially, the formulation of the BIE cannot distinguish the motion of the two surfaces, resulting in a singular matrix (dependent rows) in the numerical model.

Two modeling strategies have been developed and reported in Ref. [2] for circumventing the limitation. The first, and least desirable, models the flat crack as an open notch with a small but finite notch width; the obvious shortcoming of this approach is decreased accuracy and sensitivity to notch width. The second is to pass a plane through the structure including the crack plane; the limitation to accuracy here is the size of the elements required to describe the traction singularity on the plane ahead of the crack. This second strategy has been quite successfully applied to three-dimensional surface cracks, as described in Section 3.2.

A modified formulation of the BIE method, referred to herein as BIE/CRX, has been successfully developed for two-dimensional problems. The method, described in Ref. [22], incorporates a modified fundamental solution or Green's function which contains the full elastic representation of a flat, traction-free crack. The formulation results in an automatic inclusion of the presence of the crack without the need to model the crack surface geometry or crack-opening displacements. The BIE/CRX model is, of course, more expensive than a standard BIE model for a given number of boundary nodes. Some results for the BIE/CRX method are reviewed in Section 3.3.

### 3.2 Three-Dimensional Problems

The earliest study of a three-dimensional fracture mechanics problem (Ref. [23, 24]) showed the BIE method was capable of discerning variations in the stress intensity factor, but was not capable of adequately predicting the magnitudes. Further work, with a higher-order version (Ref. [1, 14, 26]) has greatly improved the BIE capability for surface cracks. The method has been used as the basis of fatigue life predictions for gas turbine engine disks as discussed in Ref. [27].

Figure 3 compares data from the BIE corner crack results in Ref. [26], together with results from the alternating technique by Kobayashi, Ref. [28] and from singularity finite element analysis by Tracey, Ref. [29]. It is seen that the BIE and finite element results, which both use crack opening displacement data, agree quite well; the alternating results, based on the crack front stresses show greater variability, particularly at the free surface. Earlier BIE data based on stress results (Ref. [23]) show the same type of free surface variation.

### 3.3 Two-Dimensional Problems

The essential attribute of the BIE/CRX formulation reported in Ref. [22] is that accuracy as good as handbook data, generally obtained by collocation techniques, can be obtained with ease. Work since the original publication has focused on a higher-order (linear) data model with improved efficiency and accuracy for mixed mode and edge crack problems (Ref. [30]).

Table I summarizes some of the features of the BIE/CRX method by comparison to various other techniques for a plate (6 in. long by 2 in. wide) with a crack ($a = \sqrt{2}/2$ in.) at 45° to the axial, applied tensile load. A fully converged BIE/CRX analysis is used as the reference solution. Two BIE/CRX models were chosen to compare low-cost results to the reference solution. In addition, the path-independent integral for $K_I$, $K_{II}$ analysis (PIIKA) from Ref. [22] was used to obtain $K_I$, $K_{II}$ results from two different finite element program results. The two finite element programs are CHILES (Ref. [17]) which has a singularity element, and F768, a standard, constant-strain finite element program in use at PWA. The finite element map used is shown in Fig. 4. It is seen that the BIE/CRX program is very cost effective.

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Table I: Mixed-Mode Stress Intensity Factor Data

| Program | Breakup | $K_I/\sigma\sqrt{\pi a}$ | $K_{II}/\sigma\sqrt{\pi a}$ | % Deviation $K_I$ | % Deviation $K_{II}$ | CPU Time (sec.) |
|---|---|---|---|---|---|---|
| Reference | 40 nodes | .712 | .590 | – | – | 46 |
| BIE/CRX | 16 nodes | .671 | .562 | +5.7 | +4.8 | 8 |
| | 24 nodes | .710 | .577 | +0.2 | +2.3 | 17 |
| PIIKA | F768 Outer Boundary | .708 | .580 | +0.5 | +1.8 | 3+44 |
| | CHILES Outer Boundary | .728 | .590 | –2.3 | 0.0 | 3+48 |
| CHILES | F768 | .741 | .597 | –4.1 | –1.1 | 48 |
| F768 | 668 nodes 612 elements | .708 | .556 | +0.5 | +5.8 | 44 |

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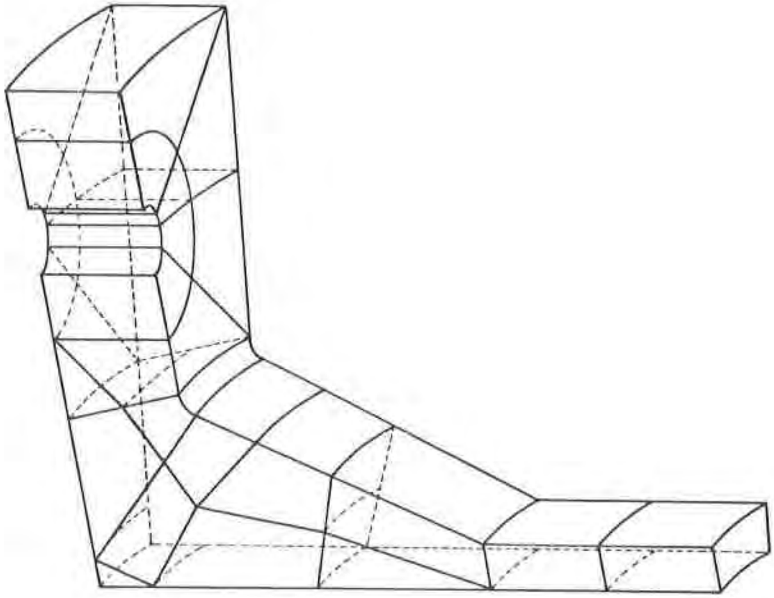

Figure 1 - BIE Model of Pipe Flange (15° Sector)

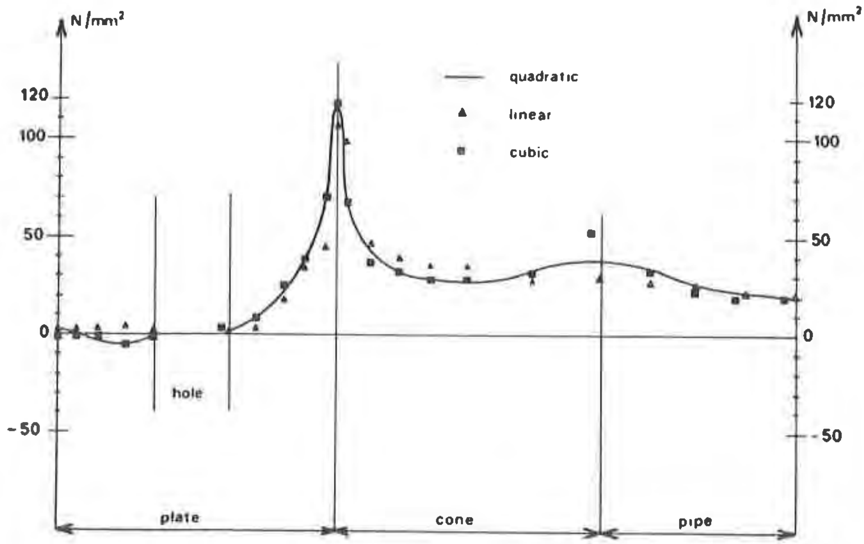


Figure 2 - Meridional Stress on Outer Surface of Pipe Flange

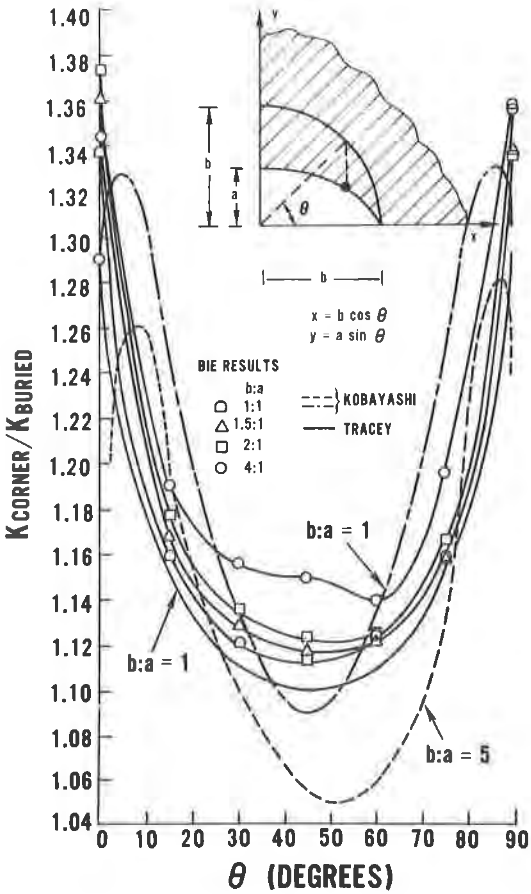


Figure 3 - Comparison of Corner Crack Stress Intensity Factor Calculations

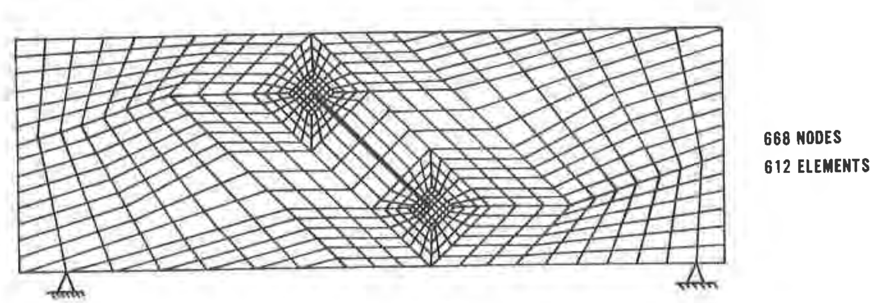


Figure 4 - F768 Finite Element Map for Angle Crack Problem